Title: Dual Theory of Decaying Turbulence

Speakers: Alexander Migdal

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Abstract: We investigate the recently found \cite{migdal2023exact} reduction of decaying turbulence in the Navier-Stokes equation in $3 + 1$ dimensions to a Number Theory problem of finding the statistical limit of the Euler ensemble.
We reformulate the Euler ensemble as a Markov chain and show the equivalence of this formulation to the quantum statistical theory of free fermions on a ring, with an external field related to the random fractions of $\pi$.

We find the solution of this system in the statistical limit $N\to \infty$ in terms of a complex trajectory (instanton) providing a saddle point to the path integral over the charge density of these fermions.

This results in an analytic formula for the observable correlation function of vorticity in wavevector space. This is a full solution of decaying turbulence from the first principle without assumptions, approximations, or fitted parameters.
We compute resulting integrals in \Mathematica{} and present effective indexes for the energy decay as a function of time Fig.\ref{fig::NPlot} and the energy spectrum as a function of the wavevector at fixed time Fig.\ref{fig::SPIndex}.

In particular, the asymptotic value of the effective index in energy decay $n(\infty) = \frac{7}{4}$, but the universal function $n(t)$ is neither constant nor linear.

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Zoom link

# Quantum Solution of Classical Turbulence

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Talk at PI colloquium:
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# Introduction



The turbulence problem looks deceptively simple: find the limit of the solution of the Navier-Stokes equations when viscosity goes to zero at a fixed energy dissipation rate.

$$\partial_t \vec{v} = -\nu \vec{\nabla} \times \vec{\omega} + \vec{v} \times \vec{\omega} - \vec{\nabla} \left( p + \frac{\vec{v}^2}{2} \right); \quad (1)$$

$$\vec{\nabla} \cdot \vec{v} = 0; \quad (2)$$

In this limit, the Navier-Stokes equation tends to the Euler equation everywhere except some singular regions: Vortex sheets and lines, where large velocity gradients could compensate the factor of $\nu$. These regions are randomly distributed in space, making velocity and vorticity stochastic variables at every point, with local vorticity values divergent in the turbulent limit.

# Dissipation anomaly



In the original theory

$$\mathcal{E} = \lim_{\nu \to 0, r \to 0} \nu \left\langle \vec{\omega}(0) \cdot \vec{\omega}(\vec{r}) \right\rangle \propto \nu r^{2\alpha - 2} \to \textit{ const } ; \; \alpha \approx \frac{1}{3} \quad (3)$$

In the dual theory of this talk

$$\mathcal{E} \propto \lim_{\nu \to 0, N \to \infty} \nu N^2 \to \textit{ const } ; \quad (4)$$

In both cases, the UV divergences renormalize the initial viscosity to a physical (turbulent) viscosity, staying finite in the local limit.

# Hopf Functional

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Hopf outlined in 1952 the general approach to the turbulence problem using the functional

$$Z[\vec{J}] = \left\langle \exp\left( \int_{\vec{r}\in\mathbb{R}_d} \vec{J}(\vec{r}) \cdot \vec{v}(\vec{r}) \right) \right\rangle \tag{5}$$

This functional generates the correlation functions of the velocity field and satisfies the functional differential equation of the form

$$\partial_t Z[\vec{J}] = \hat{H}\left[\vec{J}, \frac{\delta}{\delta \vec{J}}\right] Z[\vec{J}] \tag{6}$$

The turbulence corresponds to a degenerate fixed point of the Navier-Stokes dynamics for $Z$, in the same way as the Gibbs distribution is a degenerate fixed point $Z = \delta(E - H(\vec{p}, \vec{q}))$ of Newton's dynamics (independent of the position at the energy surface $H(\vec{p}, \vec{q})$).

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The decaying turbulence is a degenerate fixed trajectory, slowly approaching the stable fixed point at zero velocity due to friction forces represented by viscosity.

The Hopf equation for the Navier-Stokes dynamics is compatible with such a trajectory, but it is too general and too complex to compute anything.

Its complexity is equivalent to the non-Gaussian functional integral, misplacing turbulence in the same category as critical phenomena in statistical physics.

It is much simpler in our theory.

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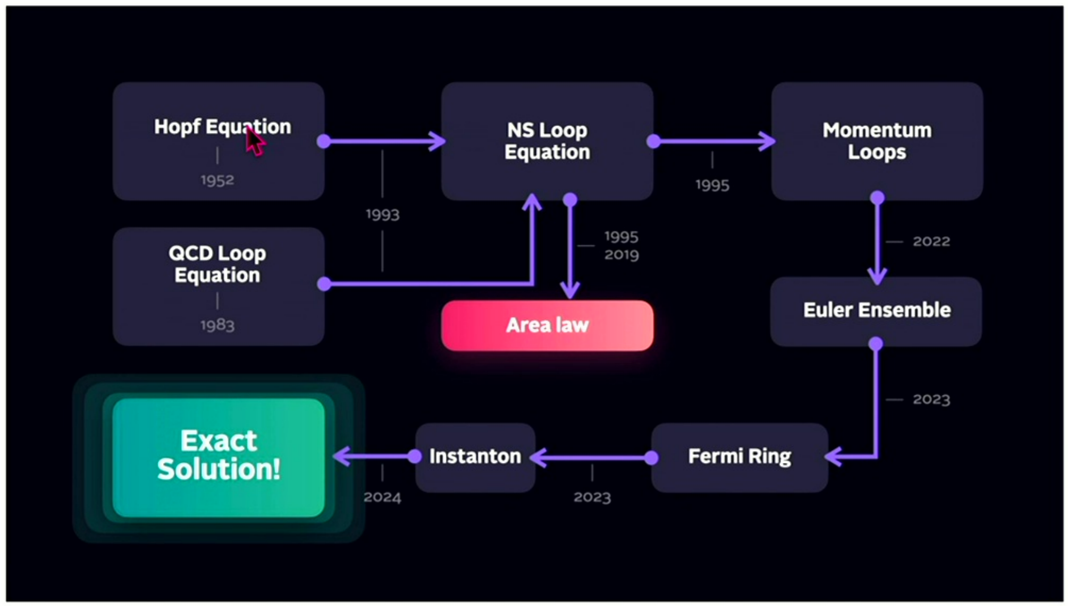
# Solution of the Hopf functional equation





# Loop Average and dimension reduction



The loop average is a particular case of the Hopf functional with the source $\vec{J}(\vec{r})$ concentrated on a fixed loop in space

$$\vec{J}_C(\vec{r}) = \frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta)\delta\left(\vec{r} - \vec{C}(\theta)\right) \tag{7}$$

The loop average is defined as

$$\Psi[\gamma, C] = \left\langle \exp\left(\int_{\vec{r}\in\mathbb{R}_d} \vec{J}_C(\vec{r}) \cdot \vec{v}(\vec{r})\right)\right\rangle =$$

$$\left\langle \exp\left(\frac{\imath\gamma}{\nu}\Gamma_C\right)\right\rangle; \tag{8}$$

$$\Gamma_C = \oint d\vec{C}(\theta) \cdot \vec{v}(\vec{C}(\theta)); \tag{9}$$

# Loop equation as dimension reduction



We derived a closed functional equation for the loop average in incompressible Navier-Stokes equation**M93, M23PR**

$$\imath\nu\partial_t\Psi[\gamma, C] =$$
$$\left\langle \gamma \oint d\vec{C}(\theta) \cdot \left(-\nu\vec{\nabla} \times \vec{\omega} + \vec{v} \times \vec{\omega}\right) \exp\left(\frac{\imath\gamma}{\nu}\Gamma_C\right)\right\rangle =$$
$$\oint d\vec{C}(\theta) \cdot \vec{L}\left[\frac{\delta}{\delta\vec{C}(.)}\right]\Psi[\gamma, C] \qquad (10)$$

The operator $\vec{L}\left[\frac{\delta}{\delta\vec{C}(.)}\right]$ only depends on the functional derivative, but does not depend on the coordinate $\vec{C}(.)$ in loop space.

This independence (translation invariance) is the key to the solution. External forces would break it.

# Loop equation as dimension reduction



This equation is equivalent to the Schrödinger equation in loop space with Hamiltonian $\hat{H}_C = \oint d\vec{C}(\theta) \cdot \vec{L}\left[\frac{\delta}{\delta \vec{C}(.)}\right]$.

A plane wave in loop space solves this Schrödinger equation

$$\Psi[\gamma, C] = \left\langle \exp\left(\frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta) \cdot \vec{P}(t,\theta)\right)\right\rangle; \tag{11}$$

$$\imath\gamma\partial_t \vec{P} = \vec{L}\left[-\imath\frac{\gamma}{\nu}\partial_\theta \vec{P}(t,\theta)\right]; \tag{12}$$

$$\nu\partial_t \vec{P} = -\gamma^2(\Delta\vec{P})^2\vec{P} + \Delta\vec{P}\left(\gamma^2\vec{P}\cdot\Delta\vec{P} + \imath\gamma\left(\frac{(\vec{P}\cdot\Delta\vec{P})^2}{\Delta\vec{P}^2} - \vec{P}^2\right)\right); \tag{13}$$

with $\Delta\vec{P} = \vec{P}(\theta+0) - \vec{P}(\theta-0), \vec{P} = \frac{\vec{P}(\theta+0)+\vec{P}(\theta-0)}{2}$.

# Duality

$$\vec{\nabla} \Rightarrow -\frac{\imath\gamma}{\nu}\Delta\vec{P} \tag{14}$$

The relation of the operators in the QCD loop equation to the discontinuities of the momentum loop was noticed, justified, and investigated in **SQM95, M98H**.

The momentum loop in QCD could be piecewise constant with an arbitrary number of such discontinuities.

In the Navier-Stokes theory, the discontinuities must be present at every point on a parametric circle $\mathbb{S}_1$, making the curve $\vec{P}(\theta, t)$ a fractal.

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We find the local limit for vorticity

$$\Omega_{\alpha\beta}(\theta, 0^+) = \frac{-\imath\gamma}{\nu} P_{\alpha\beta}(\theta); \tag{15}$$

$$P_{\alpha\beta}(\theta) = \Delta P_\alpha(\theta) P_\beta(\theta) - \{\alpha \leftrightarrow \beta\}; \tag{16}$$

$$P_\alpha(\theta) \equiv \frac{P_\alpha(\theta^+) + P_\alpha(\theta^-)}{2} \tag{17}$$

and velocity (skipping the common argument $\theta$ )

$$V_\alpha = \frac{\Delta P_\beta}{\Delta P_\mu^2} P_{\beta\alpha} = P_\alpha - \frac{\Delta P_\alpha \Delta P_\beta P_\beta}{\Delta P^2} \tag{18}$$

The Bianchi constraint is identically satisfied as it should

$$\Delta P_\alpha \left(\Delta P_\beta P_\gamma - \{\beta \leftrightarrow \gamma\}\right) + cyclic = 0 \tag{19}$$

# Exact solution



This singular nonlinear equation discouraged us back in the nineties.

Surprisingly, an infinite family of analytic solutions was found.

$$\vec{P}(t,\theta) = \sqrt{\frac{\nu}{2(t+t_0)}}\hat{\Omega} \cdot \frac{\vec{F}(\theta)}{\gamma};\ \hat{\Omega} \in O(3); \tag{20}$$

$$\vec{F}_k = \frac{\left\{\cos(\alpha_k), \sin(\alpha_k), i\cos\left(\frac{\beta}{2}\right)\right\}}{2\sin\left(\frac{\beta}{2}\right)}; \tag{21}$$

$$\theta_k = \frac{2\pi k}{N};\ \beta = \frac{2\pi p}{q};\ N \to \infty; \tag{22}$$

$$\alpha_{k+1} = \alpha_k + \sigma_k\beta;\ \sigma_k = \pm 1,\ \beta\sum \sigma_k = 2\pi p r; \tag{23}$$

The parameters $\hat{\Omega}, N, p, q, r, \sigma_0 \dots \sigma_{N-1}$ are random, making this solution for $\vec{F}(\theta)$ a fixed manifold rather than a fixed point.

# Loop equation as dimension reduction

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# Euler ensemble



This is not a toy model but an exact stochastic solution of the 3D Navier-Stokes equation.

We treat it as a quantum statistical system with a chemical potential $\mu \to 0$ (the Euler ensemble). The partition function is calculable

$$Z(\mu) = \sum_N e^{-\mu N} \sum_{2<q<N} \varphi(q) \sum_{\substack{r \\ 2|(N-qr)}} 2^{-N} \binom{N}{(N+qr)/2}; \qquad (24)$$

$$\varphi(q) = q \prod_{p|q} \left(1 - \frac{1}{p}\right); \qquad (25)$$

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$$\varphi(q) = q \prod_{p|q} \left(1 - \frac{1}{p}\right); \tag{25}$$

$$Z(\mu) \to \frac{9}{4\sqrt{2}\pi^2 \mu^{5/2}}; \tag{26}$$

# Markov process as Fermionic trace



Now, we observe that quantum Fermi statistics can represent the Markov chain of Ising variables.

Let us construct the operator $\hat{Q}(M)$ with Fermionic creation and annihilation operators, with occupation numbers $\nu_k = (1 + \sigma_k)/2 = (0, 1)$.

$$\left[a_i, a_j^\dagger\right]_+ = \delta_{ij}; \tag{29}$$

$$\left[a_i, a_j\right]_+ = \left[a_i^\dagger, a_j^\dagger\right]_+ = 0; \tag{30}$$

# Markov process as Fermionic trace



The number $n(M)$ of positive sigmas $\sum_{l=1}^{M} \delta[\sigma_l - 1]$ coincides with the occupation number of these Fermi particles.

$$\hat{n}(M) = \sum_{l=1}^{M} \hat{\nu}_l; \tag{35}$$

This relation leads to the representation

$$\hat{Q}(M) = \hat{\nu}_M \frac{\hat{n}(M)}{M} + (1 - \hat{\nu}_M) \frac{M - \hat{n}(M)}{M}; \tag{36}$$

The variables $\sigma_l$ can also be expressed in terms of this operator algebra by using

$$\hat{\sigma}_l = 2\hat{\nu}_l - 1. \tag{37}$$

# Markov process as Fermionic trace



$$\Delta \vec{C}_l = \vec{C}\left(\frac{l+1}{N}\right) - \vec{C}\left(\frac{l}{N}\right), \tag{41}$$

$$\vec{\mathcal{P}}_l(t) = \sqrt{\frac{\nu}{2\,(t+t_0)}}\frac{\vec{\mathcal{F}}_l}{\gamma}, \quad \hat{\Omega} \in O(3), \tag{42}$$

$$\vec{\mathcal{F}}_l = \frac{\{\cos(\hat{\alpha}_l), \sin(\hat{\alpha}_l), 0\}}{2\sin\left(\frac{\beta}{2}\right)}, \tag{43}$$

$$\mathcal{E}(N): \quad p, q, r \in \mathbb{Z} \quad \text{with} \quad 0 < p < q < N, \tag{44}$$

$$\mathbf{gcd}(p, q) = 1, \quad -N \leq qr \leq N, \tag{45}$$

$$\hat{\alpha}_l = \beta \sum_{k=1}^{l-1} (2\hat{\nu}_k - 1); \tag{46}$$

# Path Integral



This path integral will be dominated by the "classical history," maximizing the product of transitional probabilities if such a classical trajectory exists.

$$\max_{\phi} \Lambda_N[\phi]; \tag{47}$$

$$\Lambda_N[\phi] = \int_0^1 d\xi \left( \frac{df_+}{d\xi} \log\left(\frac{f_+}{\xi}\right) + \frac{df_-}{d\xi} \log\left(\frac{f_-}{\xi}\right)\right); \tag{48}$$

$$f_\pm(\xi) = \frac{1}{2}\left(\xi \pm \phi(\xi)\right); \tag{49}$$

$$f_\pm(\xi) \geq 0; \tag{50}$$

The solution is independent of $\phi(\xi)$:

$$\lim_{N\to\infty} \Lambda_N[\phi] = -\frac{1}{2}(1-s)\log(1-s) - \frac{1}{2}(1+s)\log(1+s) + \log(2) \tag{51}$$

# Vorticity Correlation



The correlation function reduces to the following average over the big Euler ensemble $\mathbb{E}$ **M23ES**

$$\left\langle \vec{\omega}(\vec{0}) \cdot \vec{\omega}(\vec{r}) \right\rangle = \frac{\left\langle \sum_{0 \le n < m < N} \vec{\omega}_m \cdot \vec{\omega}_n \exp\left(\imath \vec{\rho} \cdot \left(\vec{S}_{n,m} - \vec{S}_{m,n}\right)\right) \right\rangle_{\mathbb{E}(\mu)}}{4t^2}; \quad (52)$$

$$\vec{S}_{n,m} = \frac{\sum_{k=n}^{m-1} \vec{F}_k}{(m-n) \pmod{N}}; \quad (53)$$

# Vorticity Correlation



In the continuum limit, we replace summation with integration and get the following expression for the correlation function

$$\left\langle \vec{\omega}(\vec{0}) \cdot \vec{\omega}(\vec{r}) \right\rangle \propto$$

$$\frac{1}{t^2 \Phi(N)} \sum_{\text{even } q<N} \sum_{p;\,(p|q)} \frac{\cot^2(\beta/2)}{\beta^2} \int_{0<\xi_1<\xi_2<1} d\xi_1 d\xi_2$$

$$\frac{\int_{O(3)} d\Omega \int [D\alpha] \alpha'(\xi_1)\alpha'(\xi_2) \exp\left( \imath \frac{\vec{r} \cdot \hat{\Omega} \cdot \mathbf{Im}\, \vec{V}(\xi_1,\xi_2)}{\sqrt{\nu t}} \right)}{|O(3)| \int [D\alpha]}; \tag{55}$$

$$\vec{V}(\xi_1,\xi_2) = q\sqrt{X}\,\{\imath,1,0\}\,(S(\xi_1,\xi_2) - S(\xi_2, 1+\xi_1))\,; \tag{56}$$

$$S(a,b) = \frac{\int_a^b d\xi \exp(\imath\alpha(\xi))}{b-a}; \tag{57}$$

# The decaying energy in finite system



This energy pumping **SD** takes place at $t < t_0$, after which the pumping stops. At this moment, the energy spectrum is growing with wavevector by one of two possible laws (with $P$ being the net momentum of the fluid and $M$ being the rotation moment)

# Instanton computations



This estimate then requires vanishing viscosity in the local limit, at fixed turbulent viscosity

$$\tilde{\nu} = \nu N^2 \to \ \mathit{const}\ . \qquad (60)$$

This phenomenon of renormalization of viscosity by a factor of $N^2$ was already observed in our first paper M23ES.

Our Euler ensemble in the local limit $N \to \infty$ can only solve the inviscid limit of the Navier-Stokes decaying turbulence, with finite $\tilde{\nu}$ acting as a turbulent viscosity.

# The decaying energy in finite system



This energy pumping **SD** takes place at $t < t_0$, after which the pumping stops. At this moment, the energy spectrum is growing with wavevector by one of two possible laws (with $P$ being the net momentum of the fluid and $M$ being the rotation moment)

$$\begin{cases} E(k, t_0) \propto P k^2 \\ E(k, t_0) \propto M k^4 \end{cases} \tag{61}$$

At $t > t_0$, without the forcing, the pumped energy dissipates at large $k$ corresponding to smaller spatial scales of the hierarchy of vortex structures of all scales, ending with dissipative scales, or wavevectors $k \gg \pi/L$. After sufficient time, the universal regime kicks in, corresponding to the decaying turbulence.

# The decaying energy in finite system



Therefore, the decaying energy, given by the part of the spectrum $k > k_0 \sim 1/L$, has the following structure

$$t^2 \mathcal{E}(t) = 4\pi\tilde{\nu} F(\left(k_0\sqrt{\tilde{\nu} t}\right); \quad (62a)$$

$$F(\kappa) = \int_{\kappa}^{\infty} H(x) x^2 \, dx, \quad (62b)$$

$$\sqrt{t} E(k,t) = 4\pi\tilde{\nu}\sqrt{\tilde{\nu}} H\left(k\sqrt{\tilde{\nu} t}\right); \quad (62c)$$

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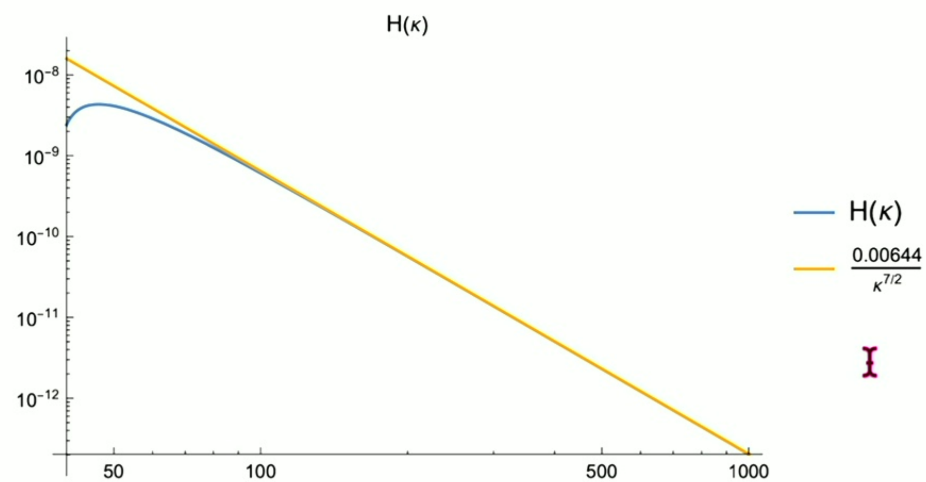

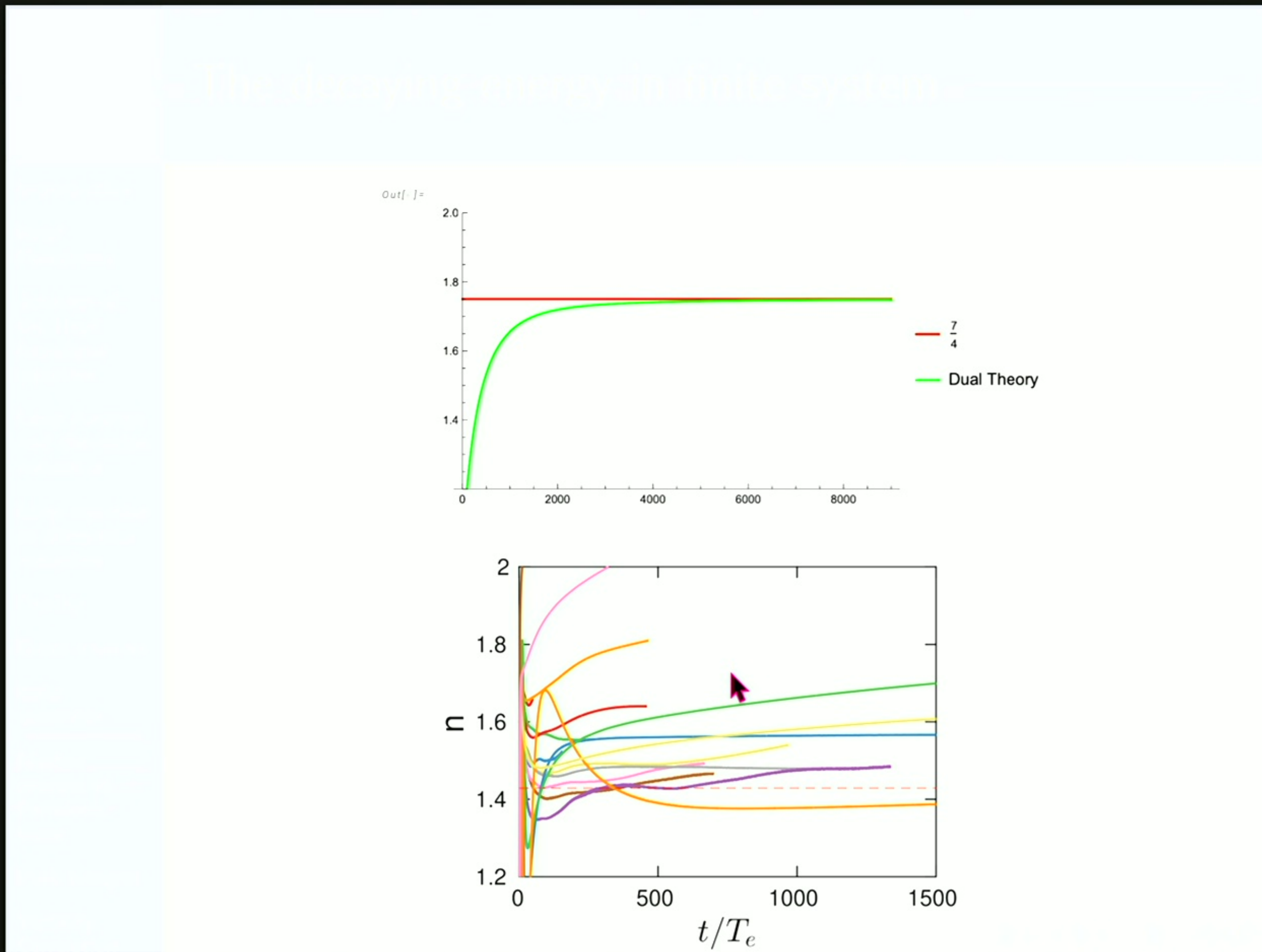
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