Title: Mathematical Physics Lecture

Speakers: Kevin Costello

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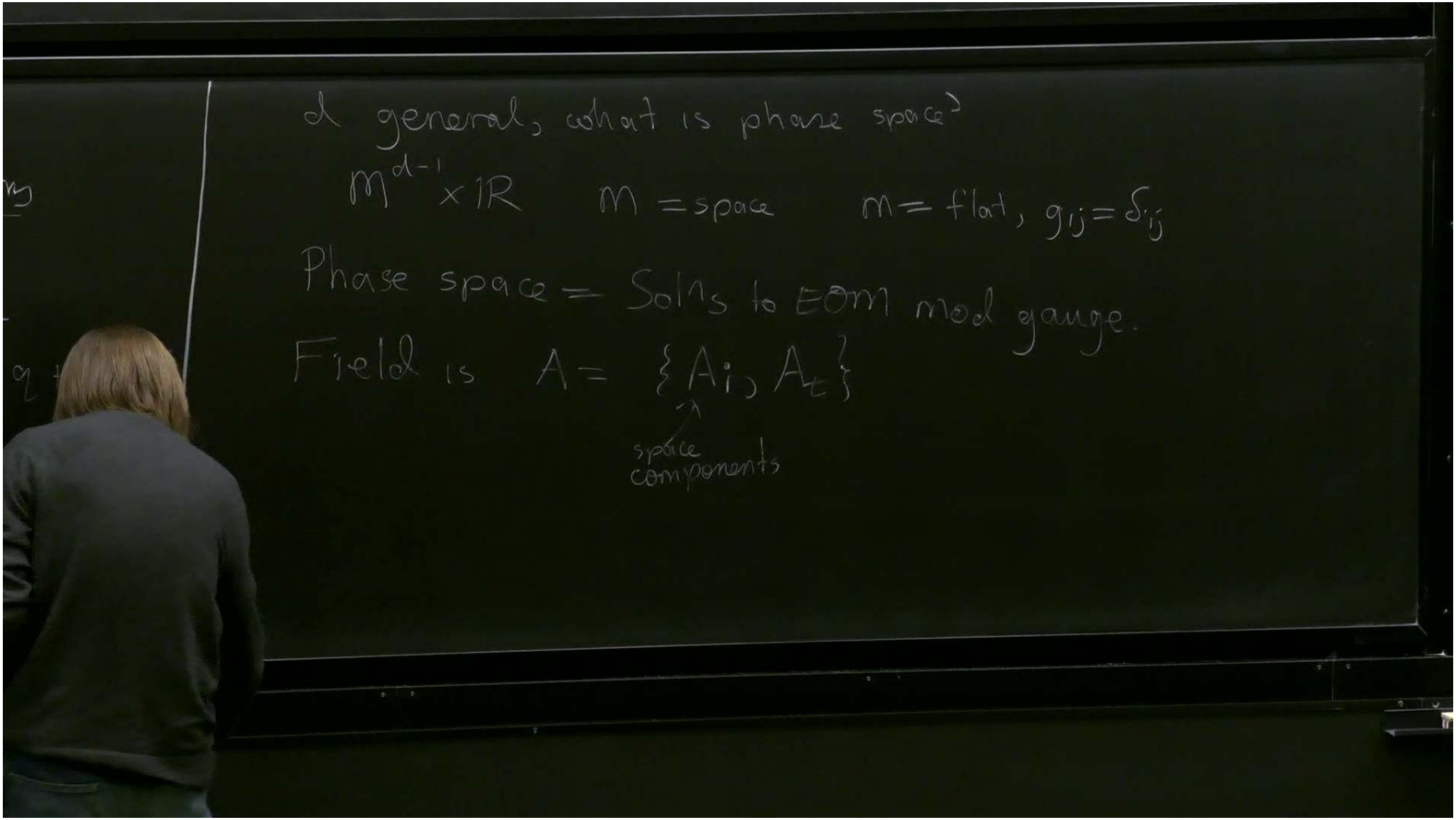
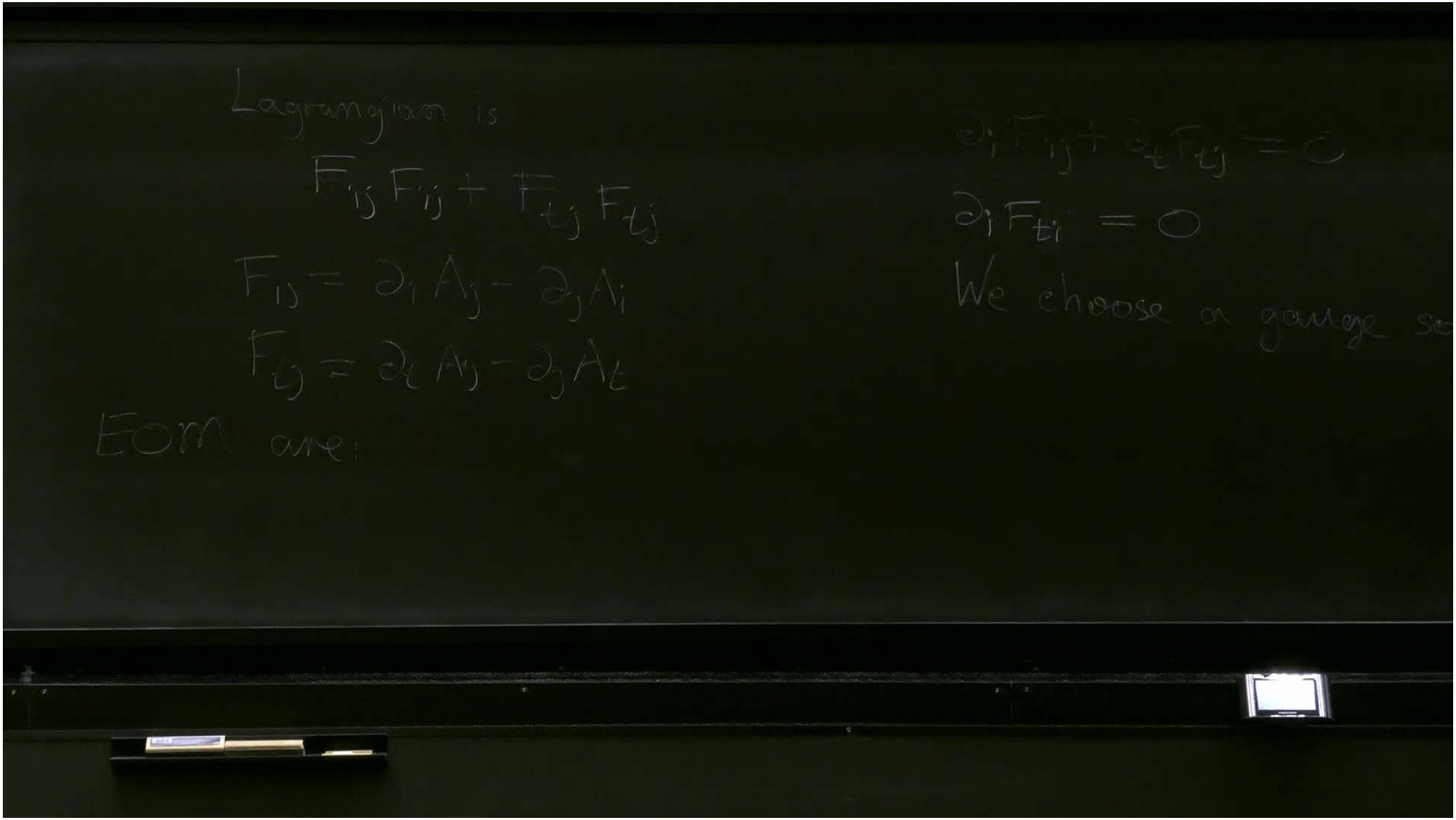
Phase space of Maxwell theory

$d = 2$ we found phase space has coords. $p \sim q \quad q \sim q + 1$

Hamiltonian $= p^2$

System = 1d particle on a circle.


d general, what is phase space?
$M^{d-1} \times \mathbb{R}$ M = space m = flat, $g_{ij} = \delta_{ij}$
Phase space = Solns to EOM mod gauge.
Field is $A = \{A_i, A_t\}$
space components


Lagrangian is
$F_{ij}F_{ij} + F_{tj}F_{tj}$
$F_{ij} = \partial_i A_j - \partial_j A_i$
$F_{tj} = \partial_t A_j - \partial_j A_t$
EOM are:
$\partial_i F_{ij} + \partial_t F_{tj} = 0$
$\partial_i F_{ti} = 0$
We choose a gauge

$\partial_i F_{ij} + \partial_t F_{tj} = 0$

$\partial_i F_{ti} = 0$

We choose a gauge so $A_t = 0$

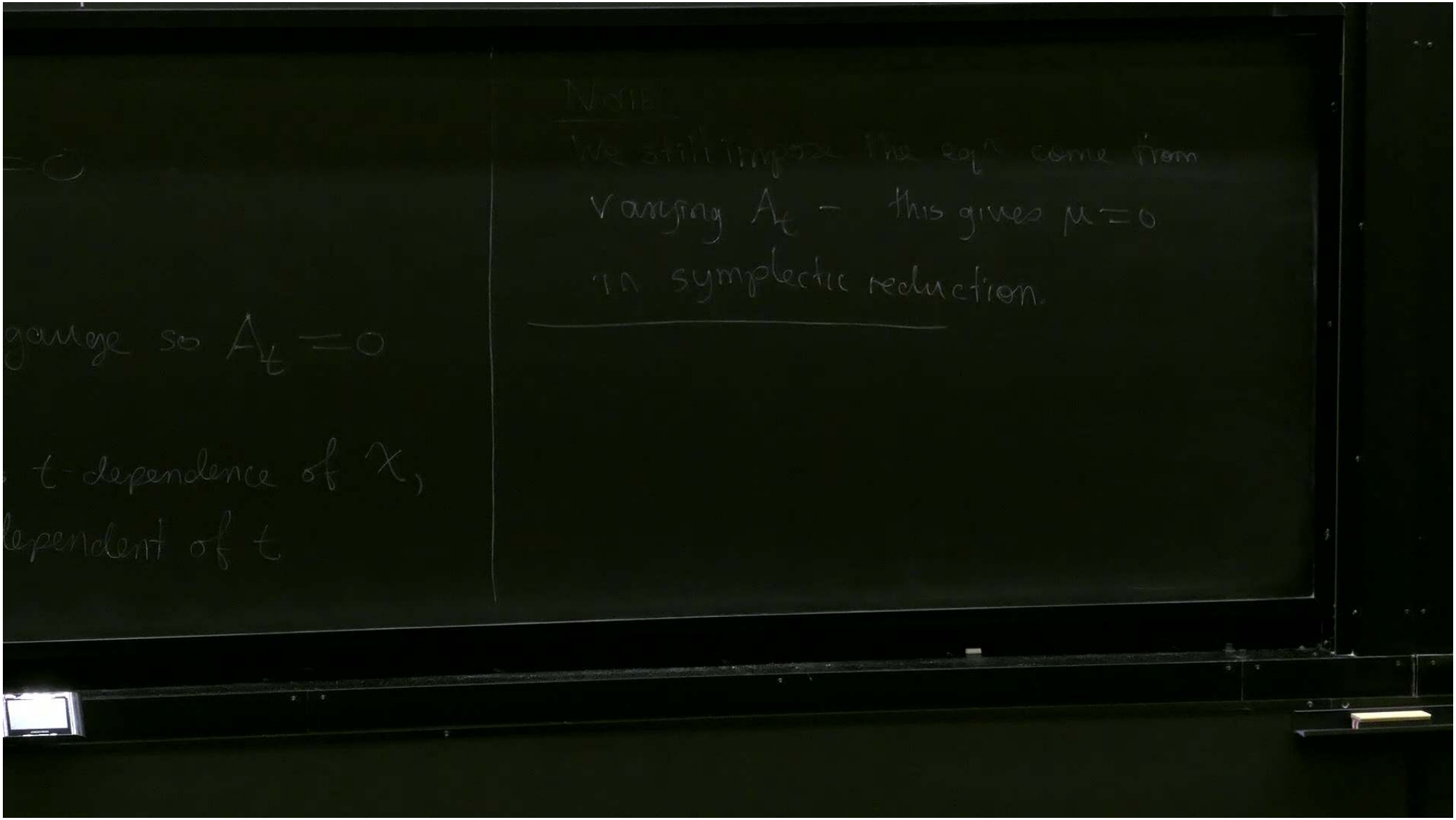
$\delta A_t = \partial_t \chi$

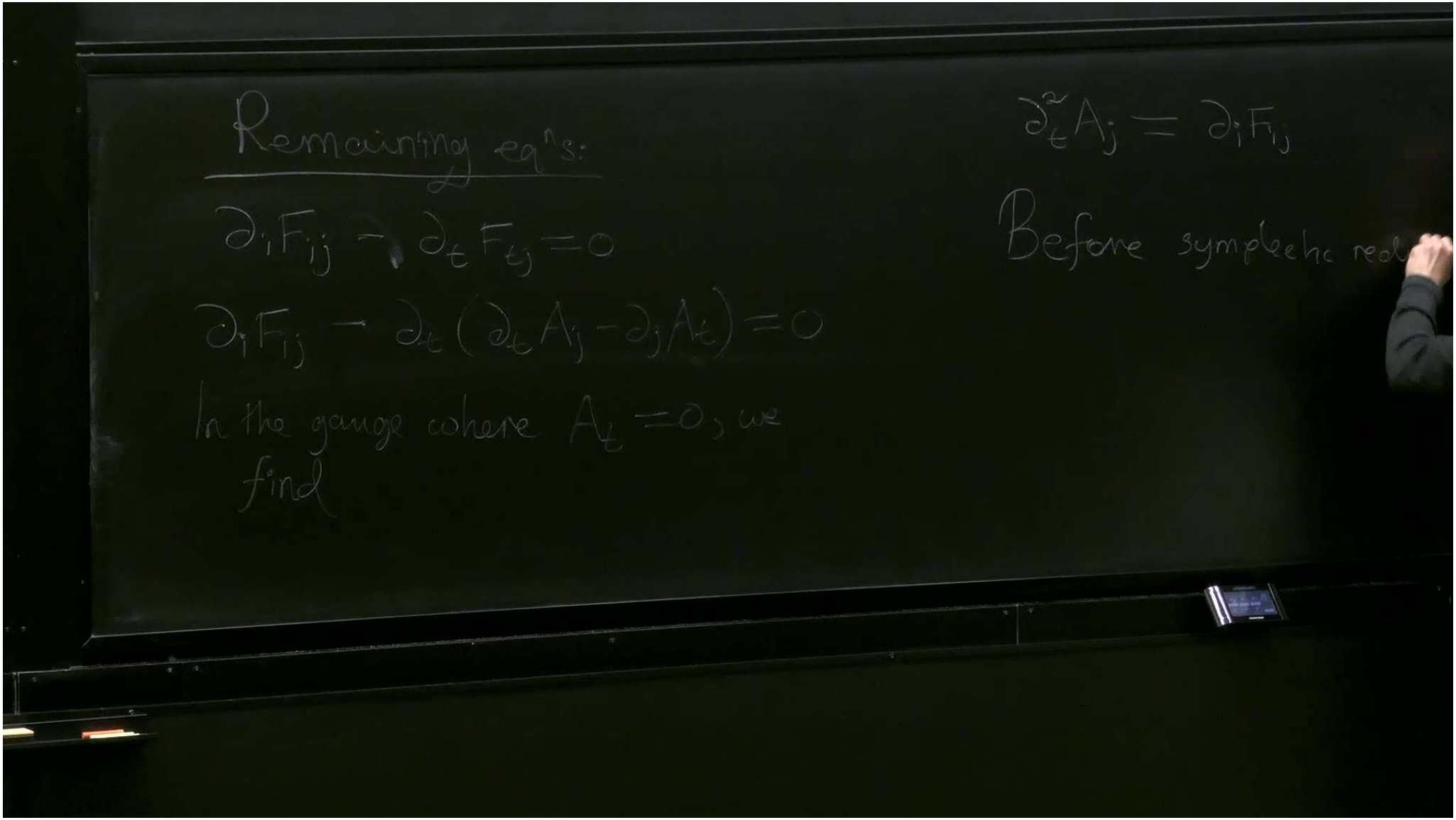
this choice absorbs t-dependence of χ,
left with χ independent of t

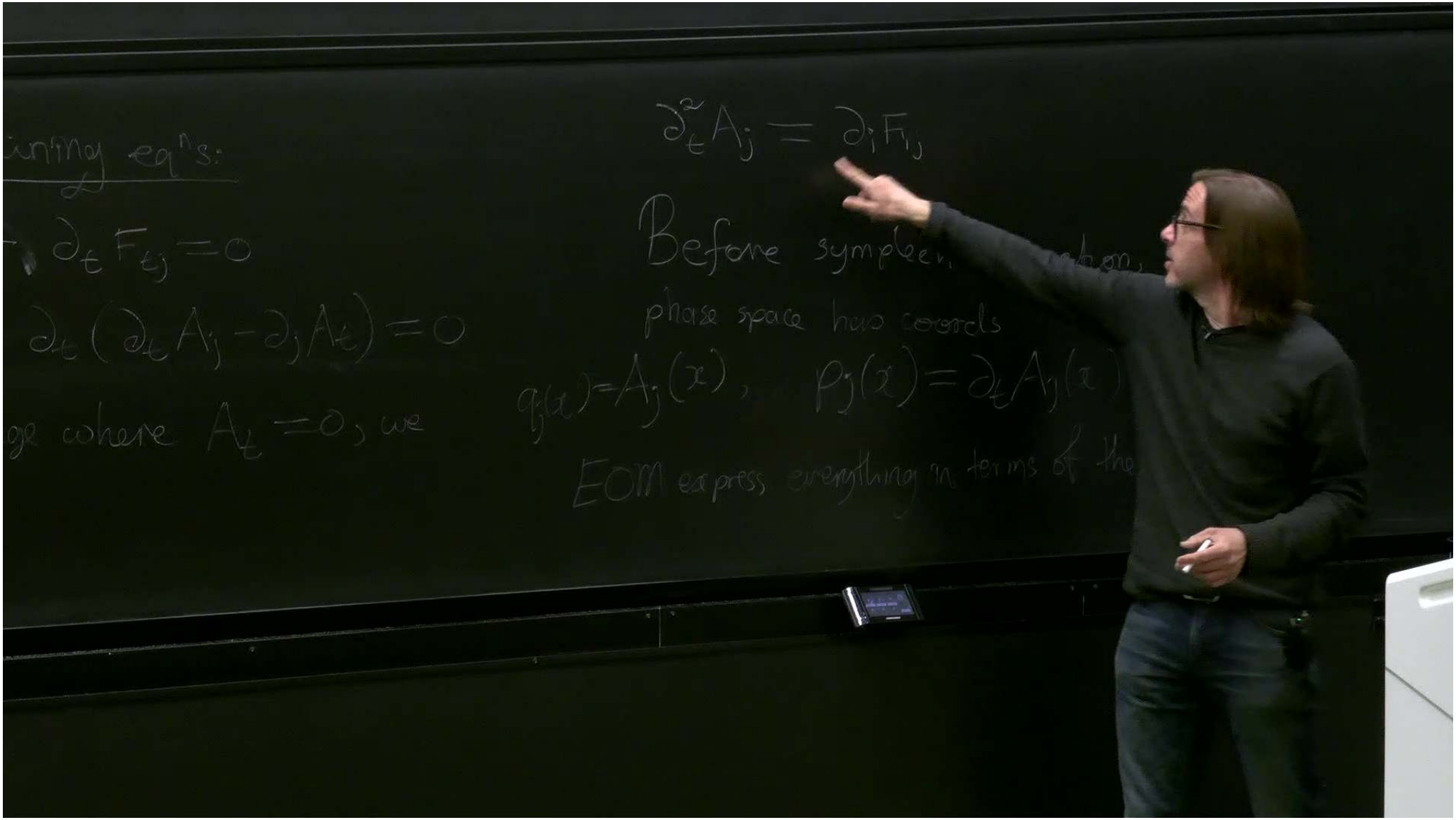
Now

Varying A_t –

in symplectic re


=0
gauge so $A_t = 0$
t-dependence of χ,
dependent of t
Note:
We still impose the eq^ come from
varying A_t - this gives $\mu = 0$
in symplectic reduction.


Remaining eq^n's:
$\partial_i F_{ij} - \partial_t F_{tj} = 0$
$\partial_i F_{ij} - \partial_t (\partial_t A_j - \partial_j A_t) = 0$
In the gauge where $A_t = 0$, we find
$\partial_t^2 A_j = \partial_i F_{ij}$
Before symplectic red


$\partial_t^2 A_j = \partial_i F_{ij}$
Before symplec
phase space has coords
$q_j(x) = A_j(x)$, $p_j(x) = \partial_t A_j(x)$
EOM express everything in terms of the
$\partial_t F_{tj} = 0$
$\partial_t(\partial_t A_j - \partial_j A_t) = 0$
where $A_t = 0$, we

$\}, q_j(x')\} = \delta_{ij}\,\delta_{x=x'}$?

propagator of YM theory.

$t \to 0$. we'll find

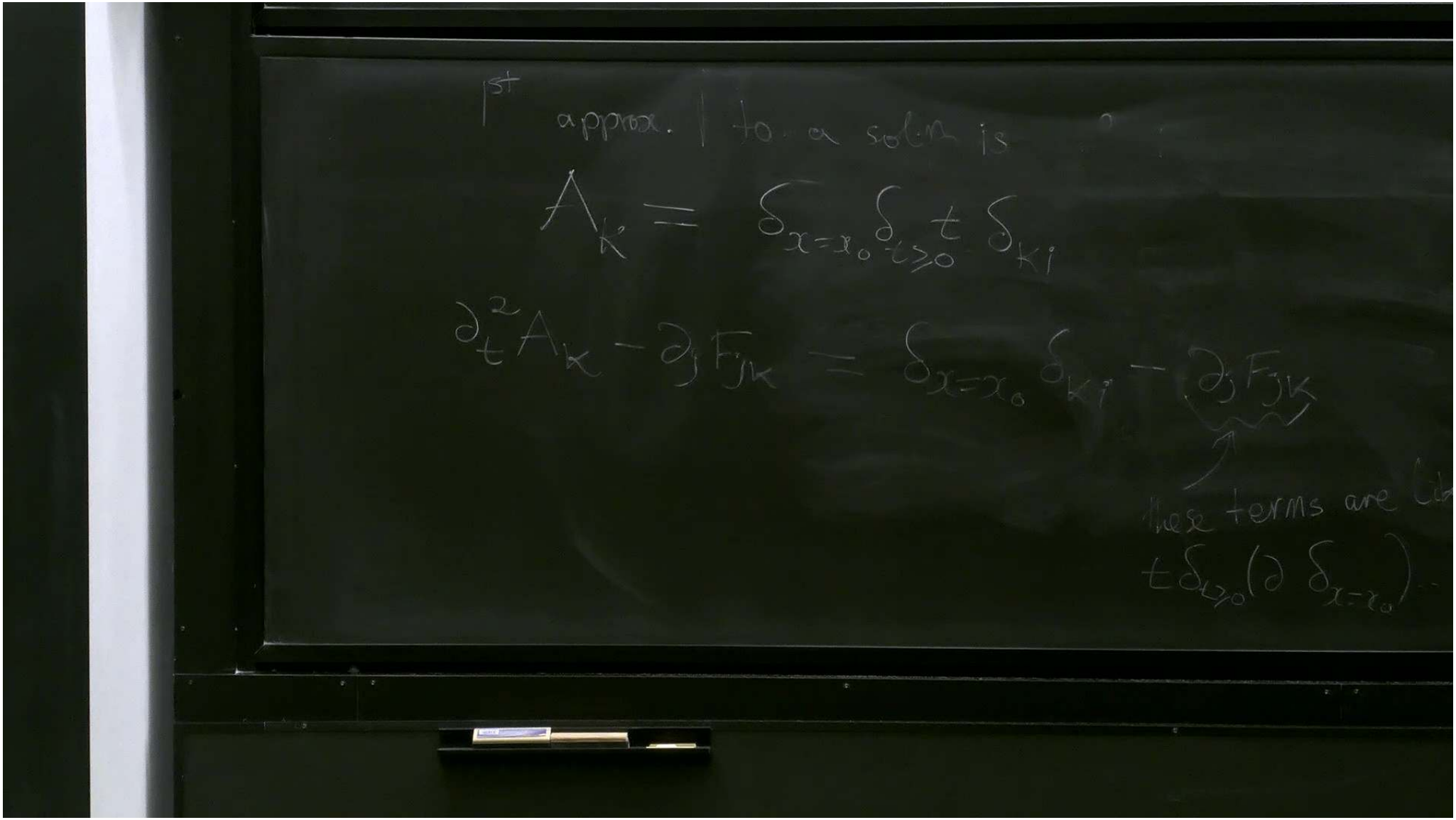
$\delta_{x=x'}\,\delta_{t>0} + \cdots$

Propagator

= Field sourced by $A_j(x,0)$ in our gauge $A_t = 0$

Solve EOM with a source term

$$\partial_t^2 A_k^{(x,t)} - \partial_i F_{ik}^{(x,t)} = \delta_{kj}\,\delta_{x=x_0}\,\delta_{t=0}$$

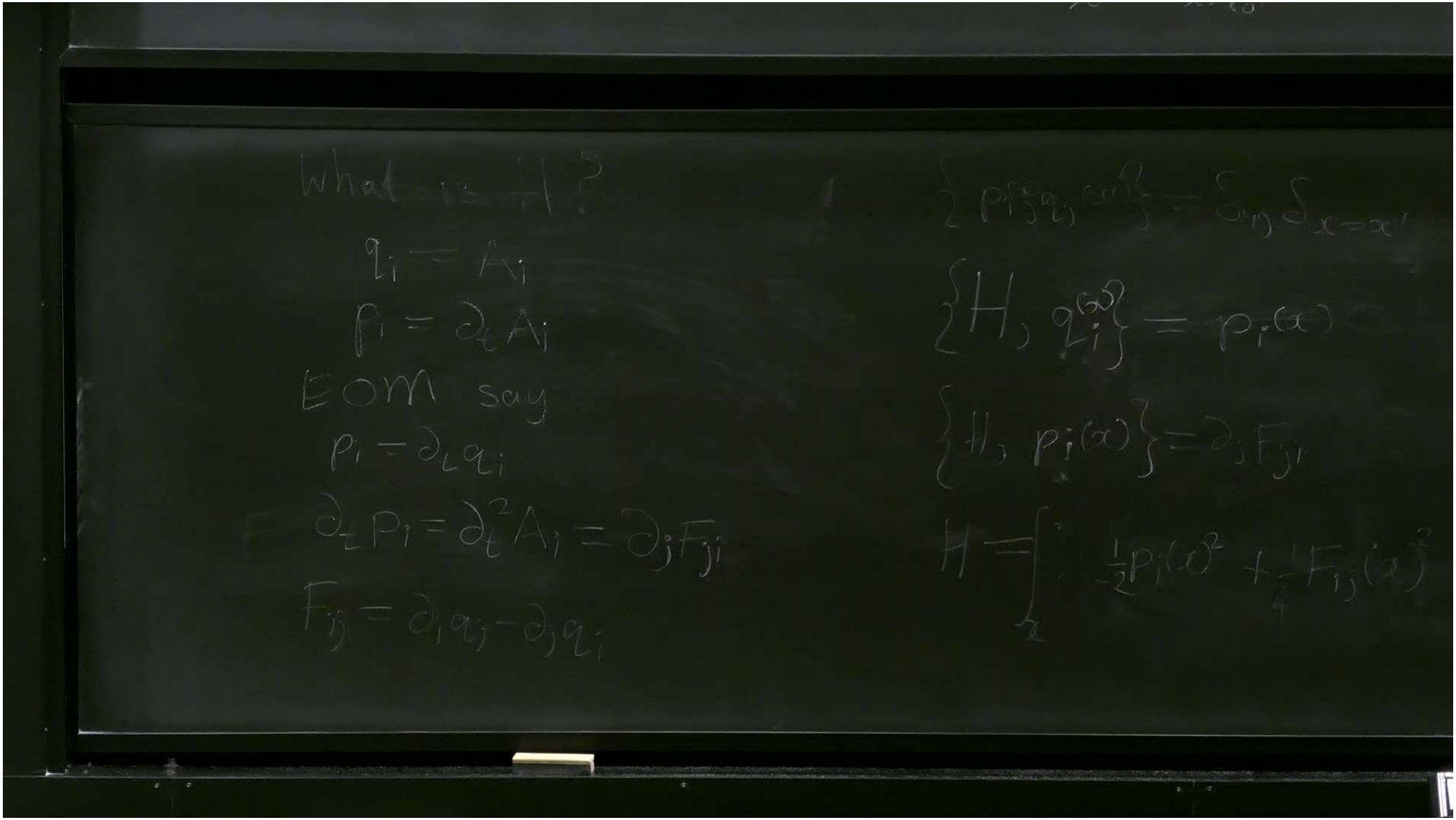

1st approx. to a soln is
$A_k = \delta_{x=x_0} \delta_{t>0} t \delta_{ki}$
$\partial_t^2 A_k - \partial_j F_{jk} = \delta_{x=x_0} \delta_{ki} - \partial_j F_{jk}$
these terms are
$t \delta_{t>0} (\partial \delta_{x=x_0})$

Full soln

$$A_k = \delta_{t>0}\delta_{k\neq 1}\left(t\delta_{x=x_0} + \frac{t^3}{6}\Delta_{space}\delta_{x=x_0} + \cdots \right)$$

2 point fn

$$A_i(x=x_0, t=0)\,\partial_t A_j(x,t) = \delta_{t>0}\,\delta_{x=x_0}\,\delta_{ij} + \text{terms that} \to 0 \text{ as } t \to 0$$

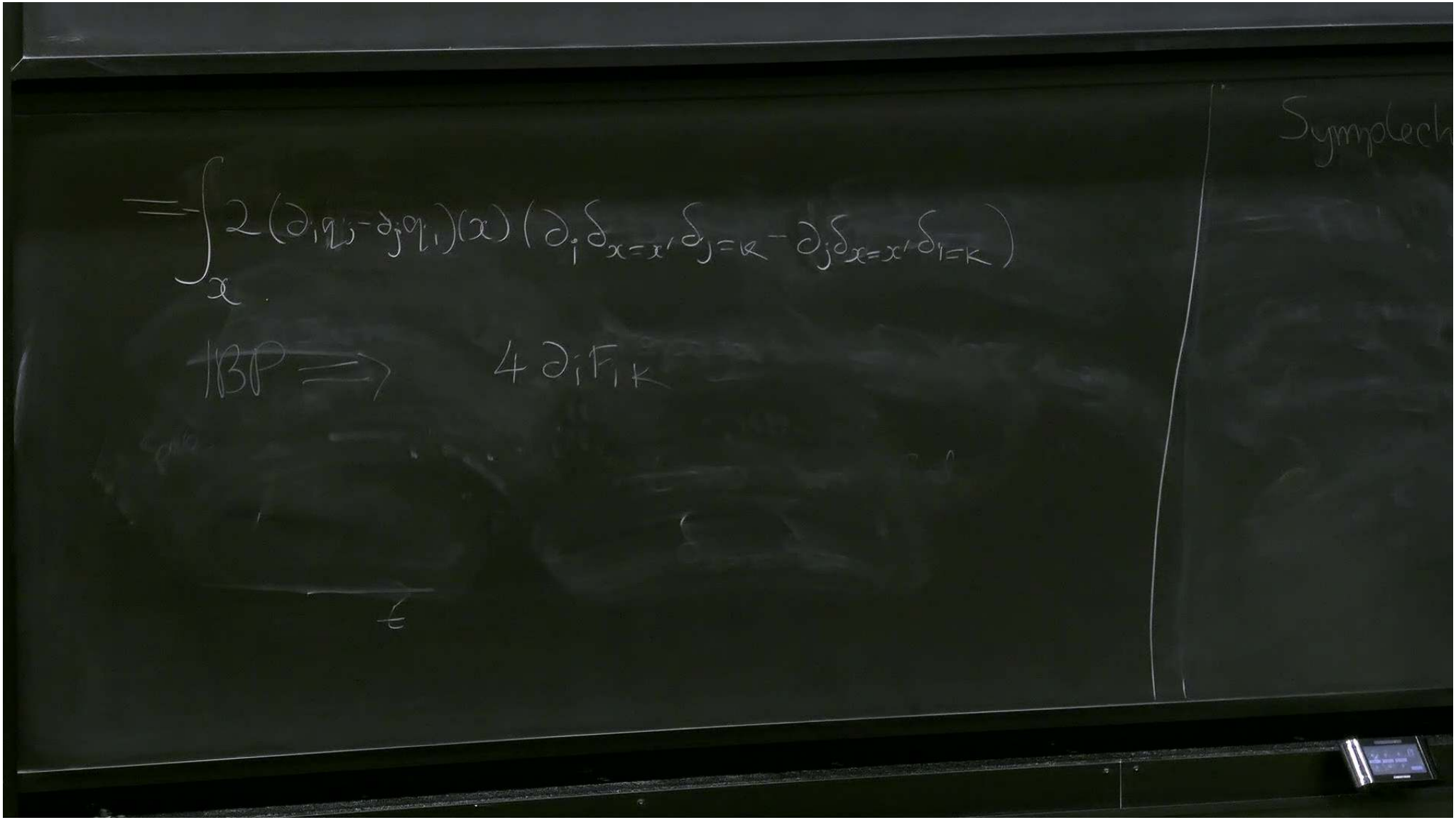

What is H?
$q_i = A_i$
$p_i = \partial_t A_i$
EOM say
$p_i = \partial_t q_i$
$\partial_t p_i = \partial_t^2 A_i = \partial_j F_{ji}$
$F_{ij} = \partial_i q_j - \partial_j q_i$
$\{p_i(x), q_j(x')\} = \delta_{ij}\delta_{x=x'}$
$\{H, q_i(x)\} = p_i(x)$
$\{H, p_i(x)\} = \partial_j F_{ji}$
$H = \int_x \frac{1}{2}p_i(x)^2 + \frac{1}{4}F_{ij}(x)^2$

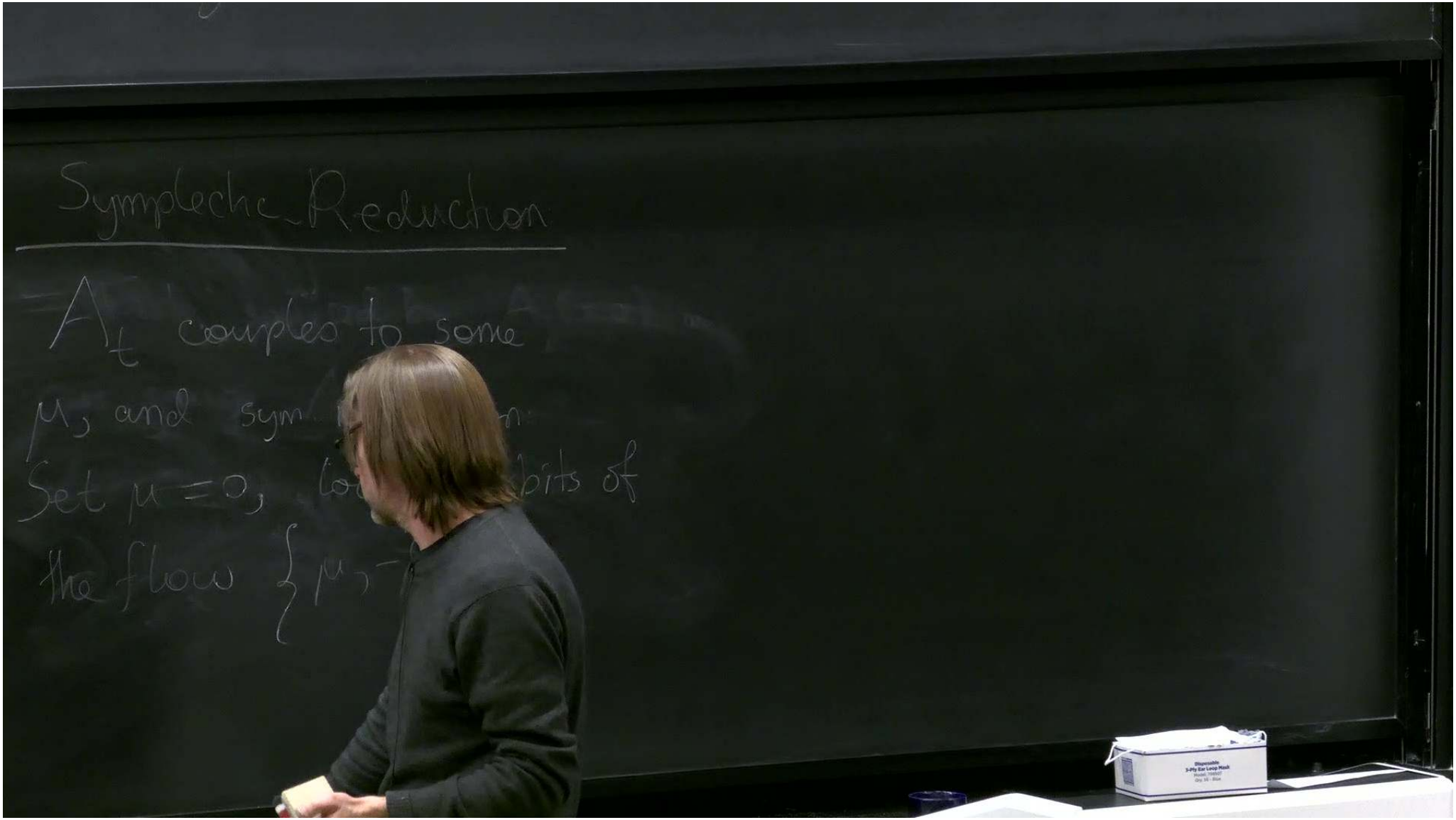
1st term is obvious

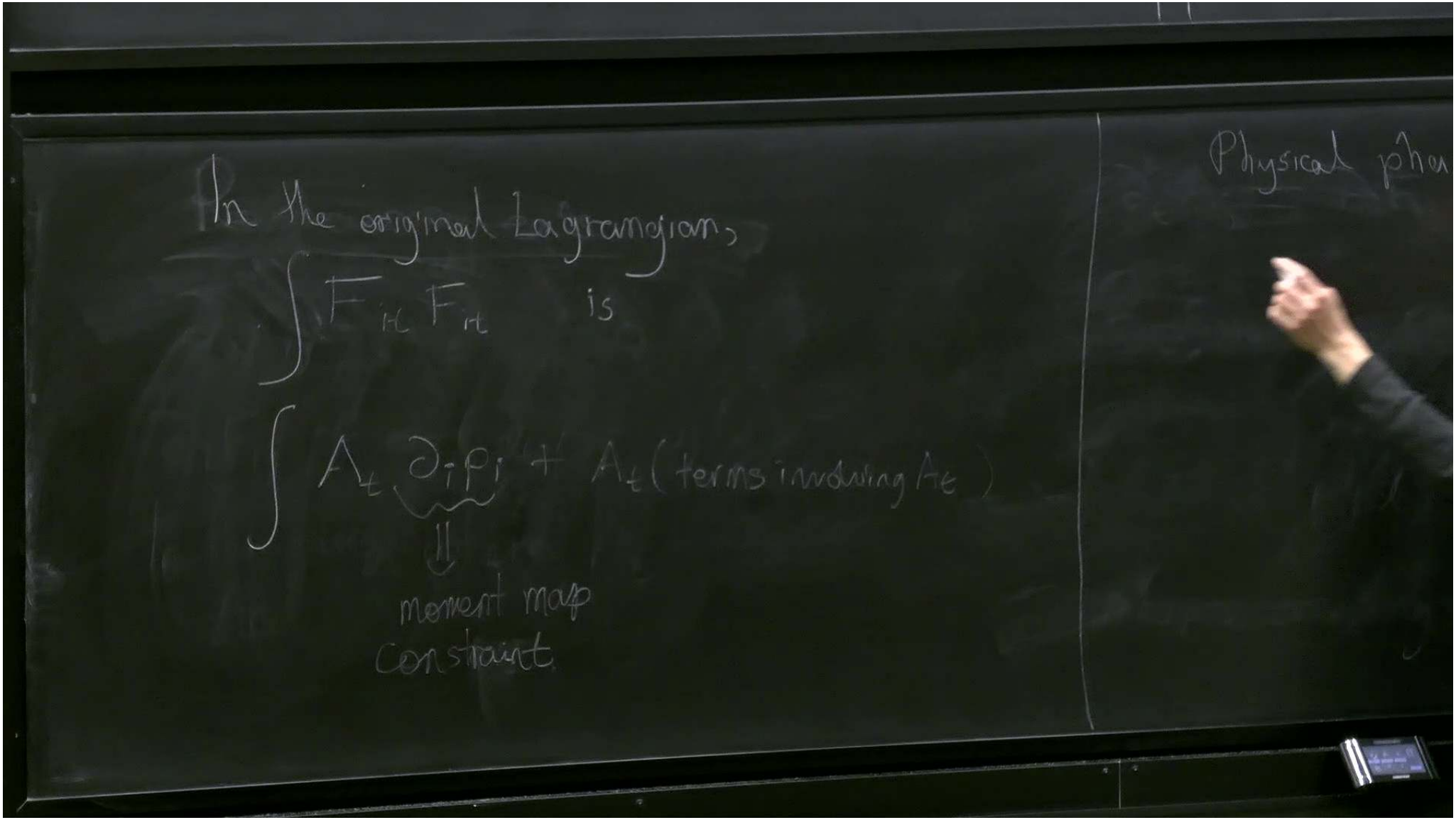
For 2nd term, we need to know

$$\left\{\int_x F_{ij}(x)^2,\ P_K(x')\right\} = 4\,\partial_i F_{iK}(x')$$

$$\int_x \left\{(\partial_i q_j - \partial_j q_i)(x)^2,\ P_K(x')\right\}$$


$= -\int_x 2(\partial_i \eta_j - \partial_j \eta_i)(x)\left(\partial_i \delta_{x=x'} \delta_{j=k} - \partial_j \delta_{x=x'} \delta_{i=k}\right)$
IBP $\Longrightarrow$
$4\,\partial_i F_{ik}$
Symplect


Symplectic Reduction
A_t coupled to some
μ_3 and sym
Set $\mu = 0$,
the flow


In the original Lagrangian,
$\int F_{it} F_{it}$ is
$\int A_t \partial_i \rho_i + A_t$ (terms involving A_t)
moment map
constraint
Physical phen

which

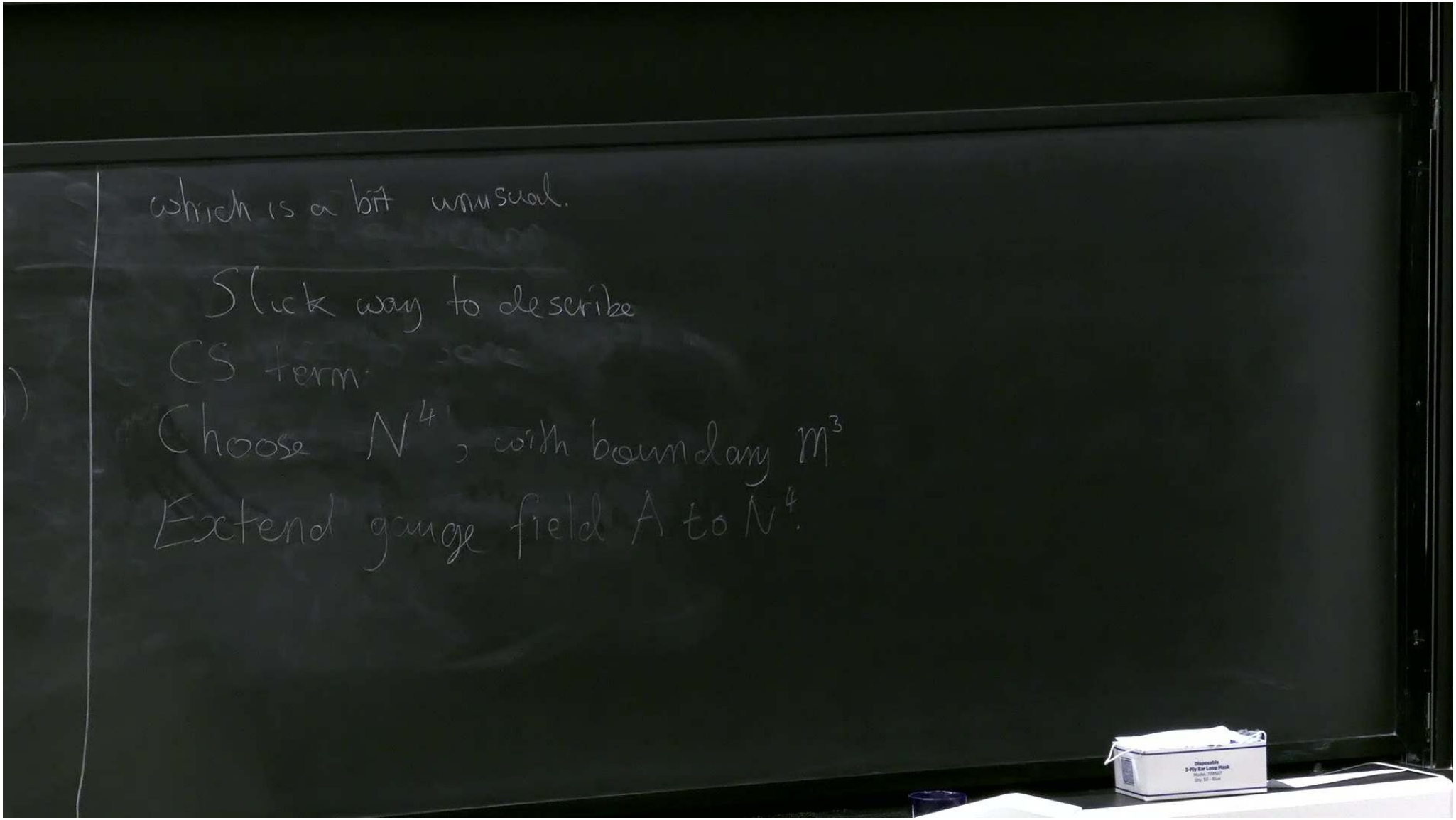
If M a manifold of dim^n 3

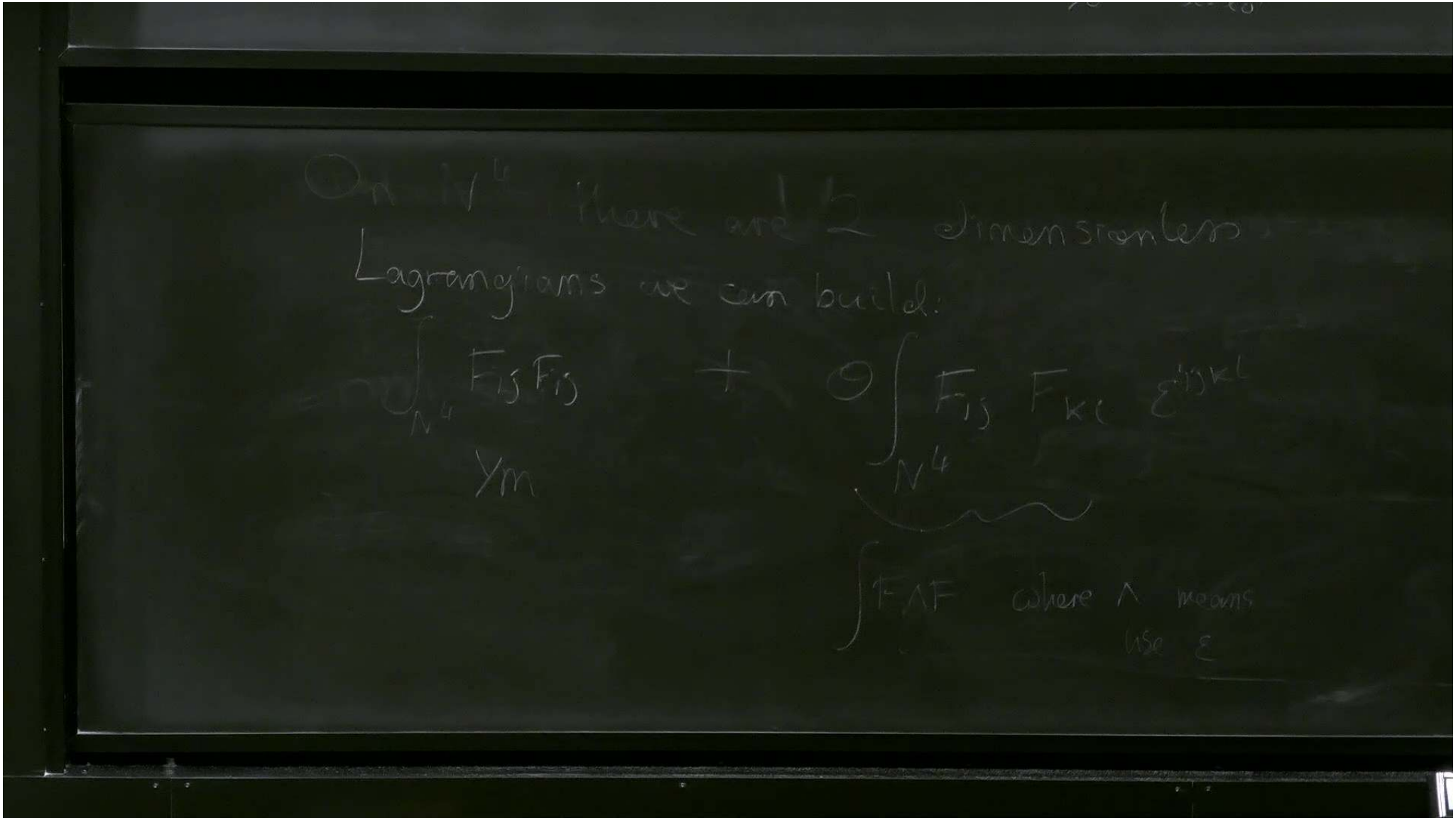
A gauge field on M (for $U(1)$ or $U(n)$)

YM Lagrangian is

$$\int F_{ij}(A) F_{ij}(A)$$

In dim^n 3 there is another term you can add;


which is a bit unusual.
Slick way to describe
CS term.
Choose N^4, with boundary M^3
Extend gauge field A to N^4.


On N^4 there are 12 dimensionless
Lagrangians we can build:
$\int_{N^4} F_{ij} F_{ij}$ + $\theta \int_{N^4} F_{ij} F_{kl} \varepsilon^{ijkl}$
YM
$\int F \wedge F$ where $\wedge$ means use ε

If N' is some other 4 manifold with $\partial N' = m$

$N \underset{m}{\cup} N'$ has no boundary

Math fact:

$\int \frac{1}{4\pi^2} F \wedge F$ is an integer.

Closed 4-manifold

In path integral, $e^{i\int \frac{k}{4\pi^2} F\wedge F} = 0$ for $k \in$ Integers $\times$ normalizing constant.

$\int \frac{1}{4\pi^2} F \wedge F$ is an integer.

Closed 4-manifold

In path integral, $e^{i\int \frac{k}{4\pi^2} F\wedge F} = 1$ for $k \in$ Integers $\times$ normalizing constant.

If N' is some other 4 manifold with $\partial N' = m$

$N \underset{m}{\cup} N'$ has no boundary

Math fact:

$$\partial A \wedge dA$$