Title: Mathematical Physics Lecture

Speakers: Kevin Costello

Collection: Mathematical Physics 2023/24

Date: April 19, 2024 - 11:30 AM

URL: https://pirsa.org/24040065

Gauge theories + Symplectic Reduction

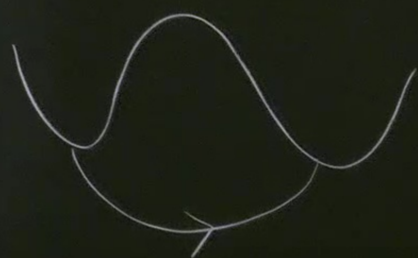
Gauge

Example

Particle moving on a circle

$$\int (\partial_t \varphi)^2$$

$$\varphi \sim \varphi + 1$$



Gauge theories + Symplectic Reduction

Example

Particle moving on a circle

$$\int (\partial_t \varphi)^2$$

$$\varphi \sim \varphi + 1$$

Gauge rotational symmetry of circle

Rotation sends $\varphi \longrightarrow \varphi + c$

Covariant derivative is

$$D_t \varphi = \partial_t \varphi + A_t$$

Action is $\int (D_t \varphi)^2 = \int (\partial_t \varphi)^2 + 2 \partial_t \varphi A_t + A_t^2$

Phase Space

= Sol^{n}s to EOM
mod gauge

Vary φ: $D_t^2 \varphi = 0$

This gives usual phase space coordinates

$$q = \varphi$$
$$p = D_t \varphi$$

$= 0$

se space

<u>Note</u> Action can be written in 1st order form

$$\int \frac{1}{2}p^2 + (p\partial_t q + pA_t)$$

integrating out $p \rightsquigarrow (\partial_t q + A_t)^2$

The function (in this case p) that couples to A_t is called the "moment map"

If we vary A_t:

2^{nd} order form: $D_t \varphi = 0$

1^{st} order: $p = 0$

Then, p

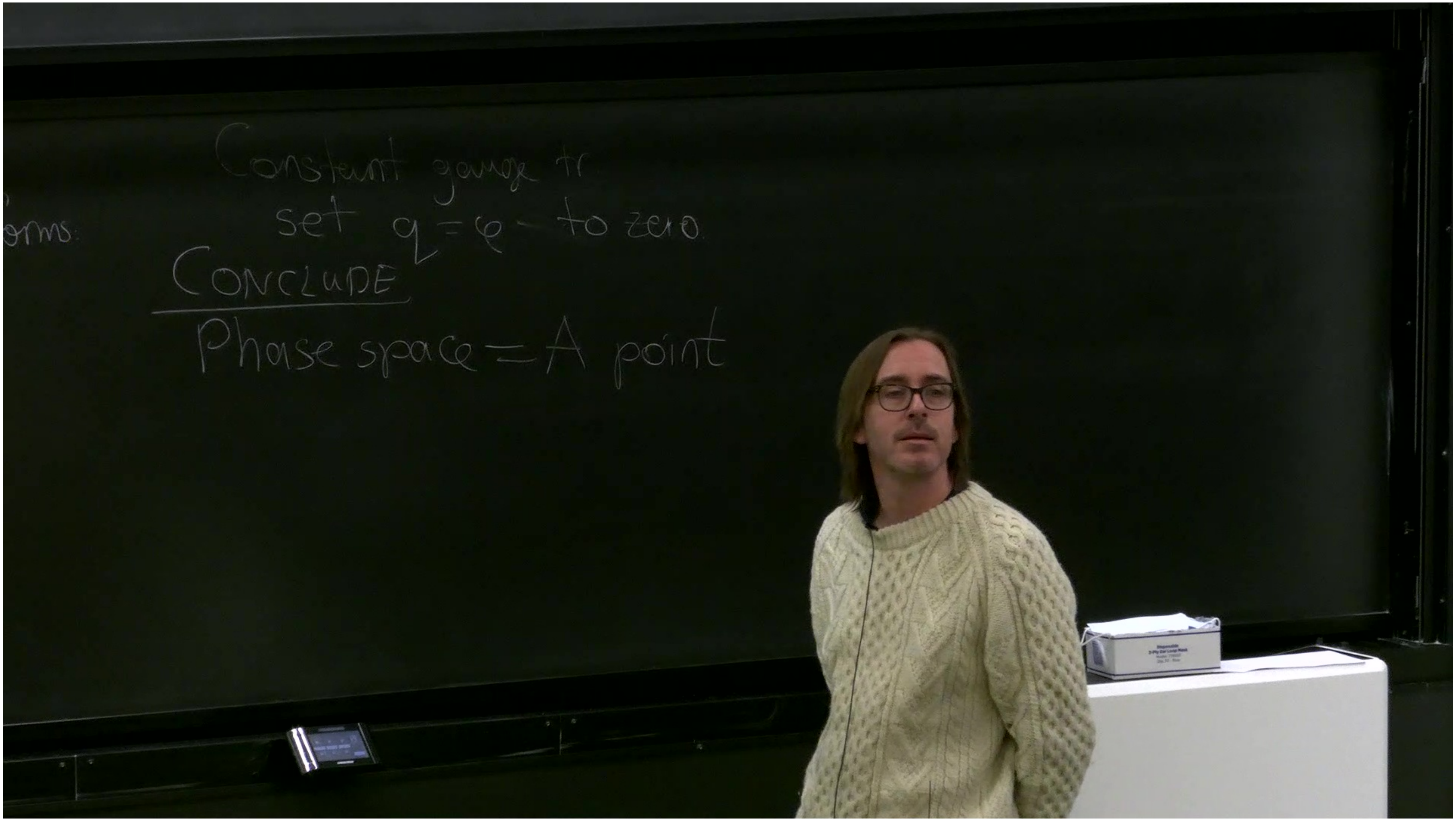
Constr

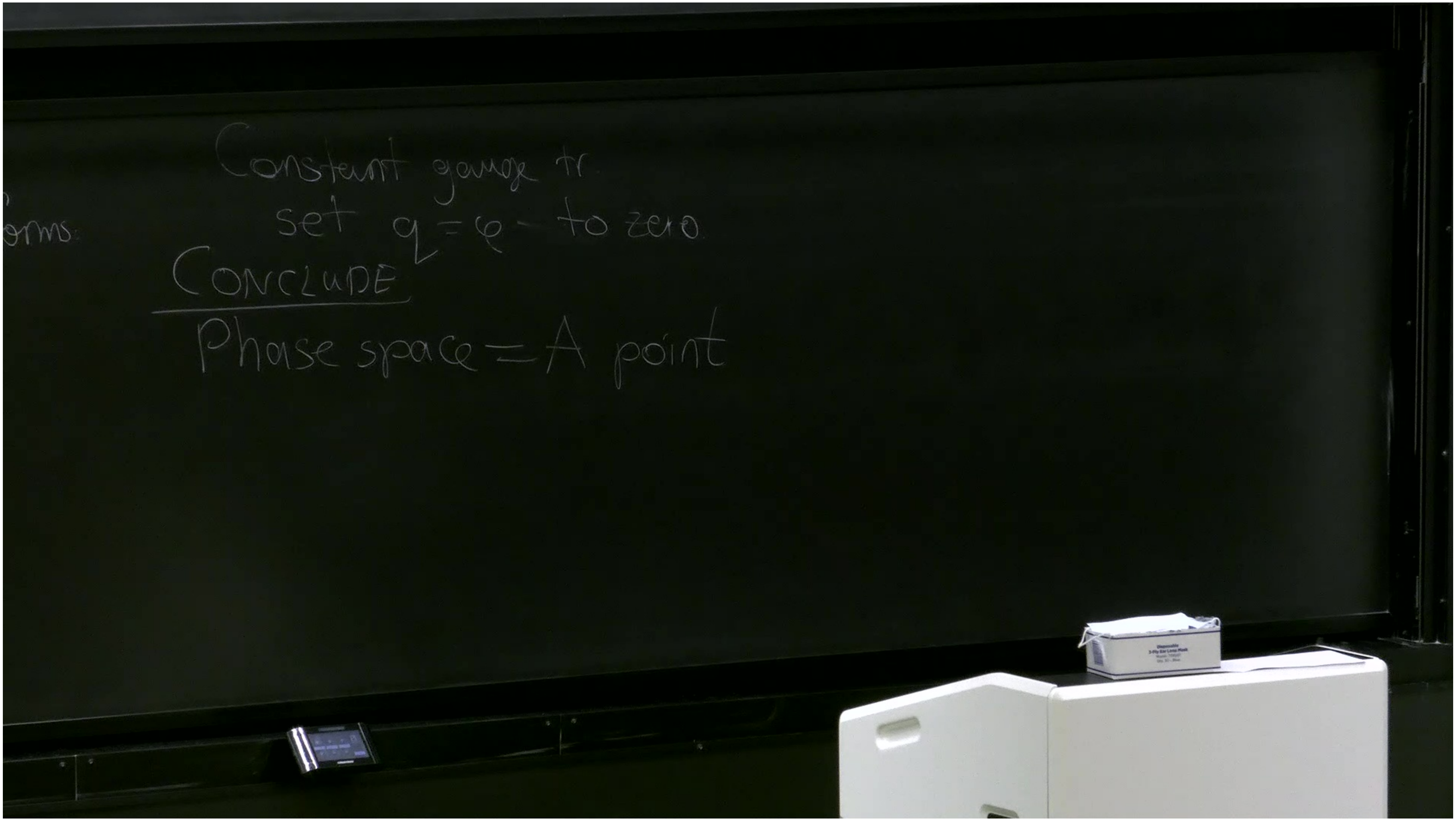
Infinitesimally, gauge symmetry transforms:

$$\delta A_t = \partial_t \chi$$

$$\delta \varphi = \chi$$

Set $A_t = 0$ by gauge transform


orms:
Constant gauge tr
set q = φ to zero
CONCLUDE
Phase space = A point

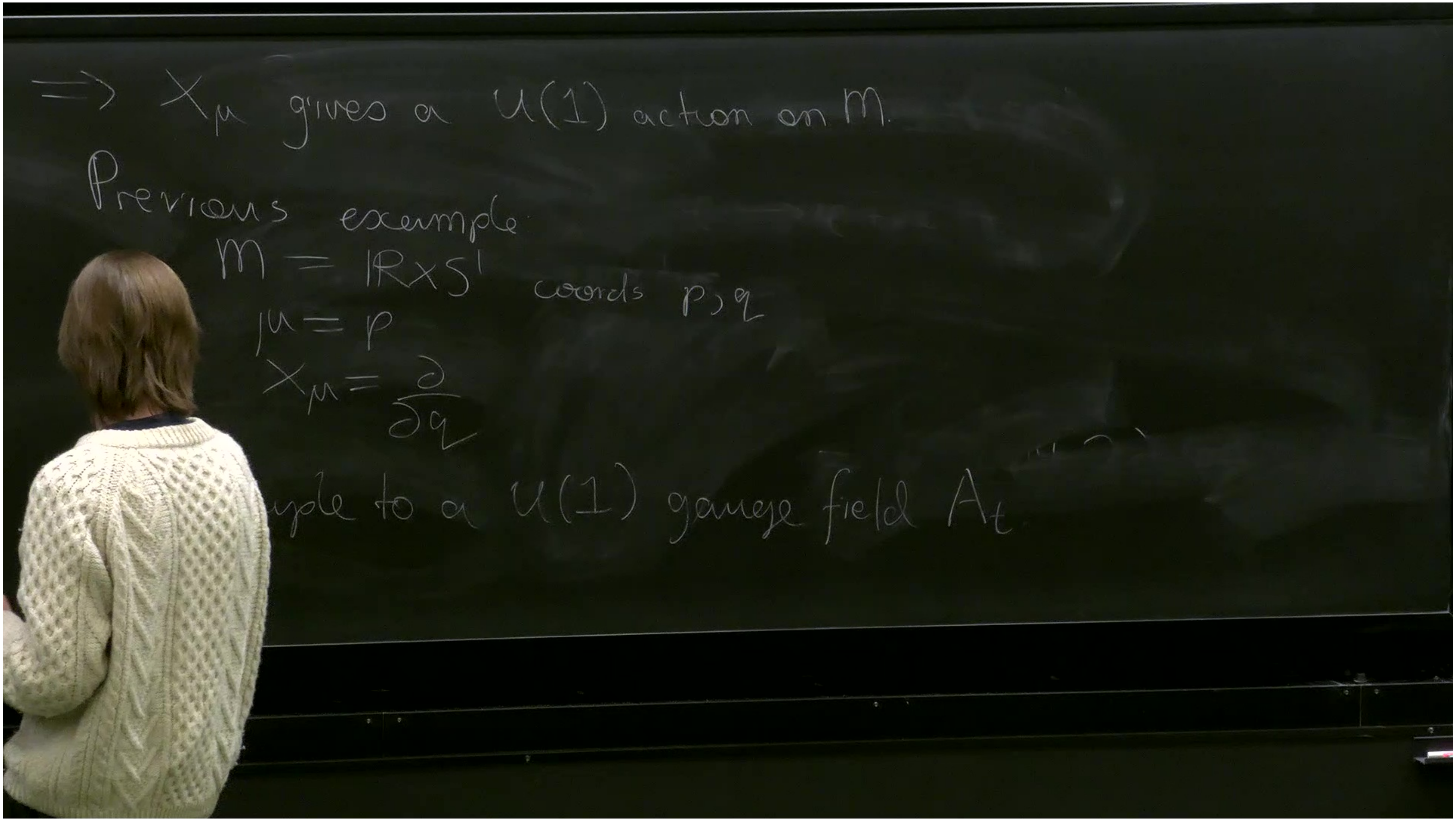

orms:
Constant gauge tr.
set q = φ to zero
CONCLUDE
Phase space = A point

M is a symplectic manifold

μ is a function on M

$X_\mu = \{\mu, -\}$ corresponding vector field.

Suppose flows of X_μ are periodic period 1


$\Longrightarrow$ X_μ gives a $U(1)$ action on M.
Previous example
$M = \mathbb{R} \times S^1$ coords p, q
$\mu = p$
$X_\mu = \frac{\partial}{\partial q}$
uple to a $U(1)$ gauge field A_t.

In words p_i, q_i, the Lagrangian is

$$\int_{\mathbb{R}} p_i(t)\, dq_i(t) + \mu(p(t), q(t)) A_t$$

(for a particle moving in M).

Gauge trans. are

$\delta p = \{\mu, p\} \chi$

$\delta q = \{\mu, q\} \chi$

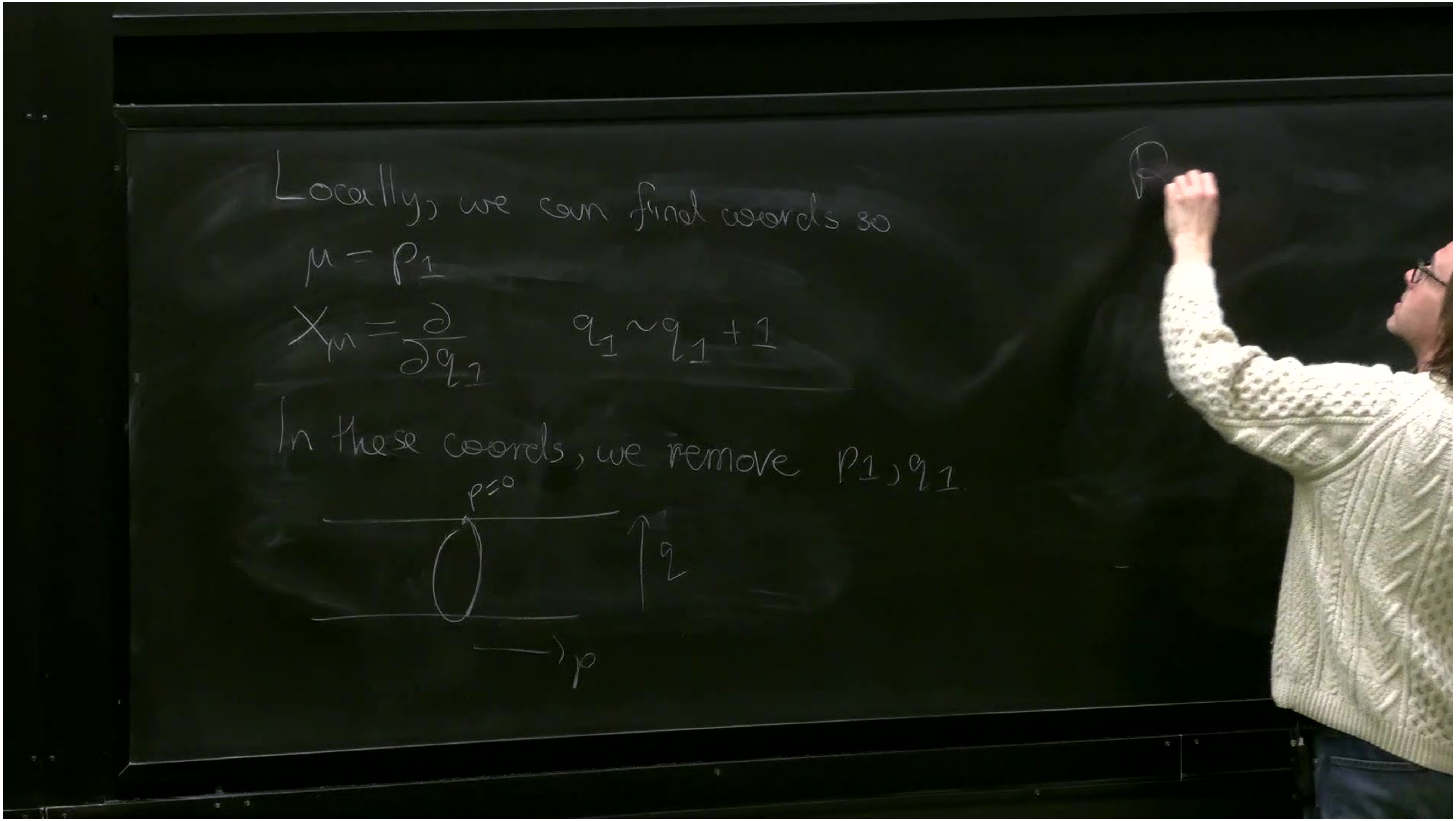
$\delta A = \partial_t \chi$

Phase Space

p, q covariant constant
determined by $p(0), q(0)$

Vary A_t : $\boxed{\mu = 0}$

Phase space =
Quotient of $\{\mu = 0\}$ by $U(1)$ action generated by X_μ


Locally, we can find coords so
$\mu = p_1$
$X_\mu = \frac{\partial}{\partial q_1}$ $\quad q_1 \sim q_1 + 1$
In these coords, we remove p_1, q_1
$p=0$
q
p

Result is again symplectic

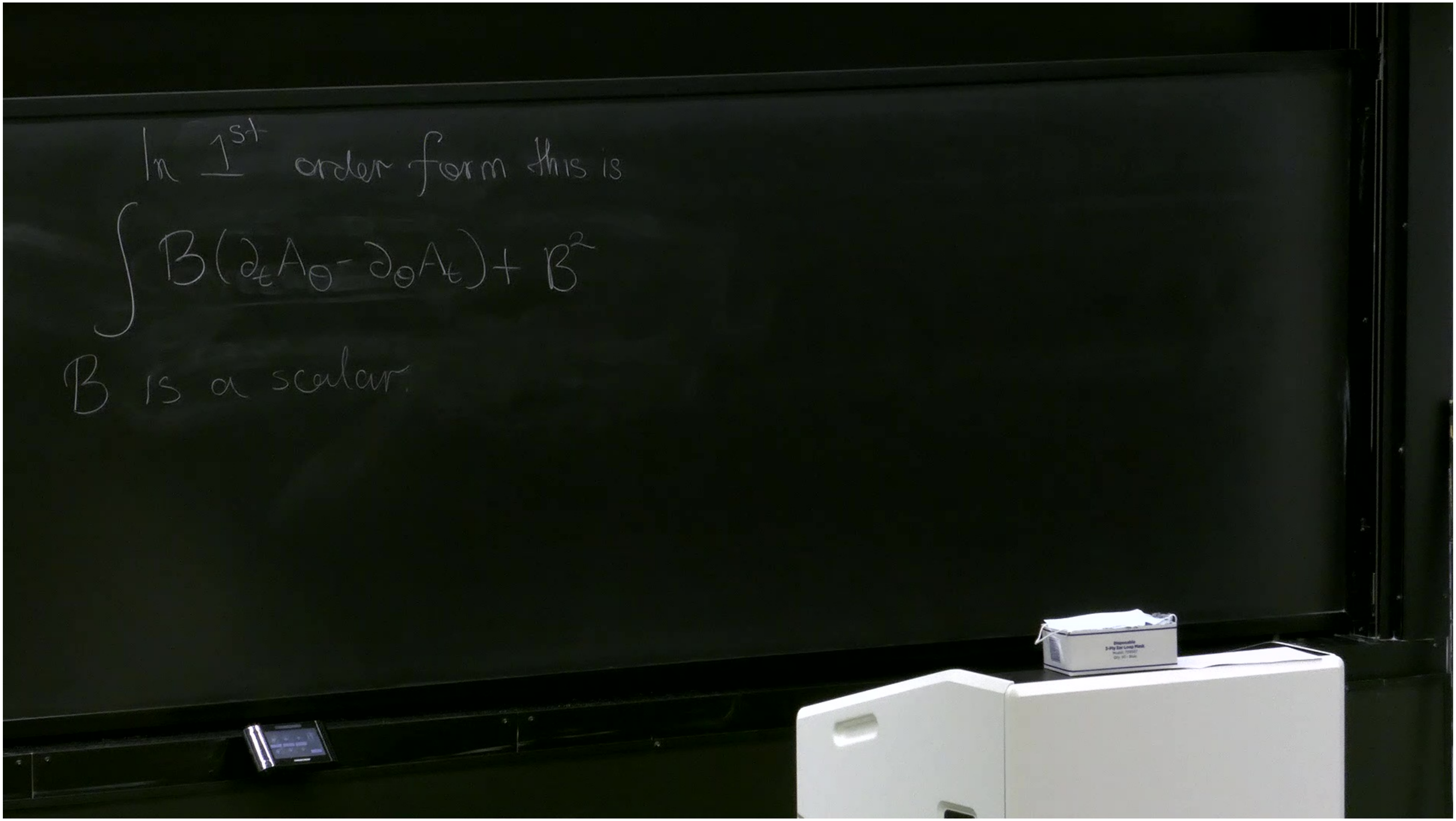
This procedure is called symplectic reduction

Example

Phase space of YM on $\underset{t}{\mathbb{R}} \times \underset{\theta}{S^1}$

Action is

$$\int \left(\partial_t A_\theta - \partial_\theta A_t\right)^2$$


In 1st order form this is
$\int B(\partial_t A_\theta - \partial_\theta A_t) + B^2$
B is a scalar.

Equations of motion:

[Looks like $\int \rho(\theta)\partial_t q(\theta) + \rho(\theta)^2 + \underbrace{\partial_\theta \rho(\theta)}_{\text{moment map}} A_t$]

Vary A_θ: $\partial_t B = 0$

Vary B: $\partial_t A_\theta - \partial_\theta A_t + 2B = 0$

Vary A_t moment map equation

$\partial_\theta B = 0$.

Gauge transformations set $A_t = 0$, leaving us with

$X(\theta)$ only depend on θ.

where w

Conclude

Phase space is the symplectic reduction of phase space for a 2d free scalar $p(\theta), q(\theta)$

where we reduce by the action of gauge trans. $\chi(\theta)$ w. moment map $\partial_\theta p(\theta)$.

This acts by
$$\delta q(\theta) = \partial_\theta \chi(\theta)$$
$$\delta p(\theta) = 0.$$

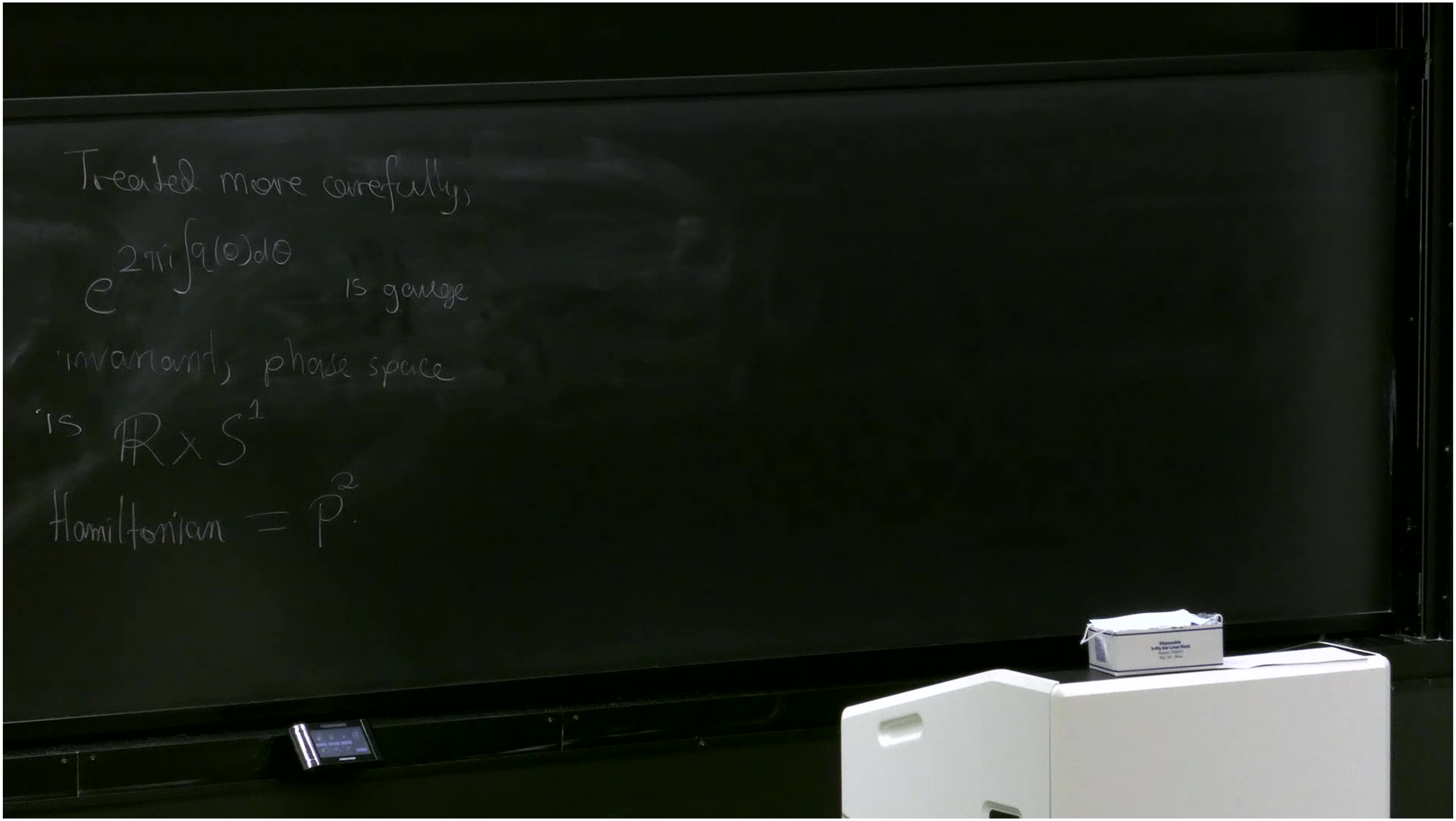
Treated more c

$p(\theta) =$ constant, say p

$q(\theta)$ is gauge equivalent to $\int q(\theta)\,d\theta$

Phase space is 2 dim^n, coords

$p = p(\theta)$, and $\int q(\theta)\,d\theta$


Treated more carefully,
$e^{2\pi i \int q(\theta) d\theta}$ is gauge
invariant, phase space
is $\mathbb{R} \times S^1$
Hamiltonian $= p^2$.