Title: Mathematical Physics Lecture

Speakers: Kevin Costello

Collection: Mathematical Physics 2023/24

Date: April 29, 2024 - 11:30 AM

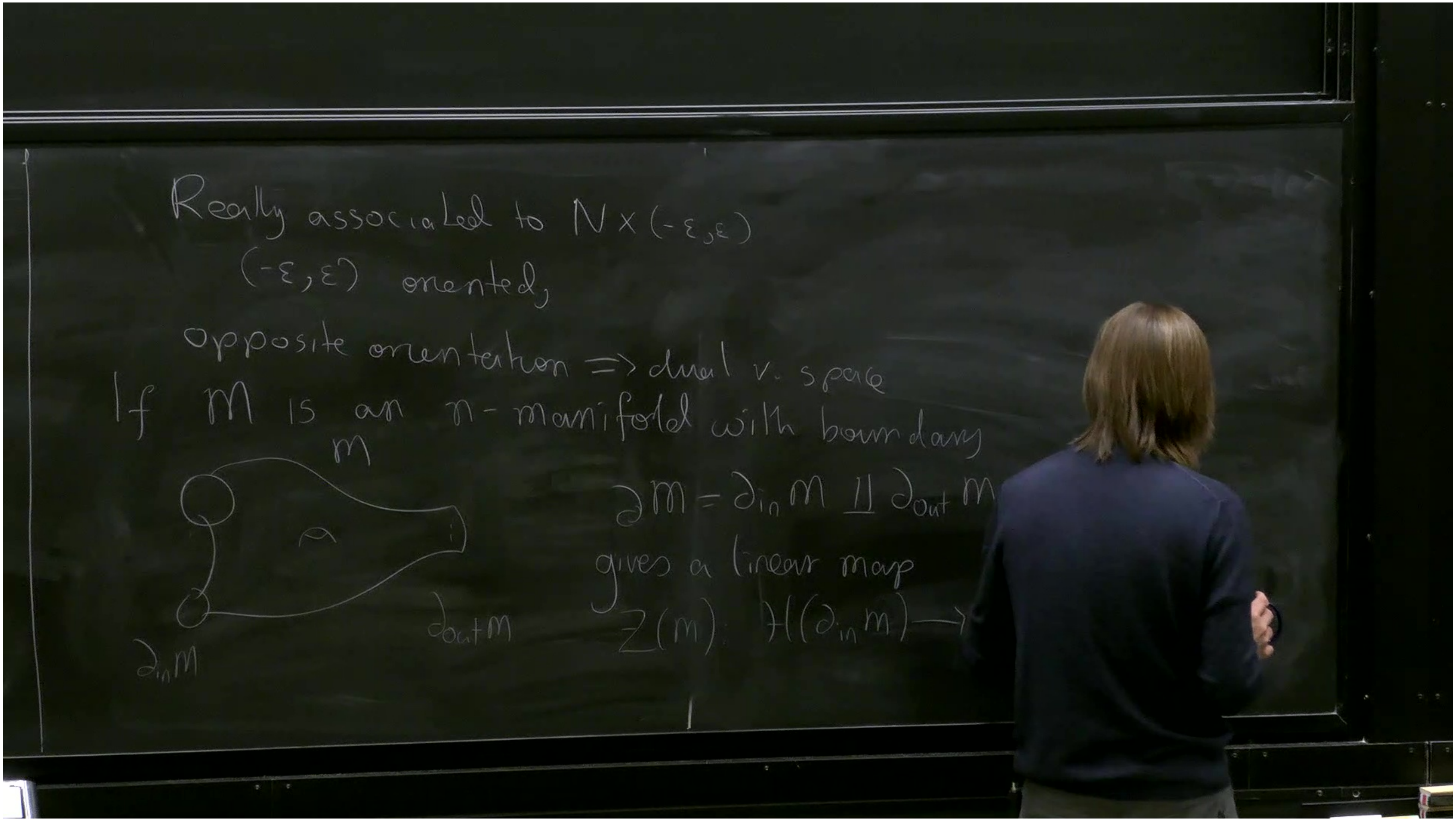
URL: https://pirsa.org/24040069

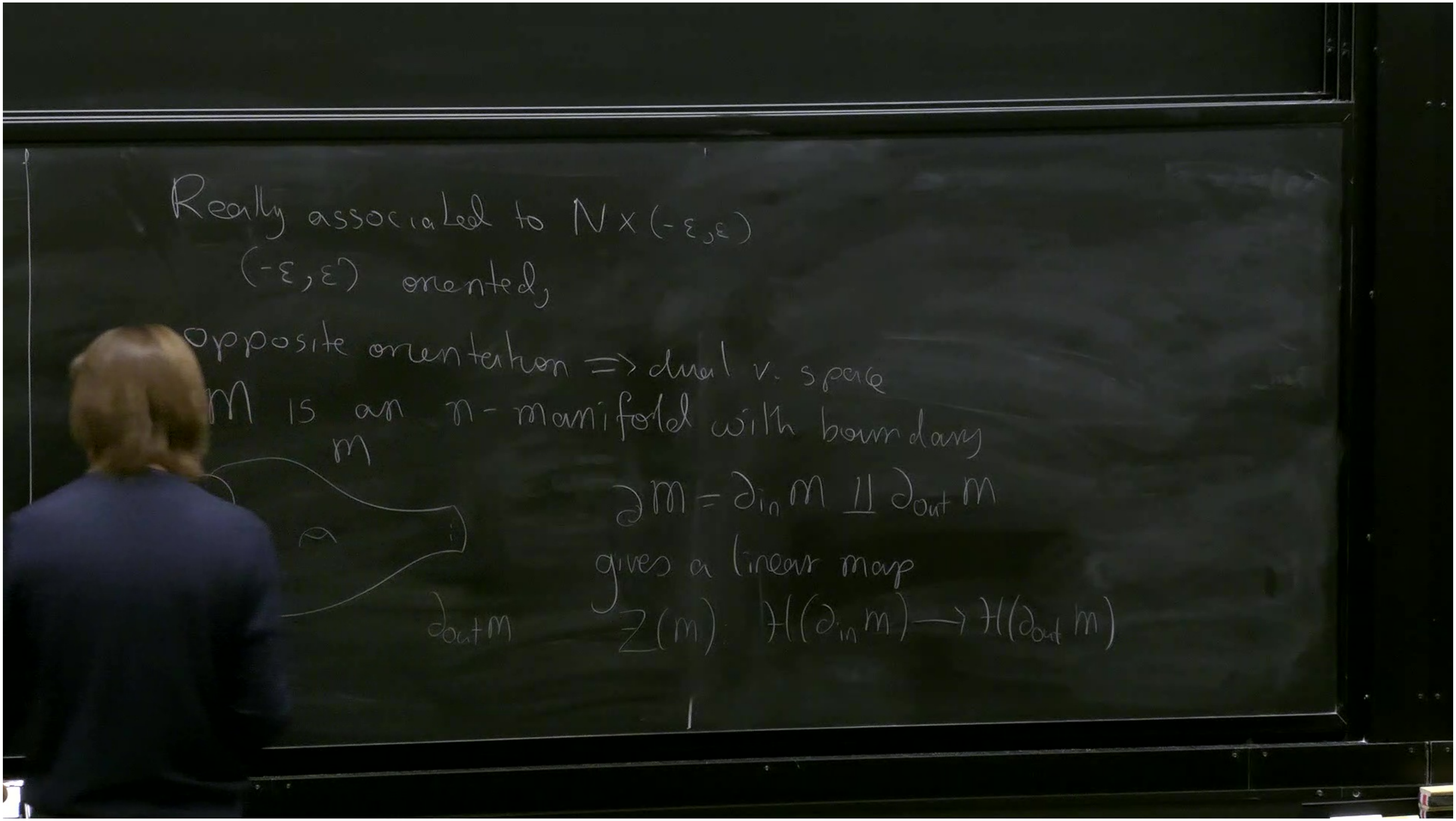
Atiyah-Segal axioms + classical limit

TFT in dim^n n (eg CS in dimension 3)
is the following data:

N^{n-1}, a compact $n-1$ manifold

$\rightsquigarrow \mathcal{H}(N^{n-1})$, a vector space (space of states)


Really associated to N × (-ε, ε)
(-ε, ε) oriented,
opposite orientation ⇒ dual v. space
If M is an n-manifold with boundary
M
∂M = ∂_in M ⊔ ∂_out M
gives a linear map
Z(M): H(∂_in M) ⟶
∂_out M
∂_in M


Really associated to N × (−ε, ε)
(−ε, ε) oriented,
opposite orientation ⇒ dual v. space
M is an n-manifold with boundary
M
∂M = ∂in M ⊔ ∂out M
gives a linear map
Z(M): H(∂in M) → H(∂out M)
∂out M

These must satisfy

1) $\mathcal{H}(\emptyset) = \mathbb{C}$

$\partial M^n = \emptyset$

$Z(M^n)$ is a number; the partition function

2) $\mathcal{H}(N \amalg N') = \mathcal{H}(N) \otimes \mathcal{H}(N')$

$Z(M \amalg M') = Z(M) \otimes Z(M')$

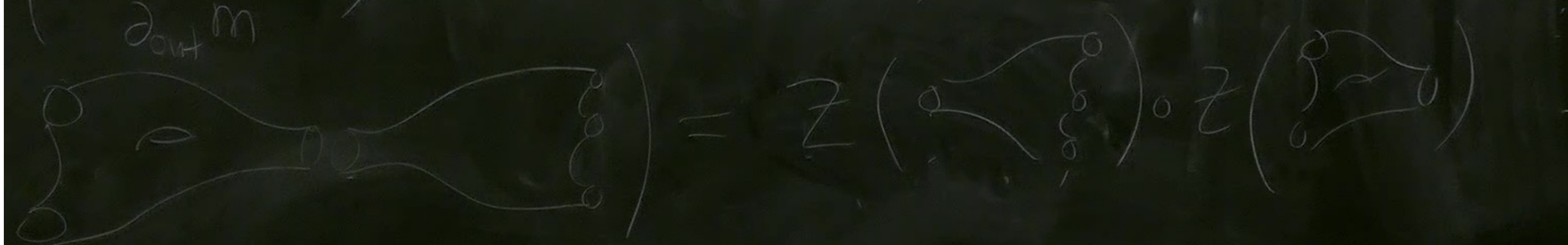
3) Glue

Glue manifolds

M, M'

$\partial_{out} M = \partial_{in} M'$

$$(M \cup_{\partial_{out} M} M') = Z(M') \circ Z(M)$$



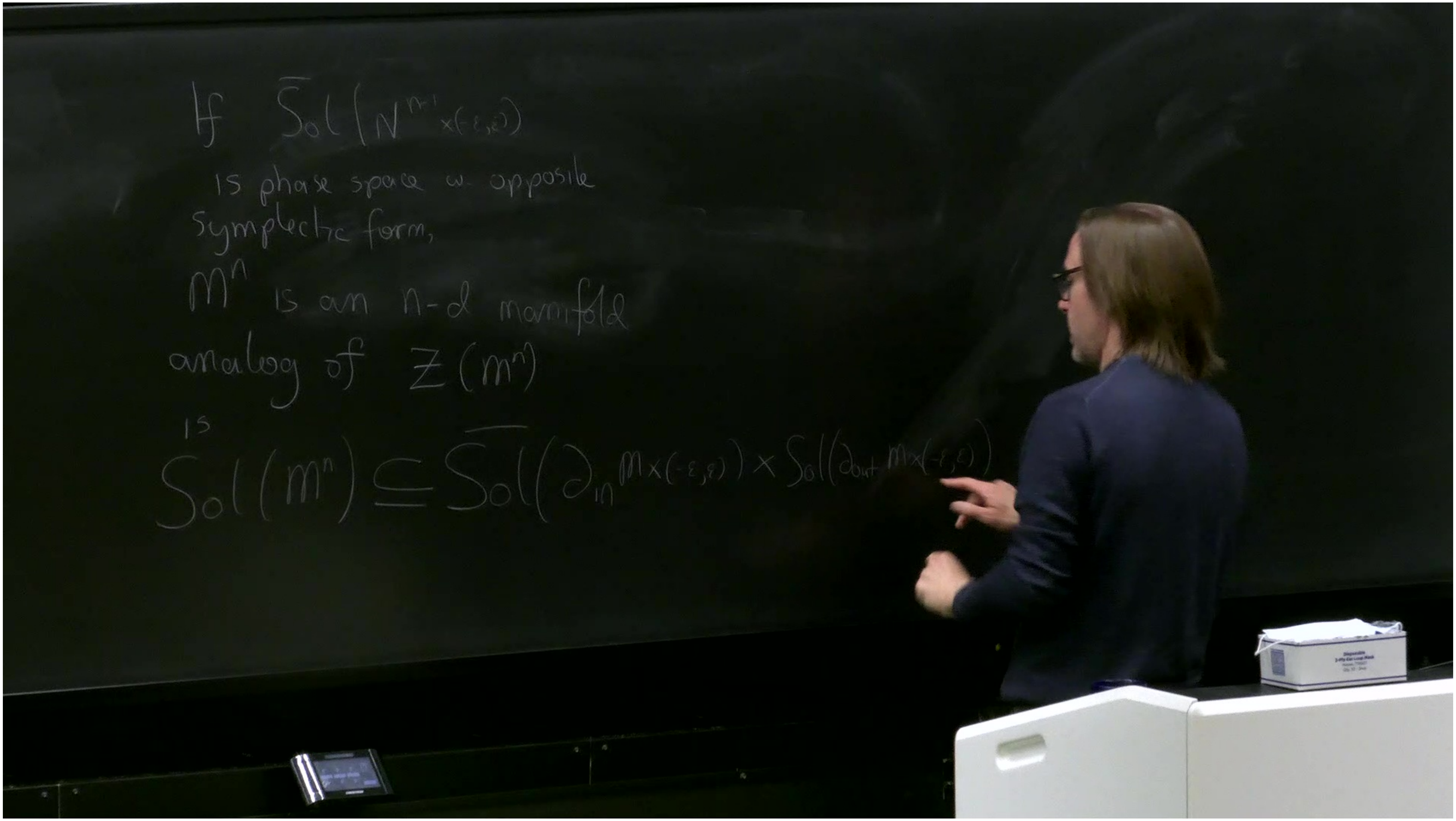
Semiclassically

$N^{n-1} \rightsquigarrow$ The phase space

$\mathrm{Sol}(N^{n-1} \times (-\varepsilon, \varepsilon))$

Hilbert space is geometric quantization of phase space

Reversing time changes sign of symplectic form


If $\overline{Sol}(N^{n-1} \times (-\varepsilon,\varepsilon))$
is phase space w. opposite
symplectic form,
M^n is an n-d manifold
analog of $Z(M^n)$
is
$Sol(M^n) \subseteq \overline{Sol}(\partial_{in} M \times (-\varepsilon,\varepsilon)) \times Sol(\partial_{out} M \times (-\varepsilon,\varepsilon))$

This is a Bohr-Sommerfeld Lagrangian

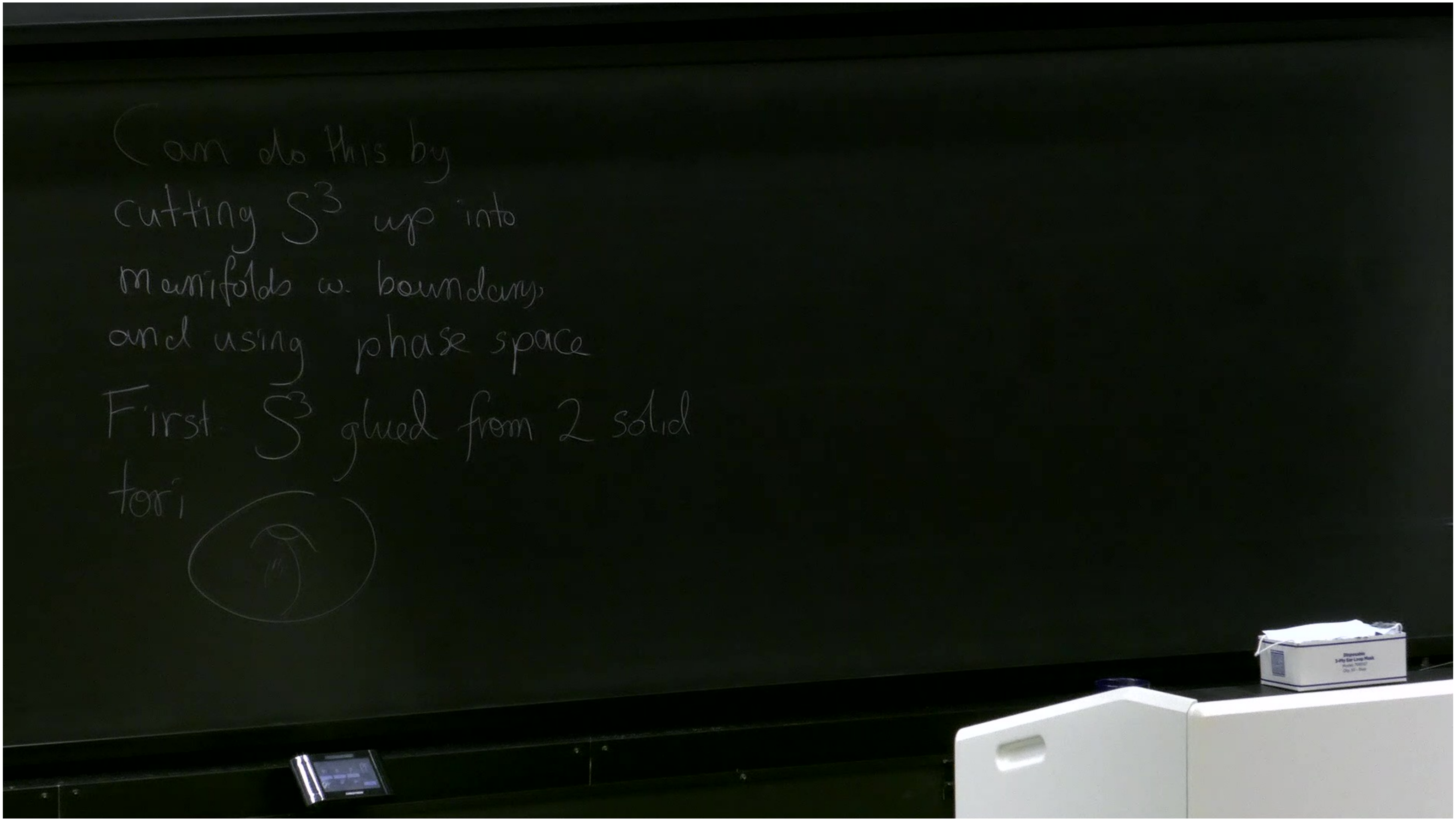
So classical limit of a state.

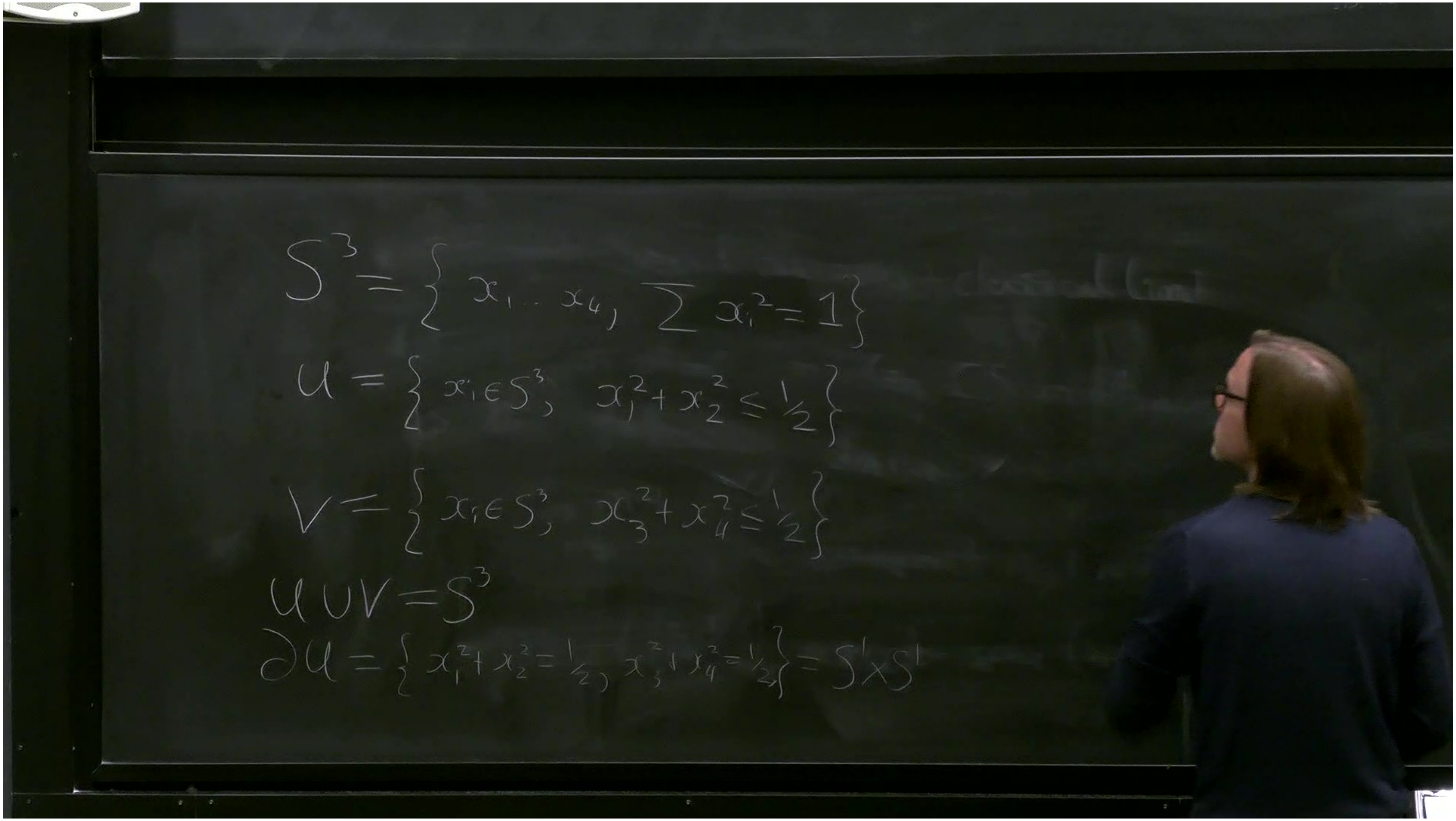
Want to understand e.g.

$$\int dA \; e^{k\int_{S^3} CS(A)} \, W_{\gamma_1}(A) \cdots W_{\gamma_n}(A)$$

CS on S^3

Wilson lines


Can do this by
cutting S^3 up into
manifolds w. boundary,
and using phase space
First S^3 glued from 2 solid
tori


$S^3 = \{ x_1 \ldots x_4,\ \sum x_i^2 = 1 \}$
$U = \{ x_i \in S^3,\ x_1^2 + x_2^2 \leq \frac{1}{2} \}$
$V = \{ x_i \in S^3,\ x_3^2 + x_4^2 \leq \frac{1}{2} \}$
$U \cup V = S^3$
$\partial U = \{ x_1^2 + x_2^2 = \frac{1}{2},\ x_3^2 + x_4^2 = \frac{1}{2} \} = S^1 \times S^1$

$$S^3 = \{ x_1 \dots x_4,\ \sum x_i^2 = 1 \}$$

$$U = \{ x_i \in S^3;\ x_1^2 + x_2^2 \le \tfrac{1}{2} \}$$

$$V = \{ x_i \in S^3;\ x_3^2 + x_4^2 \le \tfrac{1}{2} \}$$

$$U \cup V = S^3$$

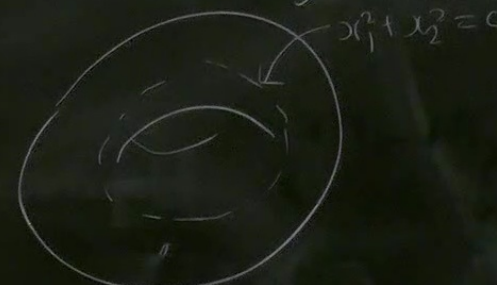
$$\partial U = \{ x_1^2 + x_2^2 = \tfrac{1}{2},\ x_3^2 + x_4^2 = \tfrac{1}{2} \} = S^1 \times S^1$$

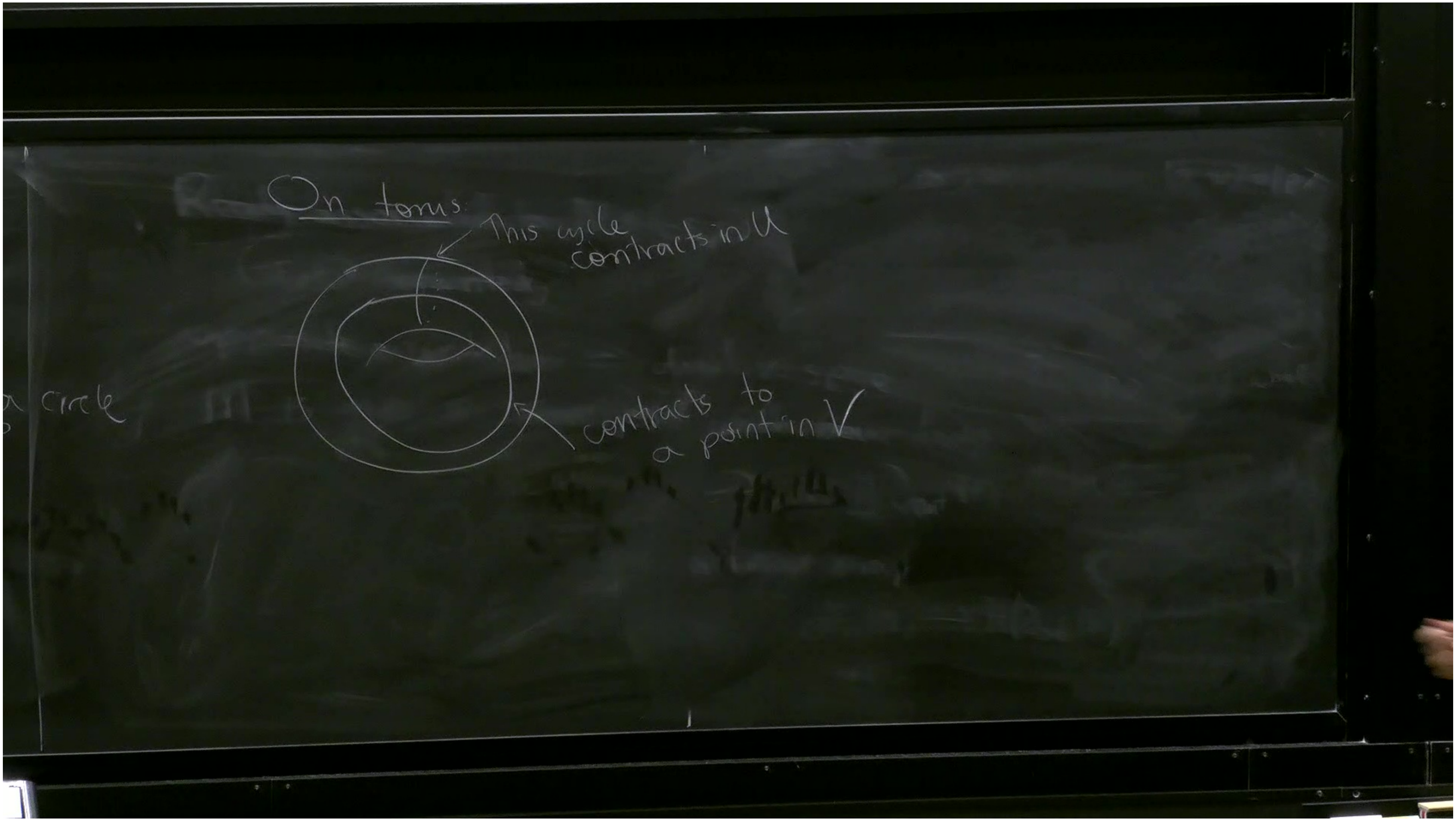
U is a solid torus

If $x_1^2 + x_2^2 = \varepsilon$, $\varepsilon > 0$

then have a torus

$\varepsilon = 0$, only have a c




On torus:
This cycle contracts in U
contracts to a point in V
circle

CS theory for $U(1)$ at level n

has operators on T^2 P, Q corresponding

to Wilson lines on the 2 cycles

of T^2, which satisfy

$$PQ = QPe^{2\pi i/n}$$

Hilbert space basis

$\langle \psi_k |$ with

$$\langle \psi_k | P = \langle \psi_{k+1} |$$

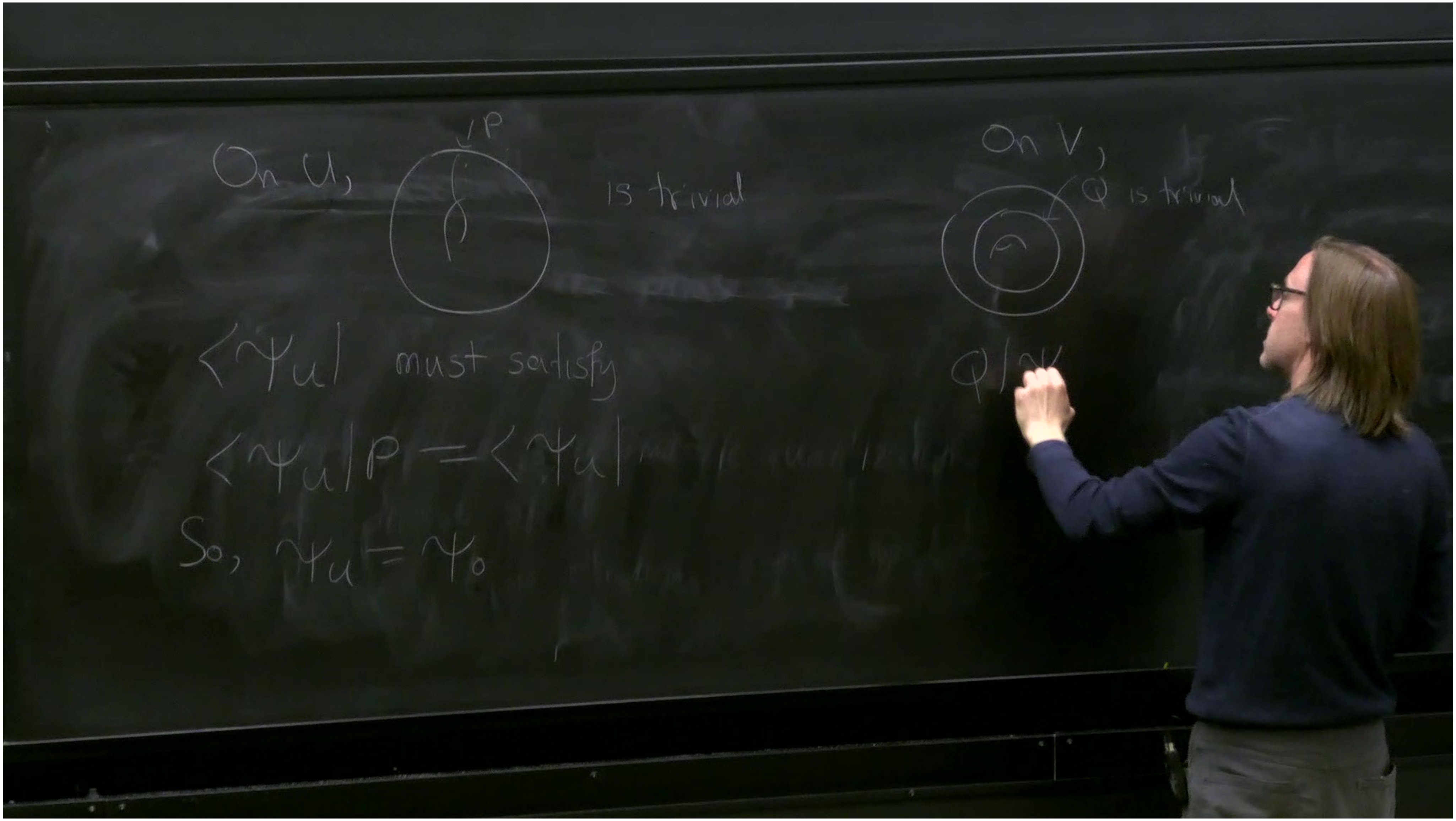
$$\langle \psi_k | Q = e^{2\pi i k/n} \langle \psi_k |$$

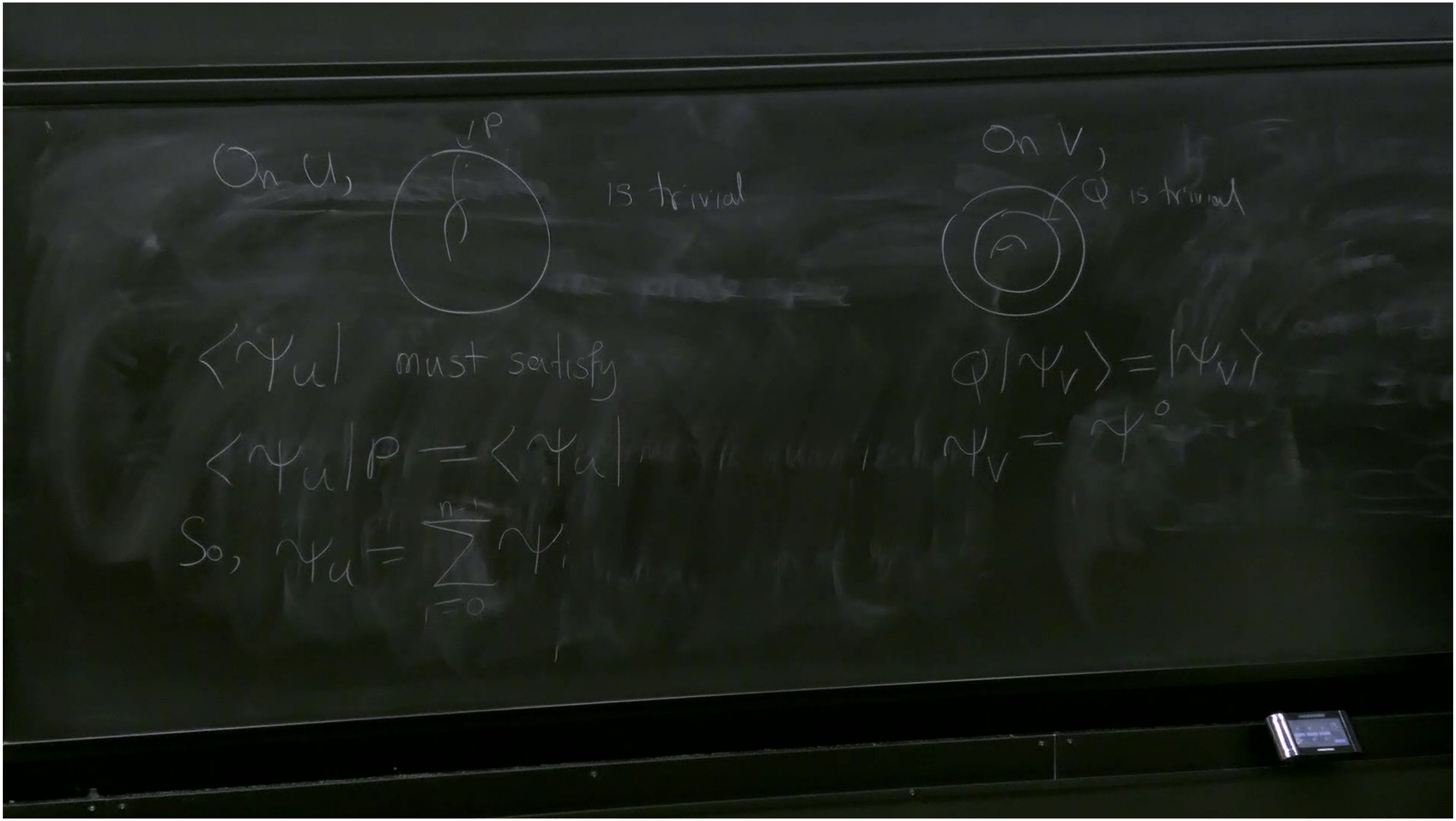
$$\langle \psi_l | \psi^k \rangle = \delta_l^k$$

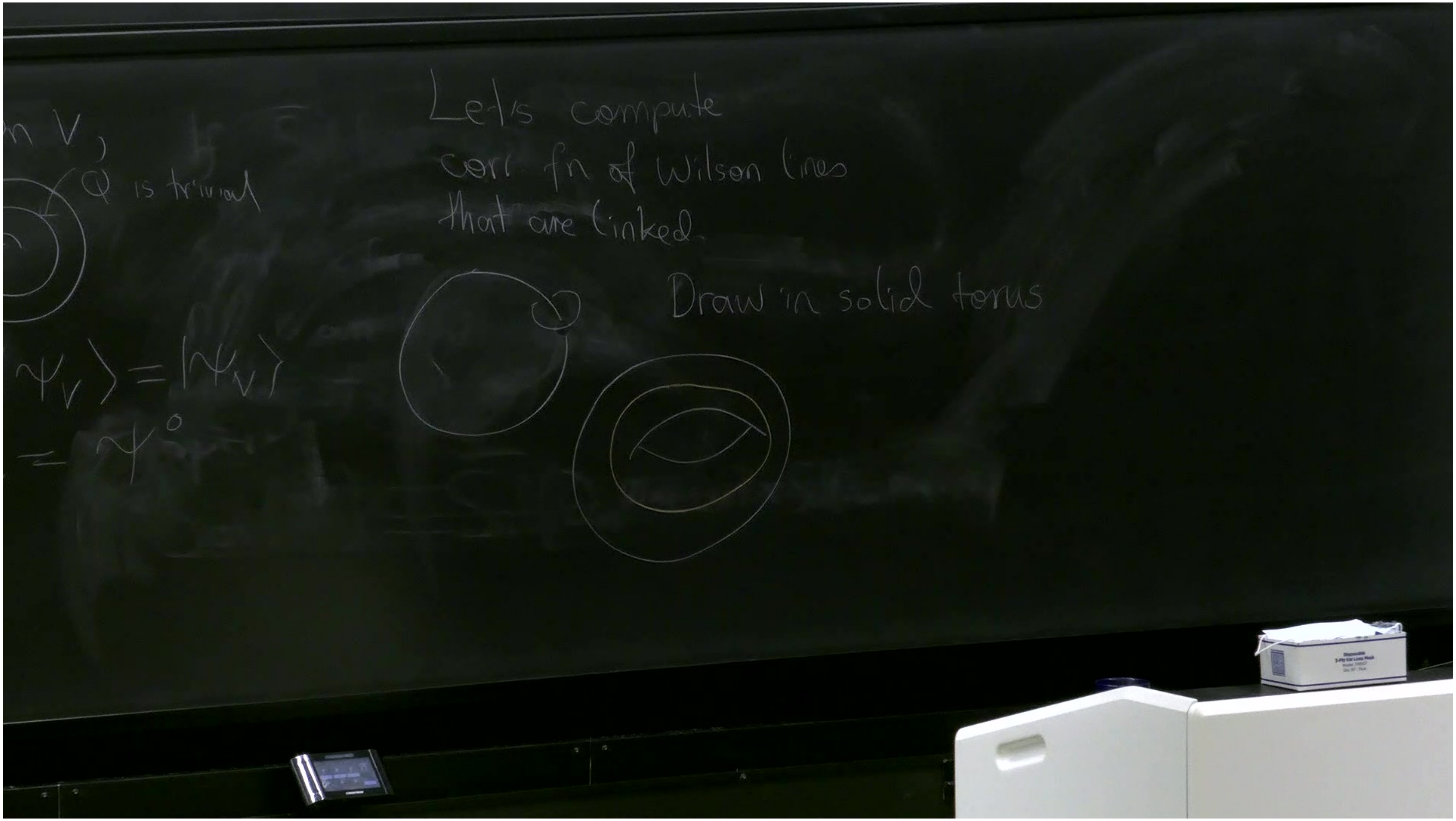
Dual basis $| \psi^k \rangle$

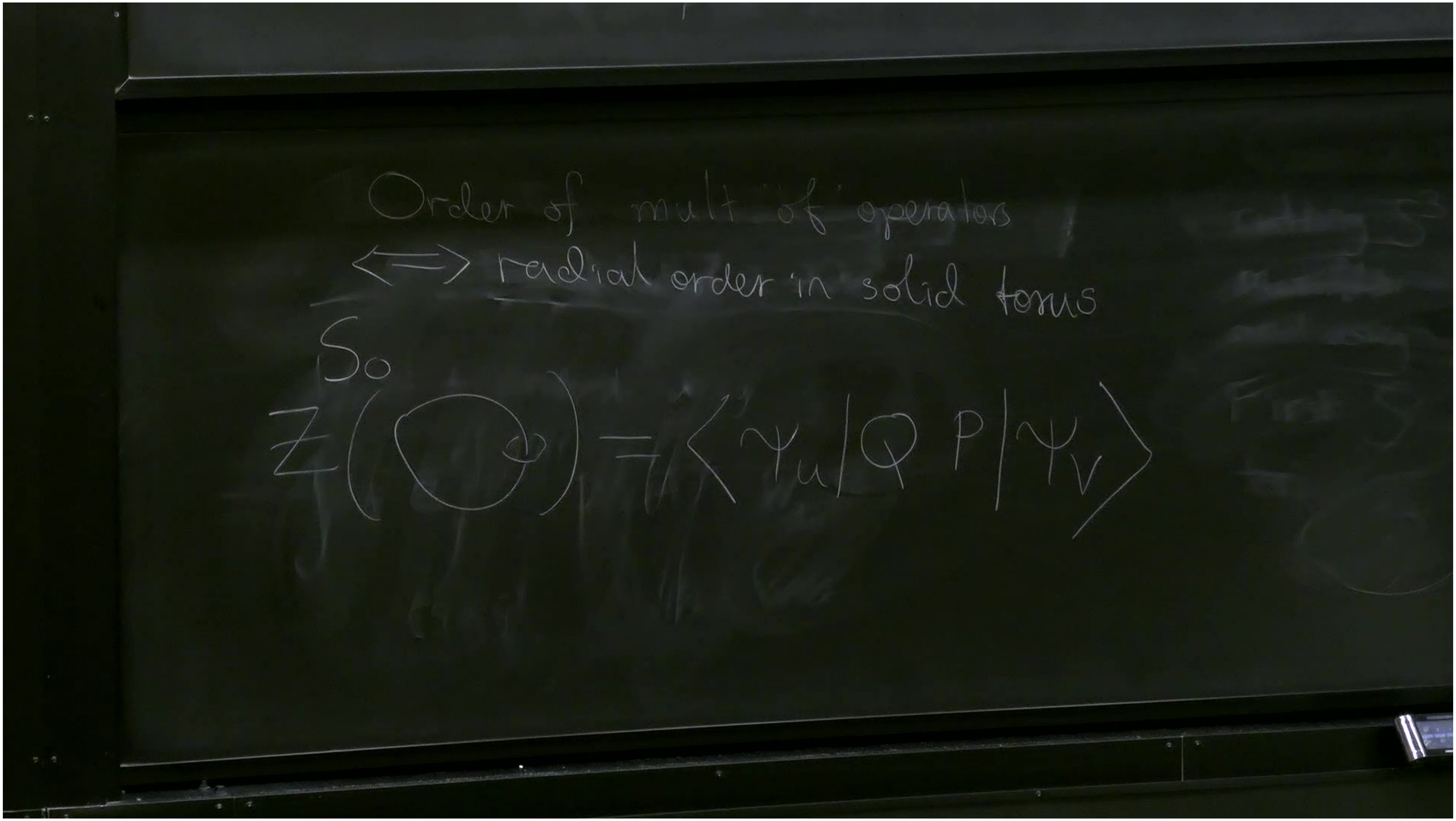
with $$Q | \psi^k \rangle = e^{2\pi i k/n} | \psi^k \rangle$$

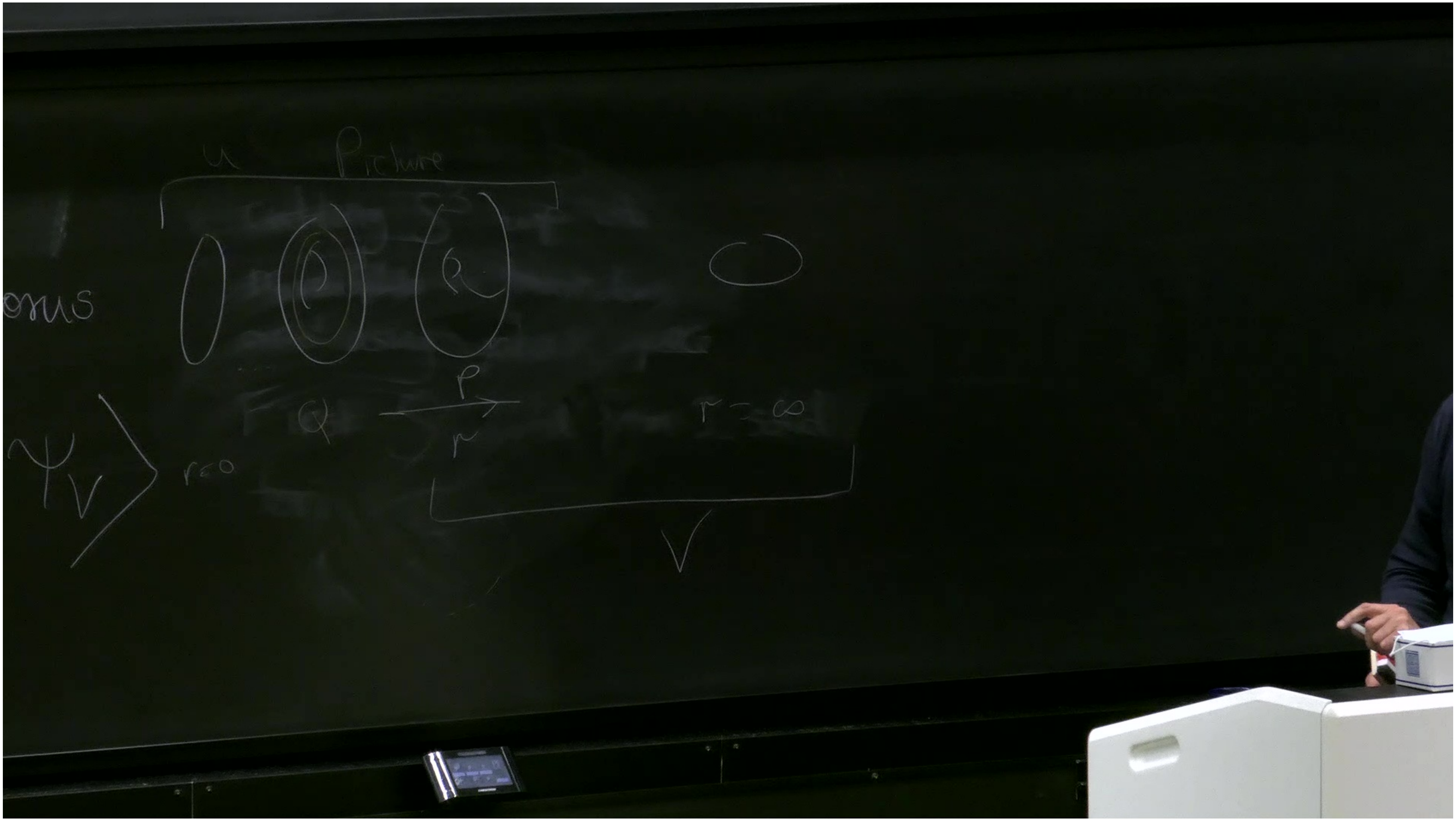
$$P | \psi^k \rangle = | \psi^{k-1} \rangle$$

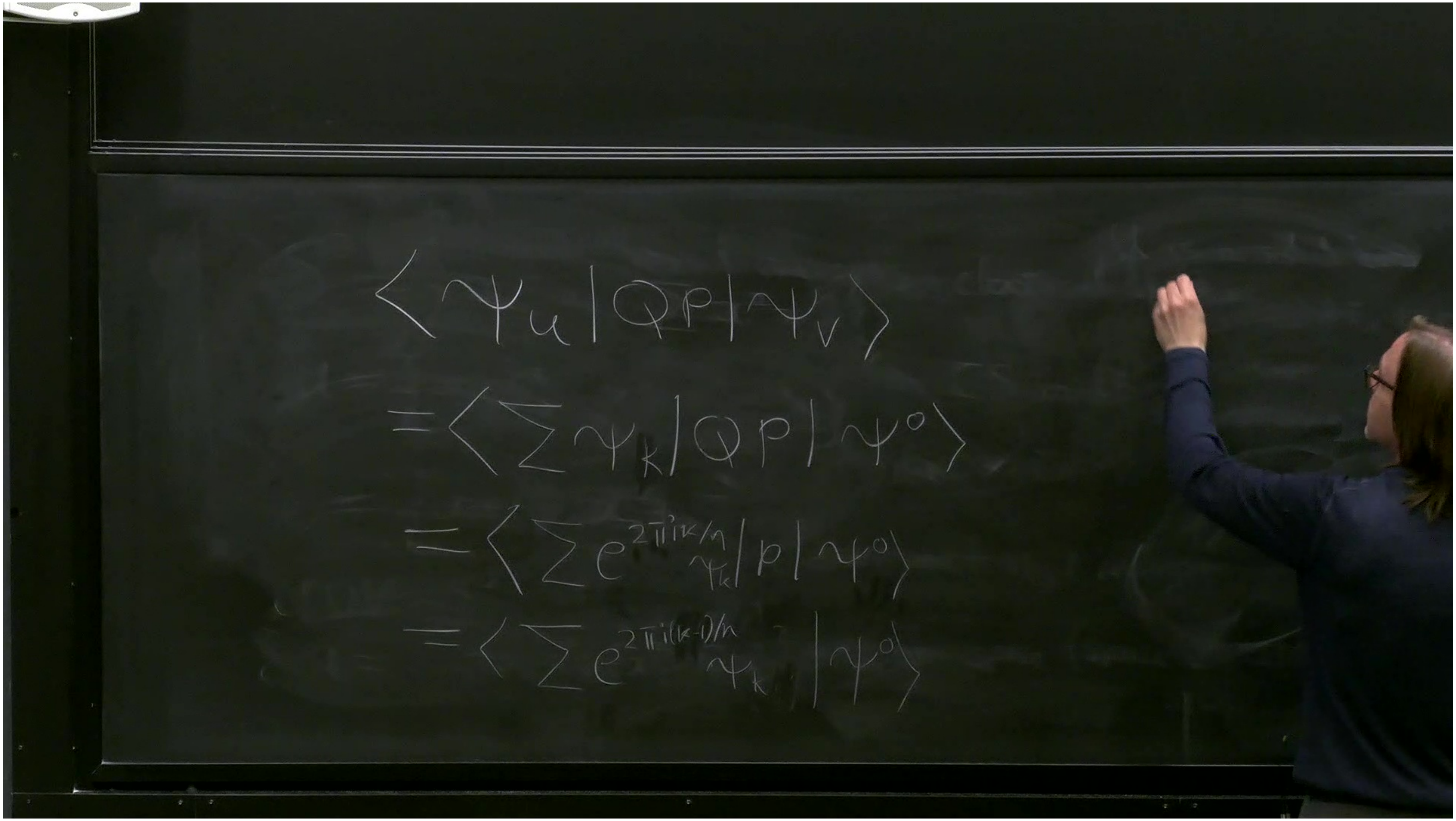

On U,
P
is trivial
On V,
Q is trivial
$\langle \Psi_u |$ must satisfy
$\langle \Psi_u | P = \langle \Psi_u |$
So, $\Psi_u = \Psi_0$

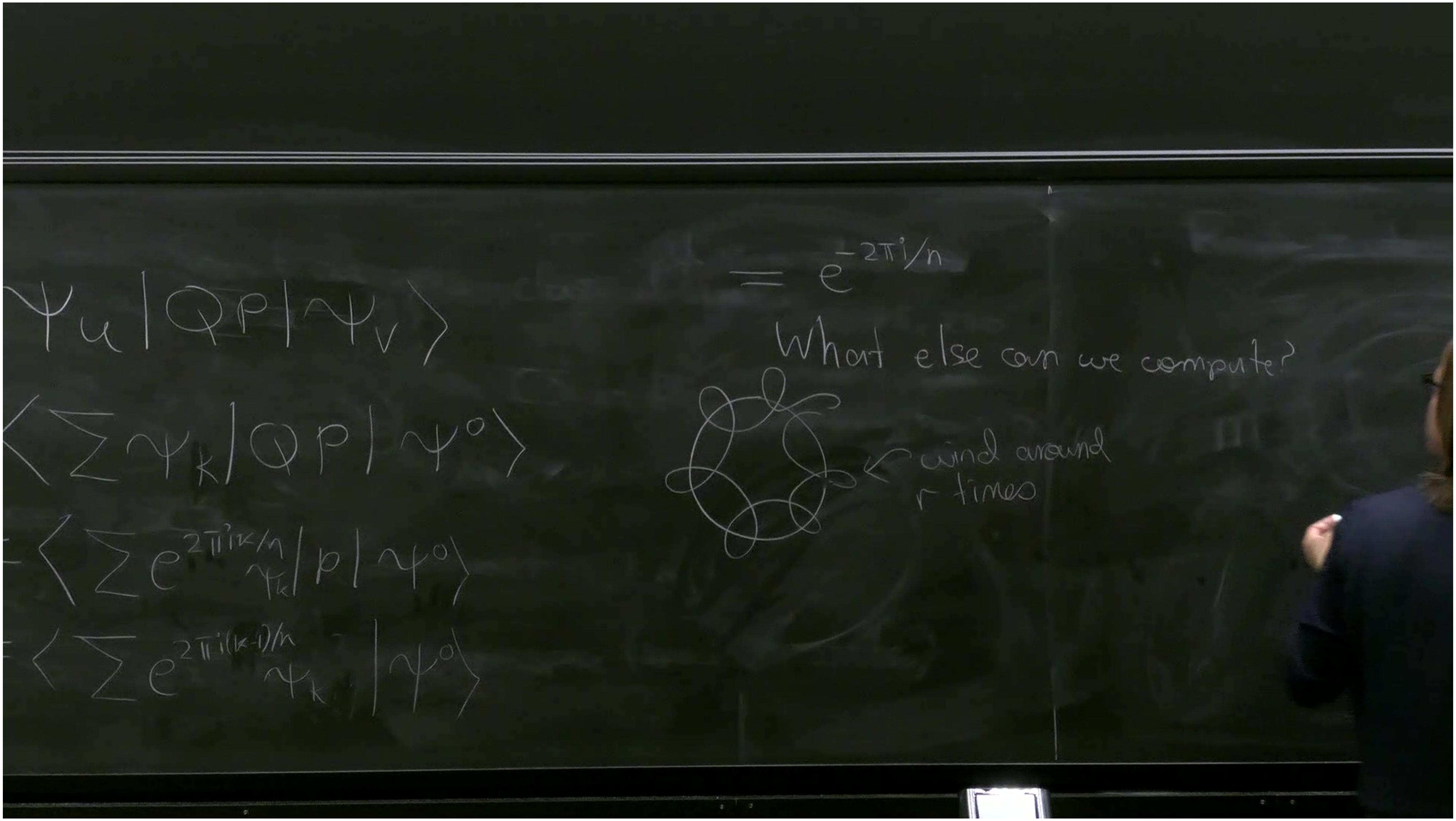

On U, P
is trivial
On V,
Q is trivial
$\langle \Psi_u|$ must satisfy
$\langle \Psi_u| P = \langle \Psi_u|$
So, $\Psi_u = \sum_{i=0}^{n-1} \Psi_i$
$Q|\Psi_V\rangle = |\Psi_V\rangle$
$\Psi_V = \Psi^0$


Let's compute
corr fn of Wilson lines
that are linked
Draw in solid torus
is trivial


Order of mult of operators
⟺ radial order in solid torus
S_0
$Z(\ ) = \langle \Psi_u | Q\, P | \Psi_v \rangle$


Picture
orus
r=0
r=∞
V


$\langle \tilde\psi_u | QP | \tilde\psi_v \rangle$
$= \langle \sum \tilde\psi_k | QP | \psi^0 \rangle$
$= \langle \sum e^{2\pi i k/n} \tilde\psi_k | P | \psi^0 \rangle$
$= \langle \sum e^{2\pi i (k-1)/n} \tilde\psi_k | \psi^0 \rangle$



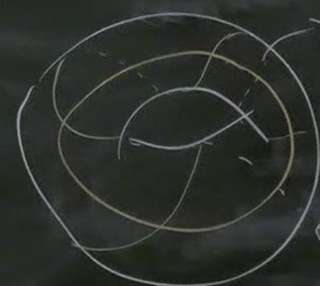
$\langle \Psi_u | Q p | \Psi_v \rangle$
$\langle \sum \Psi_k | Q p | \Psi^0 \rangle$
$\langle \sum e^{2\pi i k/n} \Psi_k | p | \Psi^0 \rangle$
$\langle \sum e^{2\pi i (k-1)/n} \Psi_k | \Psi^0 \rangle$
$= e^{-2\pi i/n}$
What else can we compute?
wind around
r times

...n we compute?

...around

...o

Same picture



→ ρ at $r<1$

at $r=1$, an operator that winds around one cycle once, the other r times

Take $r=1$

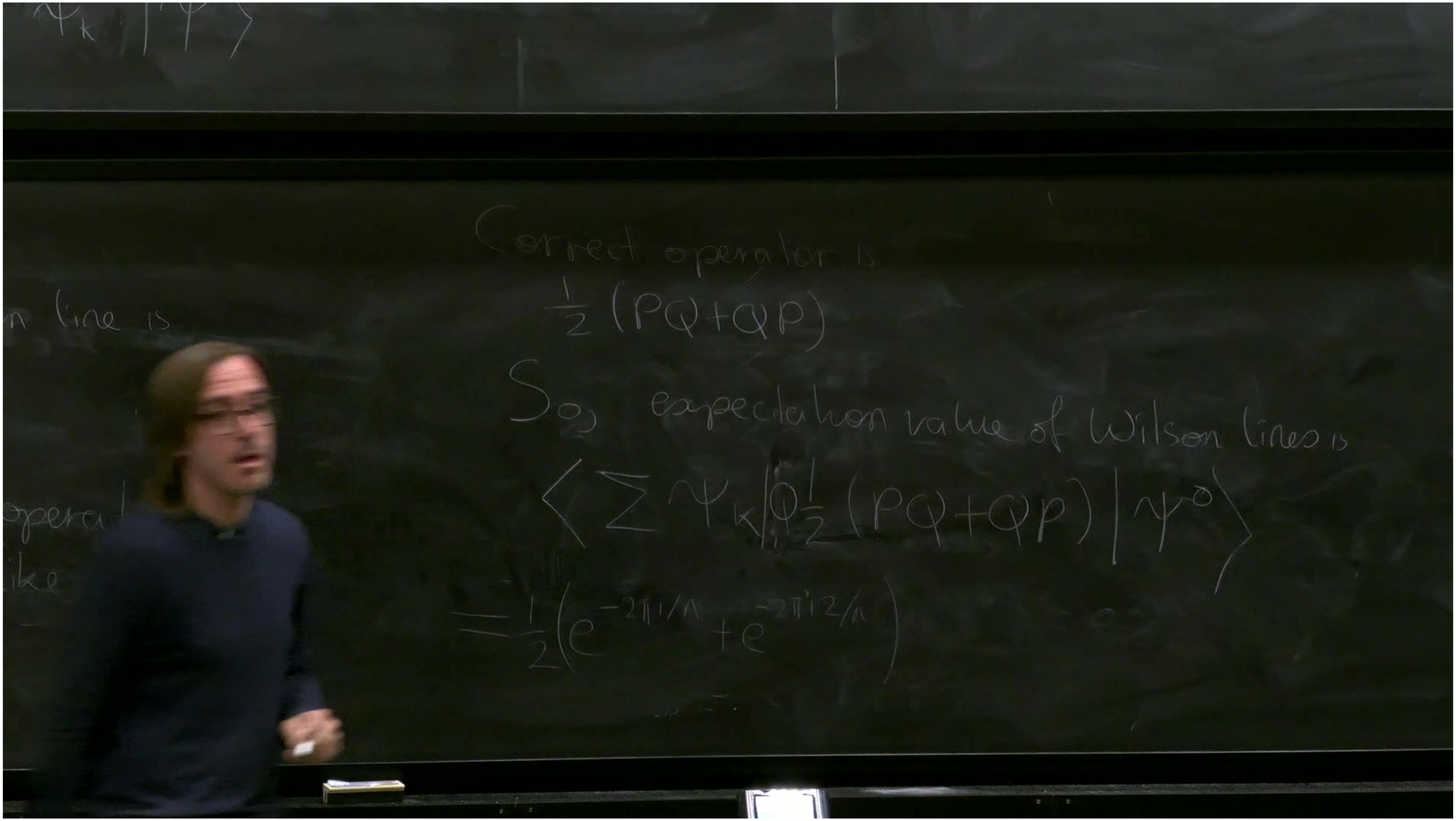
Outer Wilson line is

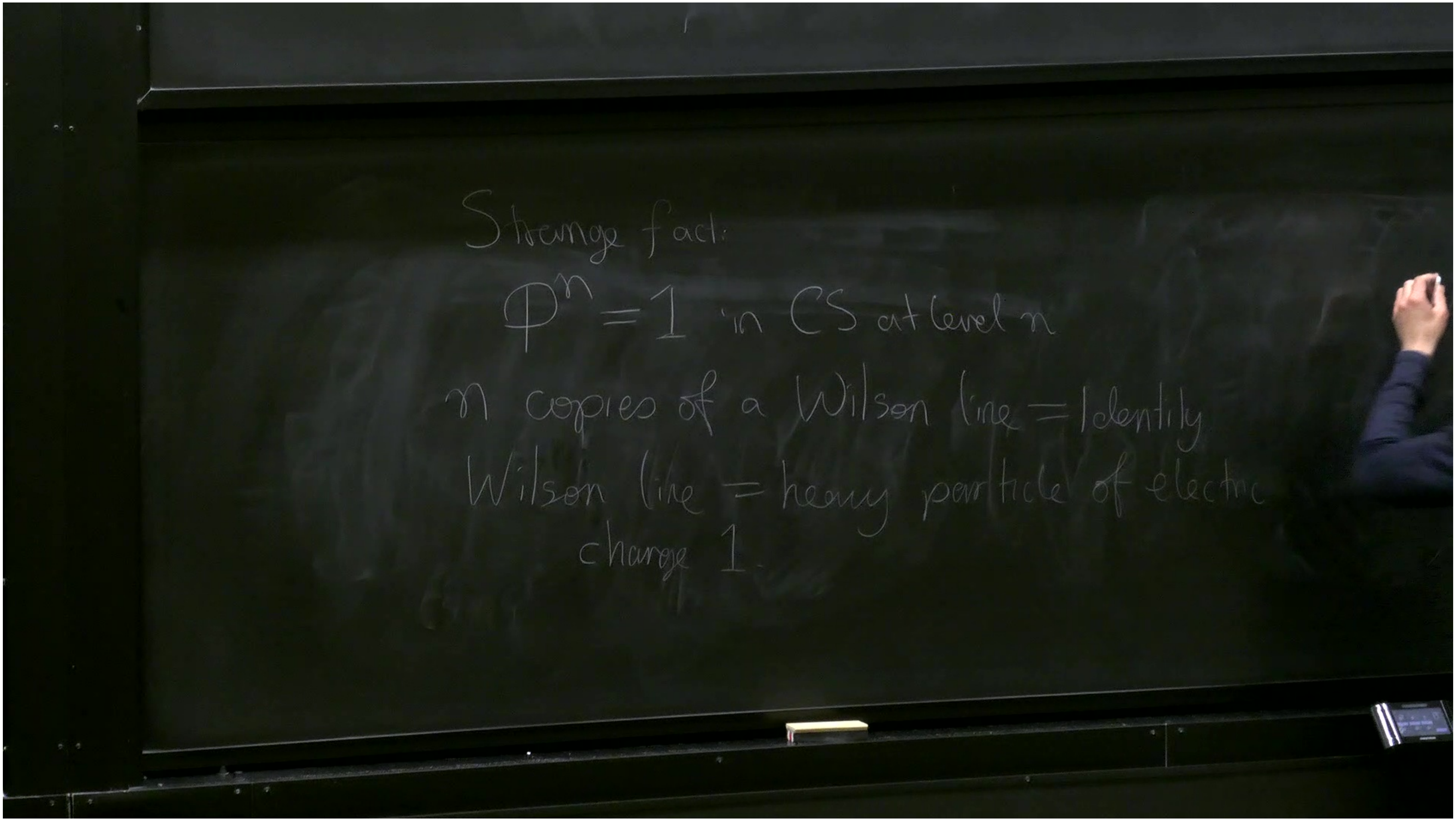
What is operator?

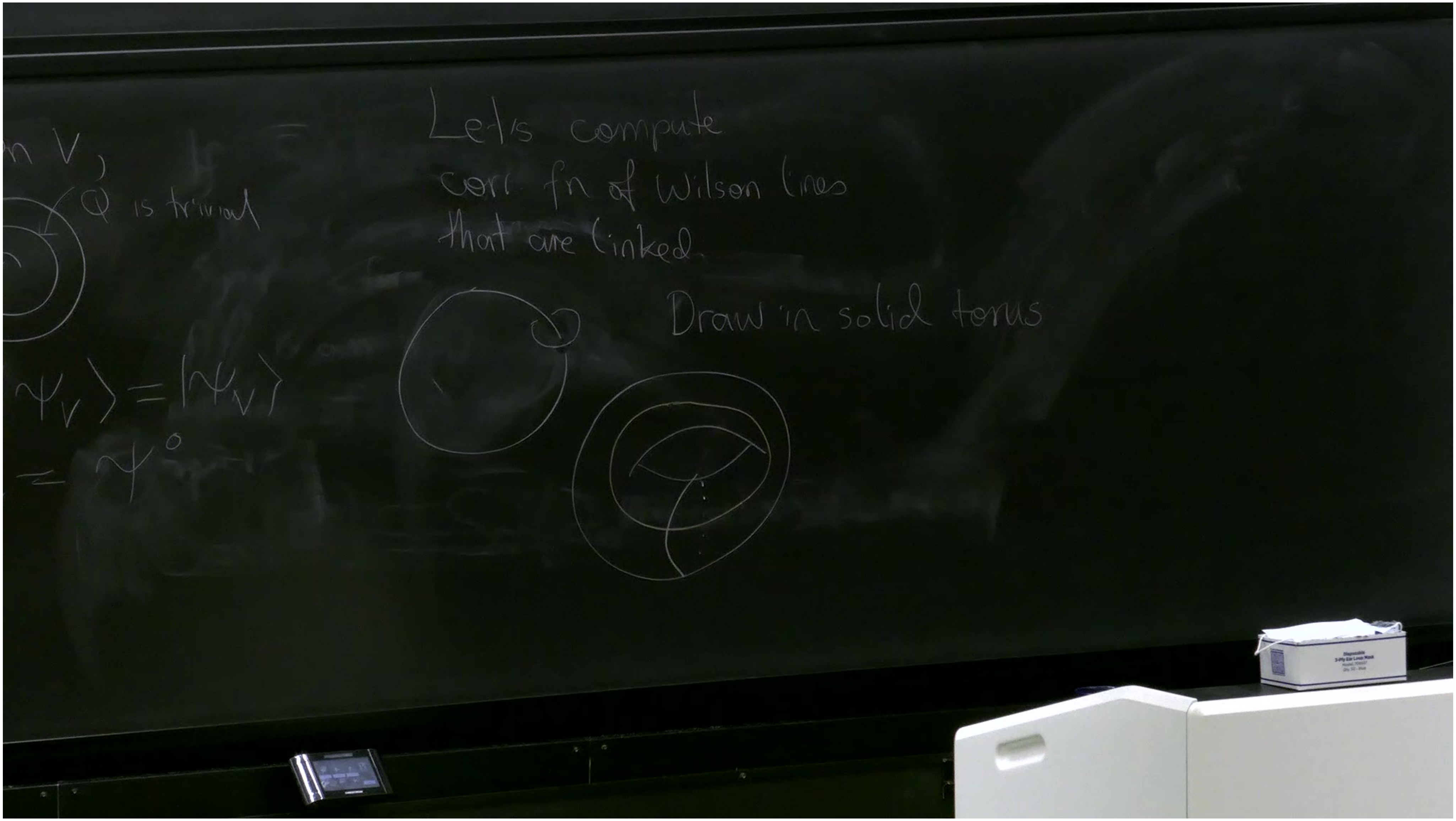
Something like PQ

Correct operator is

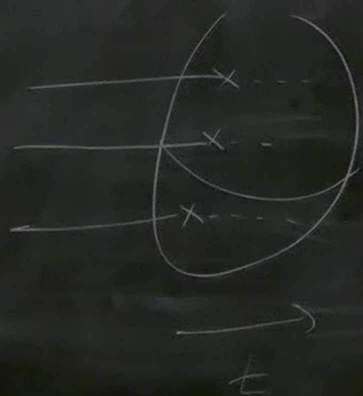
$$\frac{1}{2}(PQ+QP)$$


Correct operator is
1/2 (PQ+QP)
S_O2 expectation value of Wilson lines is
⟨Σ Ψ_k | Q 1/2 (PQ+QP) | Ψ^0⟩
= 1/2 (e^{-2πi1/n} + e^{-2πi2/n})
n line is
operat
like

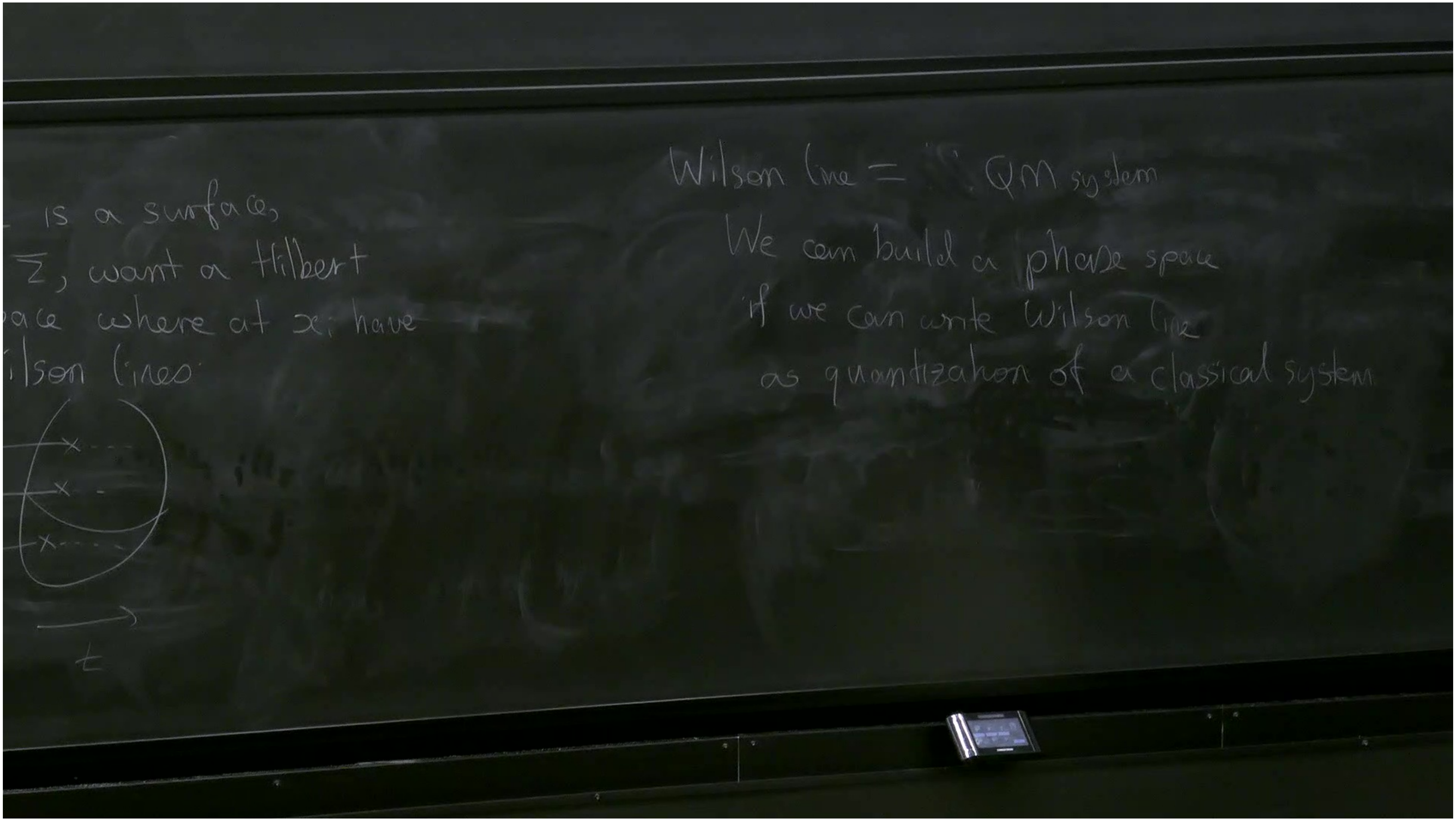

Strange fact:
$\Phi^n = 1$ in CS at level n
n copies of a Wilson line = Identity
Wilson line = heavy particle of electric
charge 1.


n V,
Φ is trivial
ψ_V⟩ = |ψ_V⟩
= ψ°
Let's compute
corr fn of Wilson lines
that are linked.
Draw in solid torus

If Σ is a surface, $x_i \in \Sigma$, want a Hilbert space where at x_i have Wilson lines



Wilson line = QM syst


is a surface,
Σ, want a Hilbert
pace where at x_i have
ilson lines
t
Wilson line = QM system
We can build a phase space
if we can write Wilson line
as quantization of a classical system