Gauge theories + Symplectic Reduction

Example

Particle moving on a circle

\[ \int (a \cdot \phi)^2 \]

\[ \phi \sim \phi + 1 \]
Gauge theories + Symplectic Reduction

Example

Particle moving on a circle

\[ \int (2 \pi \theta)^2 \]

\[ \phi \sim \phi + 1 \]
Gauge rotational symmetry of circle

Rotation sends $\varphi \rightarrow \varphi + c$

Covariant derivative is

$$D_t \varphi = \partial_t \varphi + A_t$$

Action is

$$\int (D_t \varphi)^2 = \int (\partial_t \varphi)^2 + 2\partial_t \varphi A_t + A_t^2$$
Phase Space

Solutions to EOM mod gauge

Vary $\phi$:

$D_{+}\phi = 0$

This gives usual phase space coordinates $q = \phi$

$p = D_{+}\phi$
Note: Action can be written in 1st order form

\[ \int \frac{1}{2} p^2 + (\rho \partial \xi + \rho A_\xi) \]

Integrating out \( p \) \( \implies (\partial \xi + A_\xi)^2 \)
The function (in this case $p$) that couples to $A_t$ is called the "moment map.

If we vary $A_t$:

2nd order form: $D_t \psi = 0$

1st order: $p = 0$
Infinitesimally, gauge symmetry transforms:

\[ \delta A_t = \partial_t X \]
\[ \delta \varphi = X \]

Set \( A_t = 0 \) by gauge transform
\textbf{Conclude}

Phase space = A point
Constant gmax \( tr. \)
set \( q = \varphi \to 0 \).

Conclude

Phase space = A point
$M$ is a symplectic manifold
\[ \mu \text{ is a function on } M \]
\[ X_\mu = \{ \mu \text{, corresponding vector field} \} \]
Suppose flows of $X_\mu$ are periodic period 1
\[ \Rightarrow X^\mu \text{ gives a } U(1) \text{ action on } M. \]

Previous example

\[ M = \mathbb{R} \times S^1 \]

\[ \mu = p \]

\[ X^\mu = \frac{2}{S^q} \]

implies to a \( U(1) \) gauge field \( A^\mu \).
In curvilinear coordinates, the Lagrangian is

\[ \mathcal{L} = \int_{\mathbb{R}} p_i(t) \dot{q}_i(t) + \mu(p(t), q(t)) A_t \]

(for a particle moving in \( M \)).
Gauge trans. are
\[
\begin{align*}
S\rho &= \frac{1}{2} \mu^\alpha p_\alpha \not x \\
S\phi &= \frac{1}{2} \mu^\alpha \phi_\alpha \not x \\
S\mathcal{A} &= \not D \not x
\end{align*}
\]

Phase Space
\begin{itemize}
\item $p, q$ covariant constant
\item determined by $p(0), q(0)$
\end{itemize}

Vary $A_\mu$, $\mu = 0$

Phase space = Quotient of $\{\mu = 0\}$ by $U(1)$ action generated by $x_\mu$
Locally, we can find co-ords so

\[ \mu = p_1 \]

\[ X_\mu = \frac{2}{\partial q_1} \]

\[ q_1 \sim q_1 + 1 \]

In these co-ords, we remove \( p_1, q_1 \)

\[ p = 0 \]

\[ q \]

\[ \rightarrow p \]
Result is again symplectic.
This procedure is called symplectic reduction.
Example

Phase space of YM on $R \times S^1$

Action is

$$\int (\partial_t A_\theta - \partial_\theta A_t)^2$$
In 1st order form this is

\[ \int B(\partial_t A_\theta - \partial_\theta A_\tau) + B^2 \]

\[ B \text{ is a scalar.} \]
Equations of motion:

\[
\text{Looks like } \int p(\theta) q(\theta) + p(\theta)^2 + \frac{d}{dt} p(\theta) A_{\theta} \text{ moment ineq}
\]
Vary $A_0$: $\partial_\nu B = 0$

Vary $B$: $\partial_\nu \Delta_0 A_0 - 2\partial_\nu A_0 + 2B = 0$

Vary $A_t$: moment map equation

$\partial_\nu B = 0$

Gauge transformations set $A_t = 0$, leaving us with $X(0)$ only depend on $\theta$. 
Conclude

Phase space is the symplectic reduction of phase space for a 2d free scalar $p(\theta), q(\theta)$
where we reduce by the action of gauge transforms $X(\theta)$ and moment map $\theta \in \mathfrak{p}(\theta)$.

This acts by

$$\delta q(\theta) = \partial_{\theta} X(\theta)$$

$$\delta \rho(\theta) = 0.$$
\( p(\theta) = \text{constant, say } p \)

\( q(\theta) \) is gauge equivalent to \( \int q(\theta) d\theta \)

Phase space is 2 dim, coords
\( p = p(\theta) \) and \( \int q(\theta) d\theta \)
Treated more carefully,

$$2\pi i \int_0^1 d\theta$$

is gauge invariant, phase space

is $$\mathbb{R} \times S^1$$

Hamiltonian = $$\hat{p}$$.