Title: Mathematical Physics Lecture

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Phase space of Maxwell theory

$d = 2$ we found phase space has coords. $p, q \quad q \sim q+1$

Hamiltonian $= p^2$

System $=$ 1d particle on a circle.

In general, what is phase space?

$M^{d-1} \times \mathbb{R}$ $\quad M = \text{space} \quad M = \text{flat},\ g_{ij} = \delta_{ij}$

Phase space = Sol'ns to EOM mod gauge.

Field is $A = \{A_i, A_t\}$

space components

Lagrangian is

$$F_{ij} F_{ij} + F_{tj} F_{tj}$$

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

$$F_{tj} = \partial_t A_j - \partial_j A_t$$

EOM are:

$$\partial_i F_{ij} + \partial_t F_{tj} = 0$$

$$\partial_i F_{ti} = 0$$

We choose a gauge s

$$\partial_i F_{ij} + \partial_t F_{tj} = 0$$

$$\partial_i F_{ti} = 0$$

We choose a gauge so $A_t = 0$

$$\delta A_t = \partial_t \chi$$

this choice absorbs t-dependence of χ,
left with χ independent of t

Note

We still impose th

varying A_t -

in symplectic re

$= 0$

gauge so $A_t = 0$

t-dependence of x,
dependent of t

NOTE

We still impose the eq^n come from varying A_t - this gives $\mu = 0$ in symplectic reduction.

Remaining eq^{n}s:

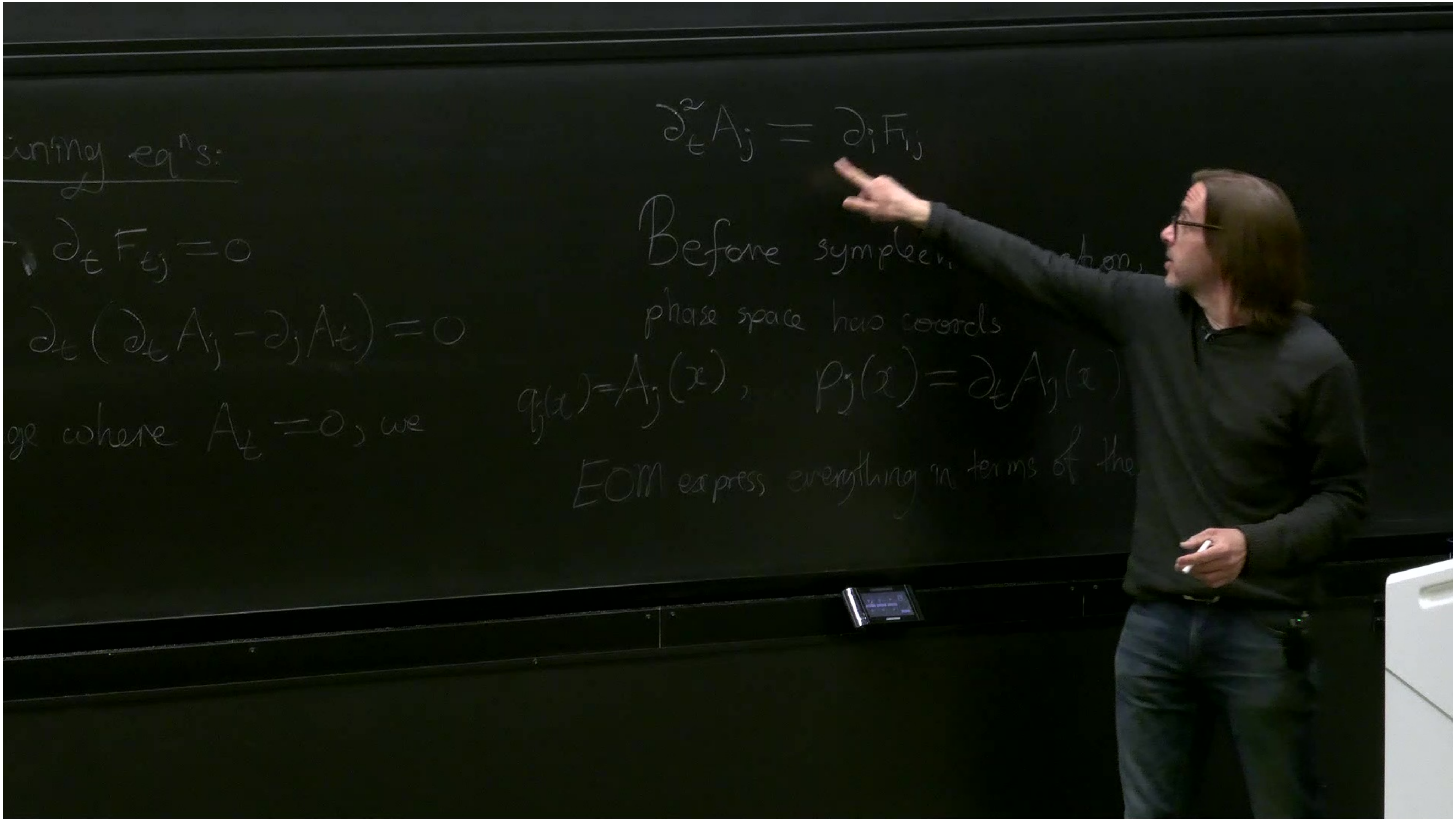
$$\partial_i F_{ij} - \partial_t F_{tj} = 0$$

$$\partial_i F_{ij} - \partial_t(\partial_t A_j - \partial_j A_t) = 0$$

In the gauge where $A_t = 0$, we find

$$\partial_t^2 A_j = \partial_i F_{ij}$$

Before symplectic redu


ining eqns:
$\partial_t F_{tj} = 0$
$\partial_t(\partial_t A_j - \partial_j A_t) = 0$
ge where $A_t = 0$, we
$\partial_t^2 A_j = \partial_i F_{ij}$
Before
phase space has coords
$q_j(x) = A_j(x)$, $p_j(x) = \partial_t A_j(x)$
EOM express everything in terms of the

$), q_j(x')\} = \delta_{ij}\delta_{x=x'}$?

propagator of YM theory.

$t \to 0$. we'll find

$\delta_{x=x'}\delta_{t>0} + \cdots$

Propagator

$=$ Field sourced by $A_j(x,0)$ in

our gauge $A_t = 0$

Solve EOM with a source term

$$\partial_t^2 A_k^{(x,t)} - \partial_i F_{ik}^{(x,t)} = \delta_{kj}\delta_{x=x_0}\delta_{t=0}$$

1st approx. to a soln is

$$A_k = \delta_{x=x_0}\delta_{t\geq 0}\, t\, \delta_{ki}$$

$$\partial_t^2 A_k - \partial_j F_{jk} = \delta_{x=x_0}\delta_{ki} - \partial_j F_{jk}$$

these terms are lik

$$t\,\delta_{t\geq 0}(\partial\,\delta_{x=x_0})\dots$$

Full sol^n

$$A_k = \delta_{t\geq 0}\,\delta_{ki}\left(t\,\delta_{x=x_0} + \frac{t^3}{6}\,\Delta_{space}\,\delta_{x=x_0} + \cdots\cdots \right)$$

2 point fn

$$A_i(x=x_0, t=0)\,\partial_t A_j(x,t) = \delta_{t\geq 0}\,\delta_{x=x_0}\,\delta_{ij} + \text{terms that} \to 0 \text{ as } t \to 0.$$

What is H?

$$q_i = A_i$$

$$p_i = \partial_t A_i$$

EOM say

$$p_i = \partial_t q_i$$

$$\partial_t p_i = \partial_t^2 A_i = \partial_j F_{ji}$$

$$F_{ij} = \partial_i q_j - \partial_j q_i$$

$$\{p_i(x), q_j(x')\} = \delta_{ij}\,\delta_{x=x'}$$

$$\{H, q_i(x)\} = p_i(x)$$

$$\{H, p_i(x)\} = \partial_j F_{ji}$$

$$H = \int_x \frac{1}{2} p_i(x)^2 + \frac{1}{4} F_{ij}(x)^2$$

1^{st} term is obvious

For 2^{nd} term, we need to know

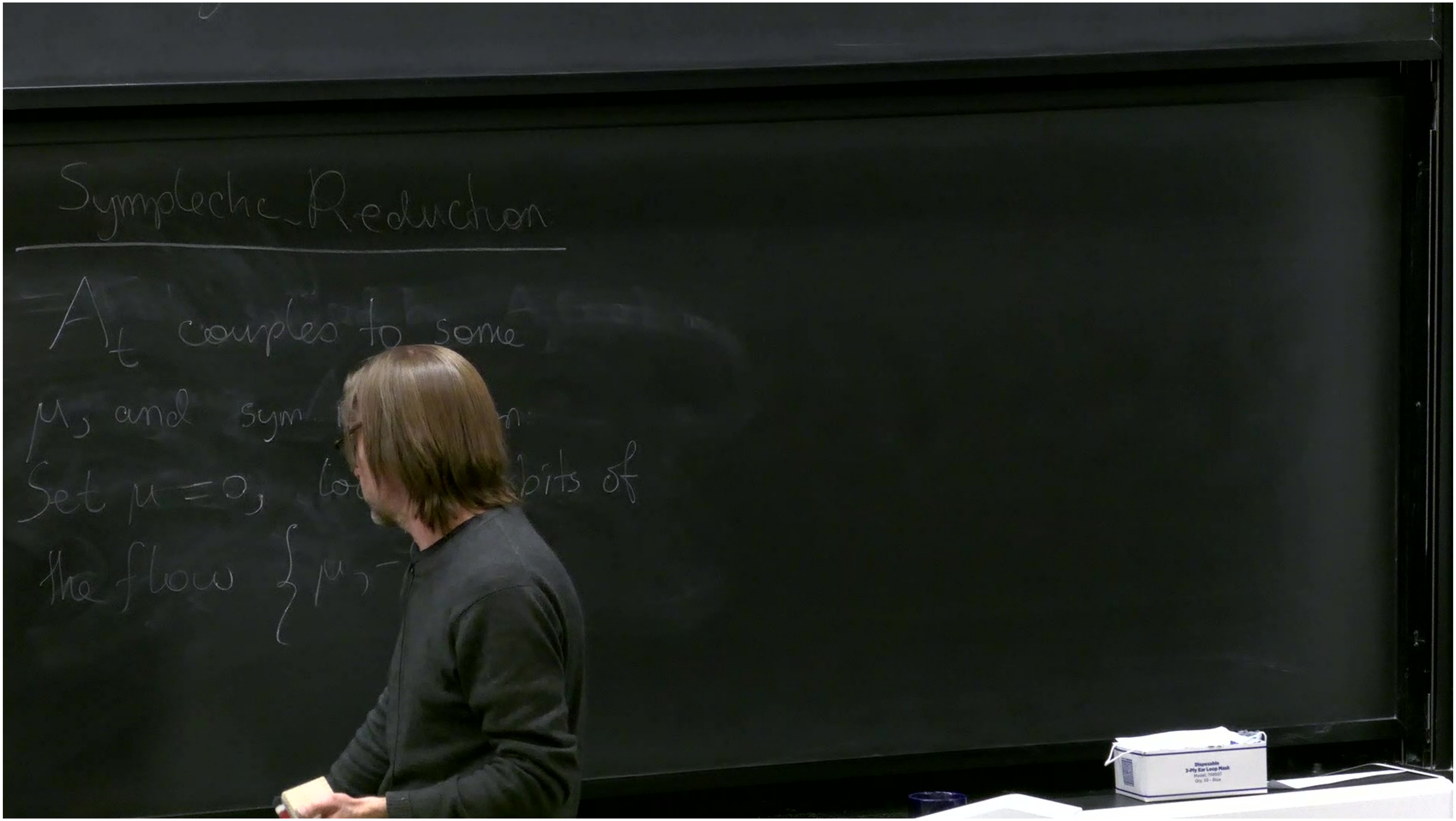
$$\left\{ \int_x F_{ij}(x)^2 , p_k(x') \right\} = 4\, \partial_i F_{ik}(x')$$

$$\int_x \left\{ (\partial_i q_j - \partial_j q_i)(x)^2 , p_k(x') \right\}$$

$$= \int_x 2(\partial_i \eta_j - \partial_j \eta_i)(x)\left(\partial_i \delta_{x=x'} \delta_{j=k} - \partial_j \delta_{x=x'} \delta_{i=k}\right)$$

$$\text{IBP} \Longrightarrow \quad 4\,\partial_i F_{ik}$$

Symplecti


Symplectic Reduction
A_t couples to some
Set μ = 0,
bits of
the flow

In the original Lagrangian,

$$\int F_{it} F_{it} \quad \text{is}$$

$$\int A_t \underbrace{\partial_i p_i}_{\Downarrow} + A_t(\text{terms involving } A_t)$$

moment map constraint

Physical phen

which

If $\mathcal{M}$ a manifold of dim^n 3

A gauge field on $\mathcal{M}$ (for $U(1)$ or $U(n)$)

YM Lagrangian is

$$\int F_{ij}(A) F_{ij}(A)$$

In dim^n 3 there is another term you can add, which is a bit unusual.

Slick way to describe

CS term:

Choose N^4, with boundary M^3

Extend gauge field A to N^4.

On N^4 there are 2 dimensionless Lagrangians we can build:

$$\underbrace{\int_{N^4} F_{ij} F_{ij}}_{YM} + \underbrace{\theta \int_{N^4} F_{ij} F_{kl} \varepsilon^{ijkl}}_{\int F \wedge F}$$

where $\wedge$ means use ε

If N' is some other 4 manifold

with $\partial N' = M$

$N \bigcup_{M} N'$ has no boundary

Math fact:

$$\int_{\substack{\text{Closed} \\ \text{4-manifold}}} \frac{1}{4\pi^2} F \wedge F \quad \text{is an integer}$$

In path integral, $e^{i\int \frac{k}{4\pi^2} F \wedge F} = 0$ for $k \in$ Integers $\times$ normalizing constant

$$\int_{\substack{\text{Closed} \\ \text{4-manifold}}} \frac{1}{4\pi^2} F \wedge F \quad \text{is an integer.}$$

In path integral, $e^{i\int \frac{k}{4\pi^2} F\wedge F} = 1$ for $k \in$ Integers $\times$ normalizing constant

If N' is some other 4 manifold
with $\partial N' = m$

$N \cup_m N'$ has no boundary

Math fact:

$$\theta \, A \wedge dA$$