Title: Mathematical Physics Lecture

Speakers: Kevin Costello

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Last time $U(1)$ CS on $S^1 \times S^1 \times \mathbb{R}$

Phase space has coords

$$p = \int_{\theta_1} A \qquad p \sim p+1$$

$$q = \int_{\theta_2} A \qquad q \sim q+1$$

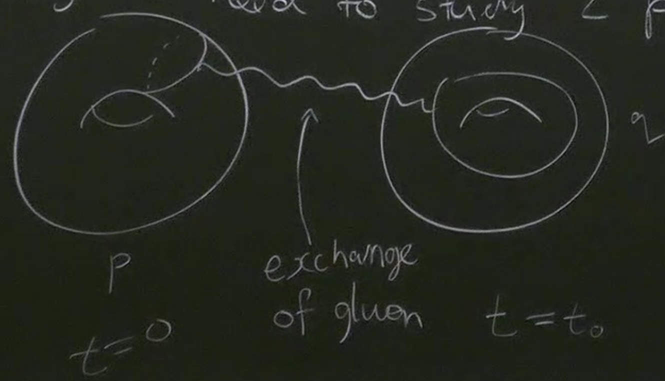
by large gauge transforms

LeA

Let's show

$$\{p, q\} = \frac{1}{k}$$

As always we need to study 2 point fn.



Discontinuity at $t_0 = 0$

= Poisson bracket

Solve the EOM in presence of operator

$$q = \int_{\Theta_2} A$$

Action is

$$\int k\, CS(A) + \delta_{t=0,\Theta_1=0}\, A \wedge d\Theta_1 \wedge dt$$

$-\int_{\theta_2} A$ by large gauge transforms

in presence of operator

$+ \delta_{t=0,\theta_1=0} \, A \wedge d\theta_1 \wedge dt$

Vary A_{θ_2} we get

$$-K F_{\theta_1 t} + \delta_{t=0,\theta_1=0} = 0$$

$$F_{\theta_1 t} = \frac{1}{K} \delta_{t=0,\theta_1=0}$$

A Solution is $A_{\theta_2} = 0, \; A_t = 0,$

$$A_{\theta_1} = +\frac{1}{K} \delta_{t \geq 0, \theta_1 = 0}$$

$t=0$ of given $t=t_0$

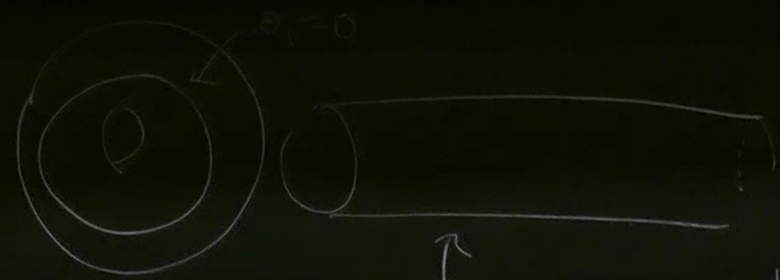
we get

$+\ \delta_{t=0,\theta_1=0} = 0$

$\frac{1}{K}\,\delta_{t=0,\theta_1=0}$

$A_{\theta_2}=0,\ A_t=0,$

$\delta_{t\geq 0,\theta_1=0}$



$\xrightarrow{t}$

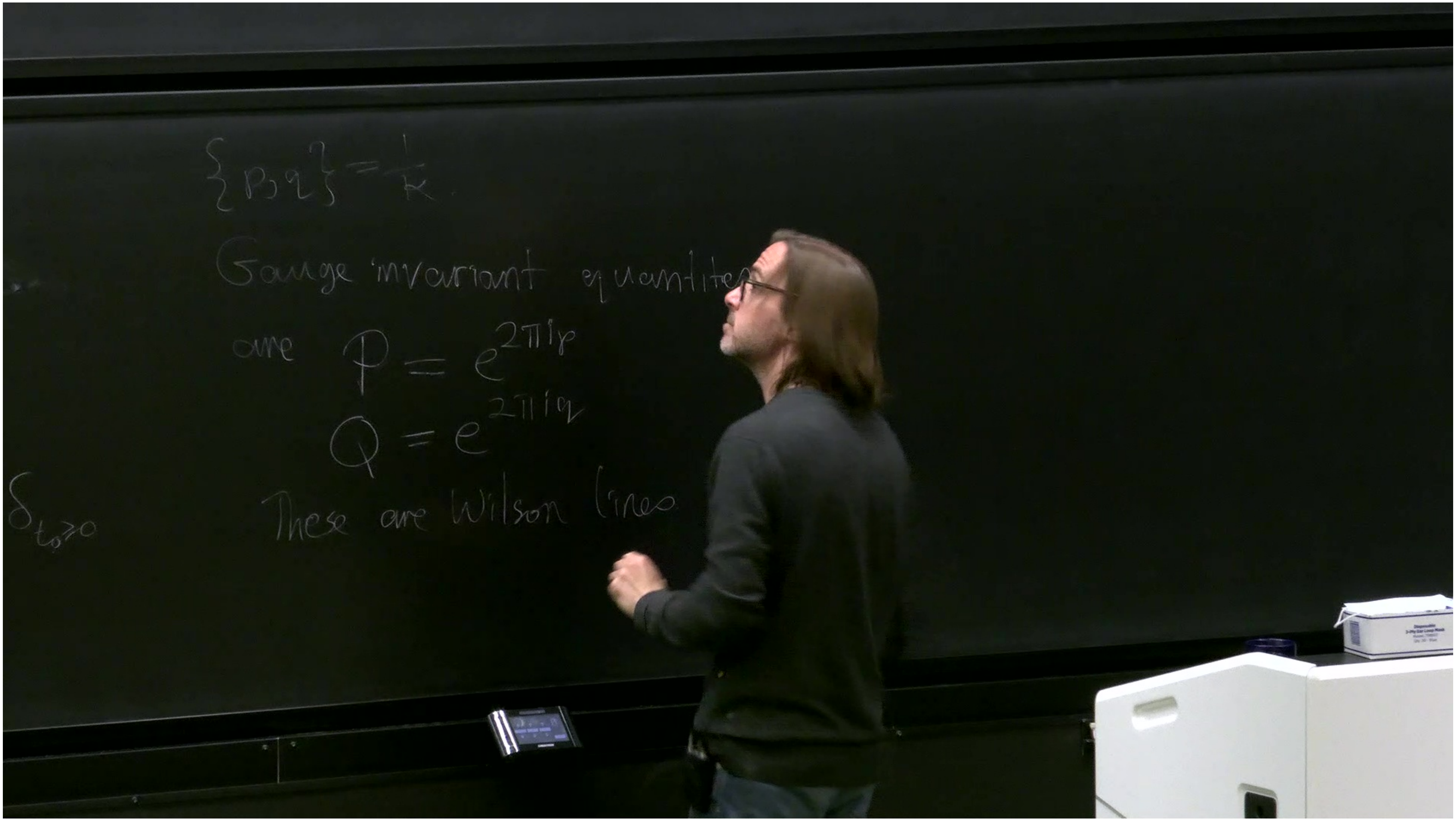
A_{θ_1} supported on this cylinder

$\{p,q\} = \frac{1}{K}$

2 point fn:

$\{p,q\} =$ measure $\int_{\Theta_1} A_{\Theta_1}$
in the field sourced
by $\int_{\Theta_2} A_{\Theta_2}$

$$\{p,q\} = \int_{\Theta_1, t=t_0} \frac{1}{K}\, \delta_{t\geq 0, \Theta_1 = 0}\, d\Theta_1 = \frac{1}{K}\, \delta_{t_0 \geq 0}$$



$\{p, q\} = \frac{1}{k}$
Gauge invariant quantities
are $P = e^{2\pi i p}$
$Q = e^{2\pi i q}$
These are Wilson lines
$\delta_{t_0 \geq 0}$

The Poisson bracket is

$$\{P, Q\} = PQ\, e^{2\pi i/k}$$

If we promote $\{\ \}$ to $[\ ]$

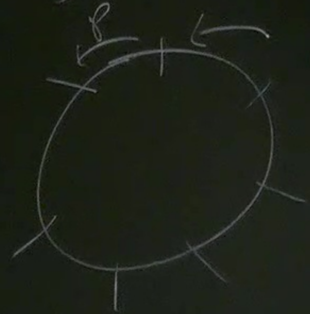
we find

$$PQ = QP\, e^{2\pi i/k}$$

K has to be an

K has to be an integer
for this algebra to have a
nice representation of
dim^n K.

Dually, area of phase space must be an integer.



Algebra acts on v. space
spanned by k^{th} roots of 1
P multiplies by $e^{2\pi i/k}$

Q acts on ψ_n
by $e^{2\pi i m/k}$

$$q = \int_{\Theta_2} A$$

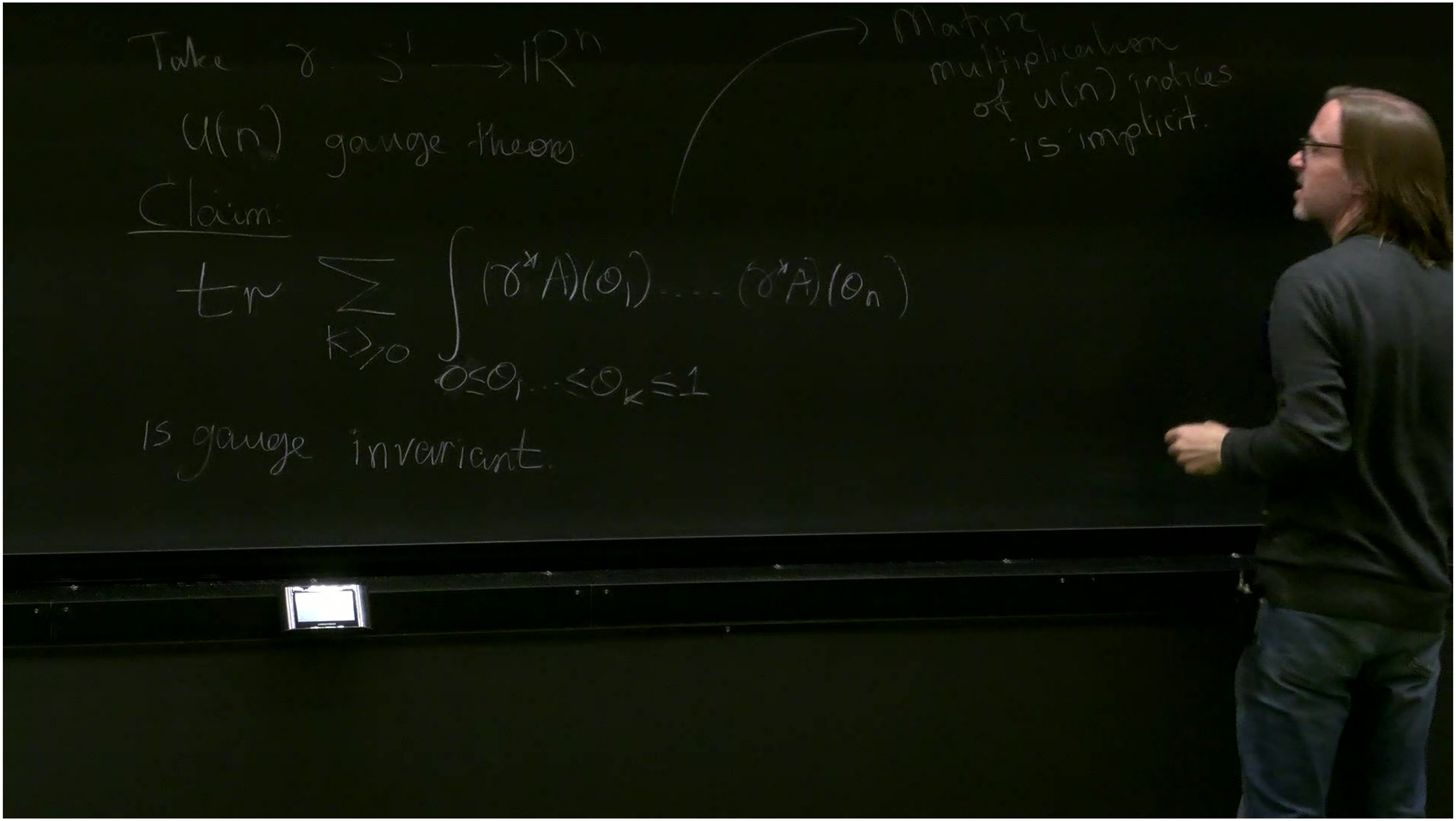
by large gauge transforms

Take γ : S^1 —

Non-Abelian Case:

1^{st} Wilson lines

In general, any gauge theory we can build a gauge inv. operator from gauge field as follows.


Take $\gamma : S^1 \longrightarrow \mathbb{R}^n$
$U(n)$ gauge theory
Claim:
$\mathrm{tr} \sum_{k \geq 0} \int_{0 \leq \theta_1 \cdots \leq \theta_k \leq 1} (\gamma^* A)(\theta_1) \cdots\cdots (\gamma^* A)(\theta_n)$
is gauge invariant.
Matrix multiplication of $u(n)$ indices is implicit.

If $n=1$, order doesn't matter

this is $\exp\left(i\int_{\sigma} \gamma^* A\right)$

Proof that it's gauge invariant:

$$\delta A = d\chi + [A,\chi]$$

Wilson loop

$W(\gamma, A)$ varies by

$\delta W(\gamma, A)$

$$\delta W(\gamma, A) = \sum_{k \geq 0} \mathrm{tr} \int_{0 \leq \theta_1 \ldots \leq 1} \gamma^* A(\theta_1) \ldots (d\chi + [\gamma^* A, \chi]) \ldots \gamma^* A(\theta_n)$$

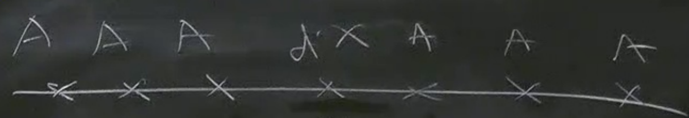
IBP: $d\chi$ doesn't contribute, except boundary terms

[using $d\gamma^* A = 0$]

$\theta_i = \theta_{i+1}$, or $\theta_1 = 0$, or $\theta_n = 1$.

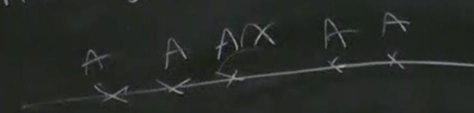
Focus on $\theta_i = \theta_{i+1}$

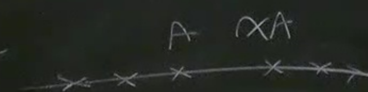
$\gamma^* A(\theta_i) \wedge \gamma^* A(\theta_i) = 0$ as these are 1 forms in 1 variable.



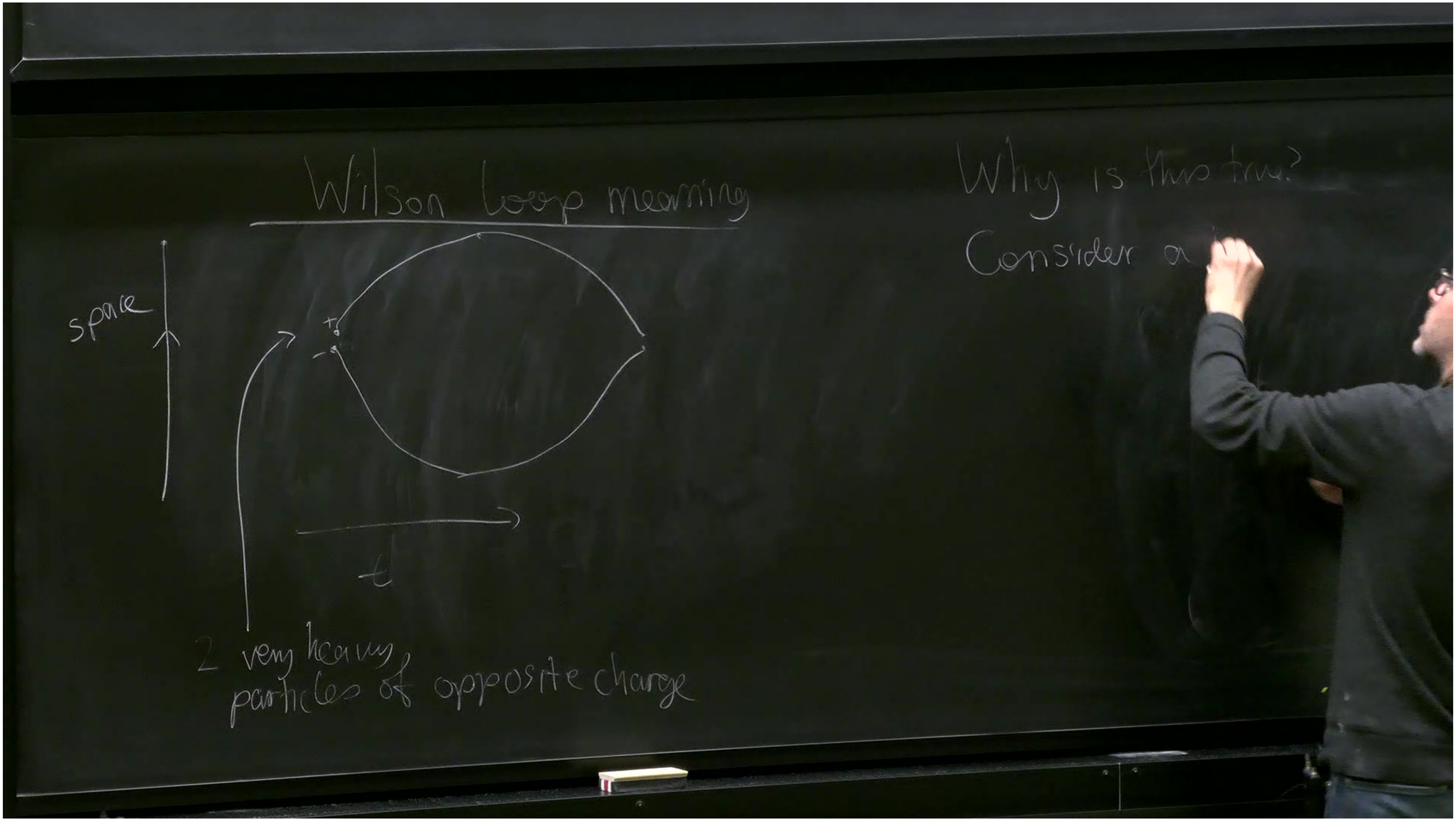
Boundary $\int$ terms:
Where this point
(labelled by dX)
hits previous, or subsequent
point.

Boundary terms

$-\sum$ 

$+\sum$ 

This is exactly the same
as contribution of
$[A, X]$ term in gauge variation.


Wilson loop meaning
space
t
2 very heavy particles of opposite charge
Why is this true?
Consider a

oup meaning

Why is this true?

Consider a heavy charged particle coupled to $U(n)$ gauge theory.

Moves on some path γ.

Hilbert space = some rep. of $U(n)$

we take it to be $\mathbb{C}^n$.

osite charge

...ge theory

...an is

Hamiltonian.

$\psi(0)$ = initial state

$$\frac{\partial}{\partial t}\psi(t) = \gamma^* A \cdot \psi(t)$$

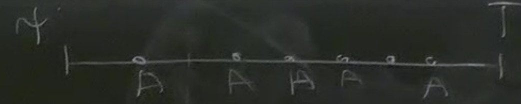
This equation is solved by

$$\psi(T) = \sum_{k \geq 0} \int_{0 \leq t_1 \dots \leq T} \gamma^*(A)(t_1) \dots (\gamma^* A)(t_k)$$

$)$

by

$\gamma A)(t_1)\psi(0)$



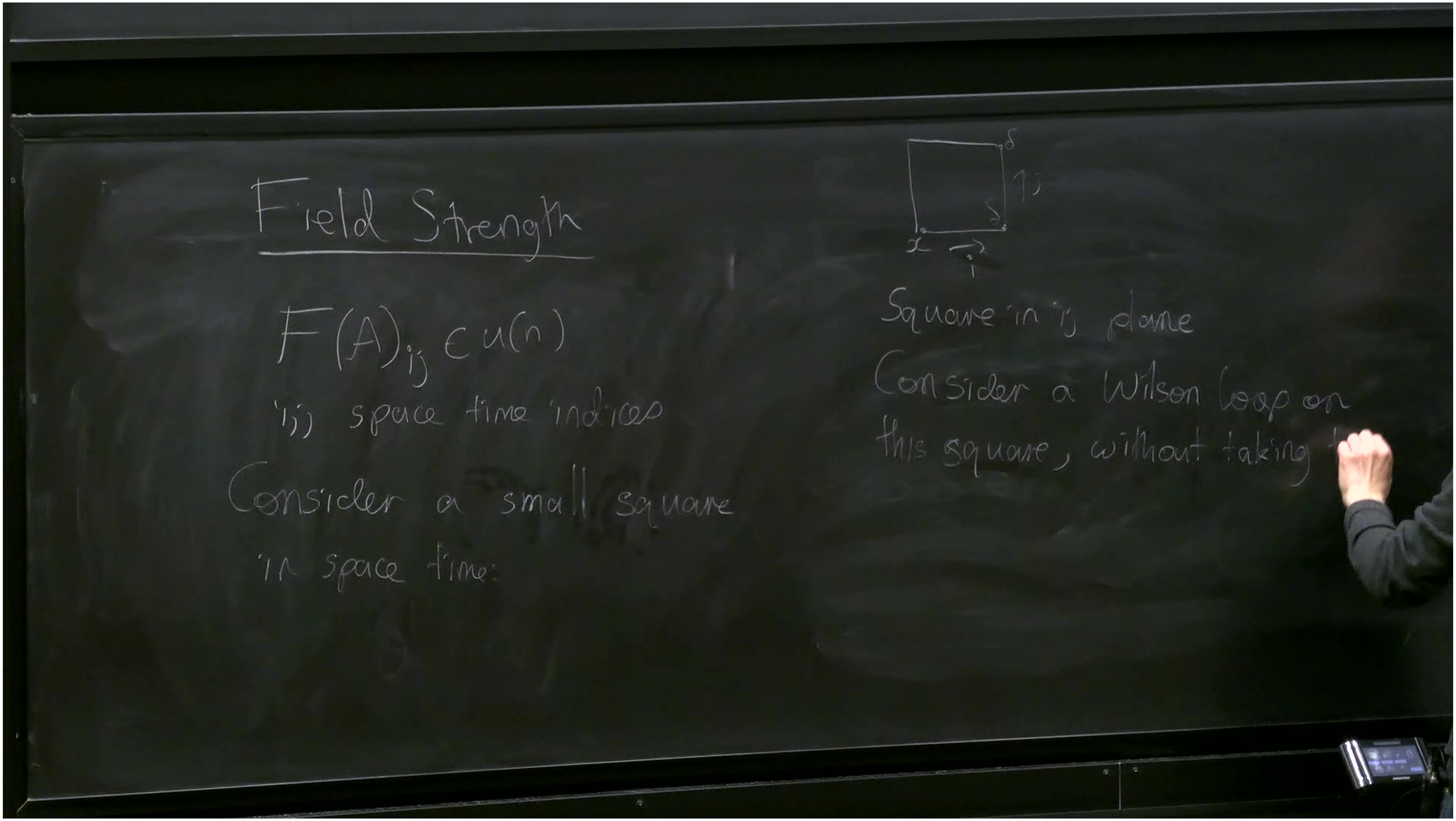
$$\frac{\partial}{\partial T}\psi(T) = \gamma^\mu A(T)\sum\int\gamma^\mu(A)(t_k)\ \cdots\ \gamma A(t_1)$$

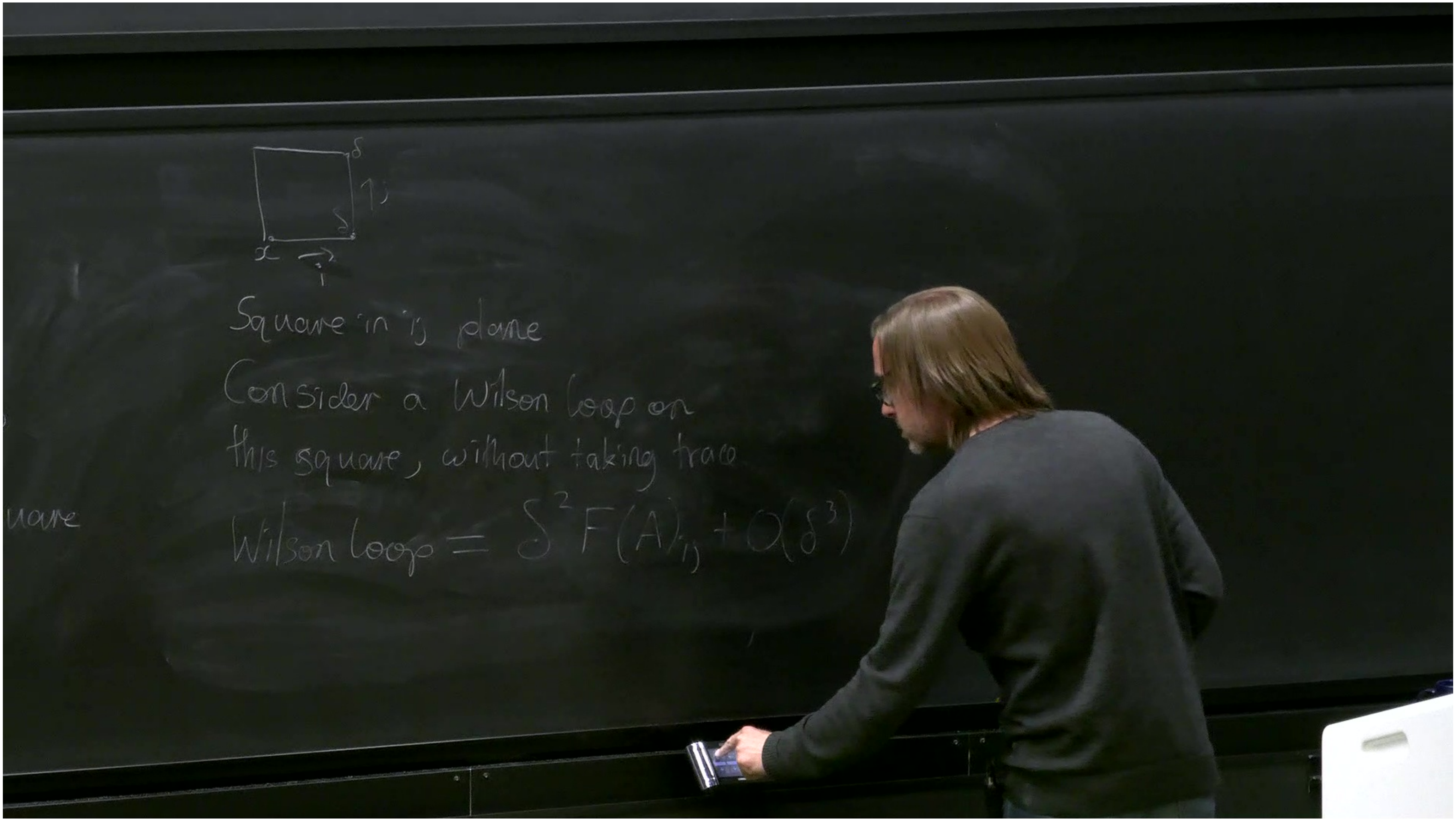
$$= \gamma^\mu A(T)\cdot\psi(T)$$

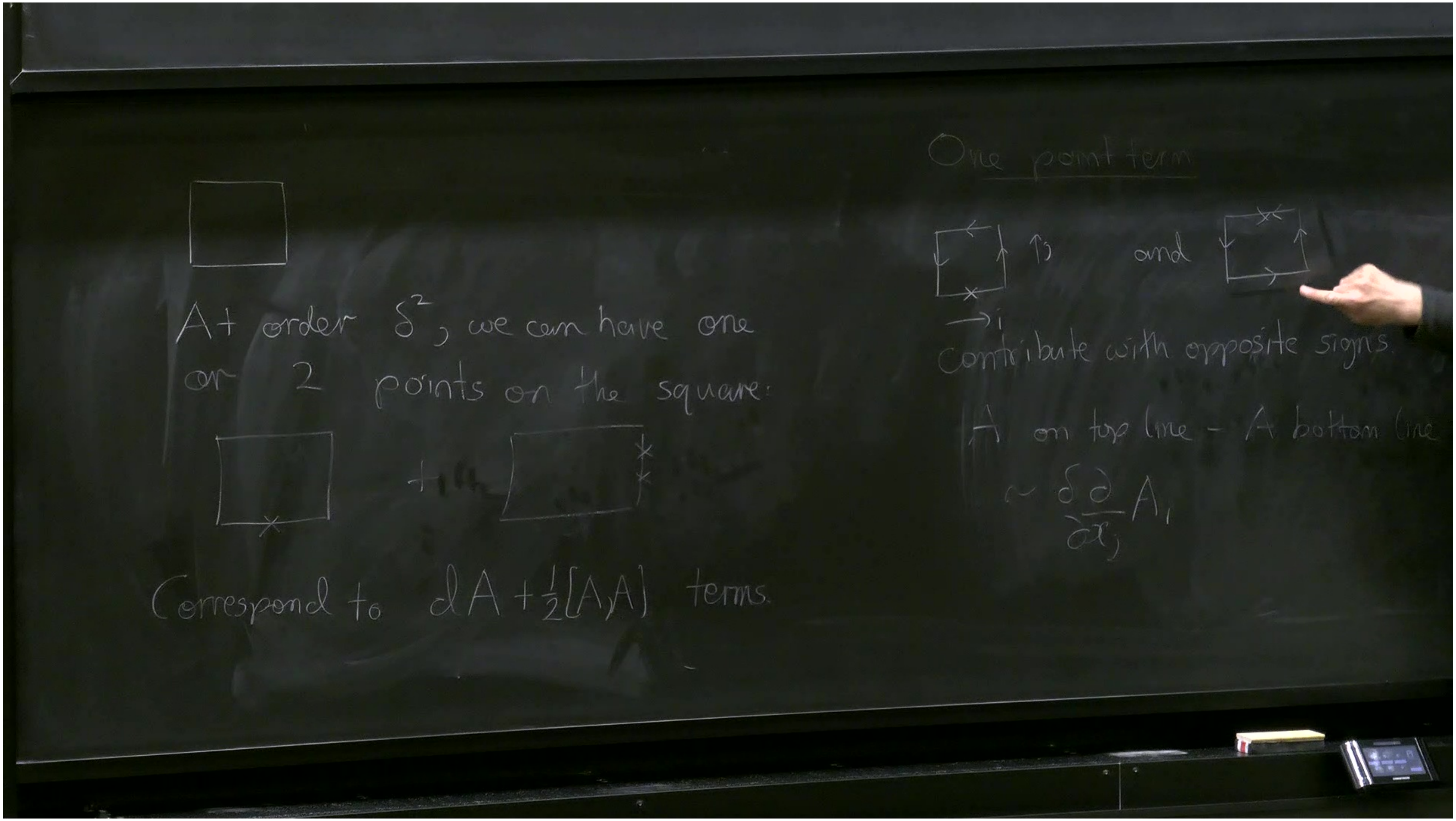
then we trace over Hilbert space which is $\mathbb{C}^n$

$\Rightarrow$ Wilson loop

spa


Field Strength
F(A)ij ∈ u(n)
i,j space time indices
Consider a small square
in space time:
δ
j
x
i
Square in ij plane
Consider a Wilson loop on
this square, without taking


Square in ij plane
Consider a Wilson loop on
this square, without taking trace
Wilson loop = $\delta^2 F(A)_{ij} + O(\delta^3)$

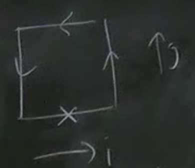

One point term
At order δ^2, we can have one or 2 points on the square:
Correspond to $dA + \frac{1}{2}[A,A]$ terms.
$\uparrow j$ and
$\rightarrow i$
Contribute with opposite signs.
A on top line - A bottom line
$\sim \delta \frac{\partial}{\partial x_j} A_i$

... can have one

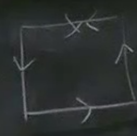
on the square:

$+\frac{1}{2}[A,A]$ terms.

One point term



and



Contribute with opposite signs.

A on top line − A bottom line

$$\sim \delta \frac{\partial}{\partial x_j} A_i$$

Side length δ, $\int$ gives

$$\delta^2 \frac{\partial A_i}{\partial x_j}$$

2 vertical lines give

$$-\delta^2 \frac{\partial A_j}{\partial x_i}$$

Total is $\delta^2\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right)$