Title: Mathematical Physics Lecture

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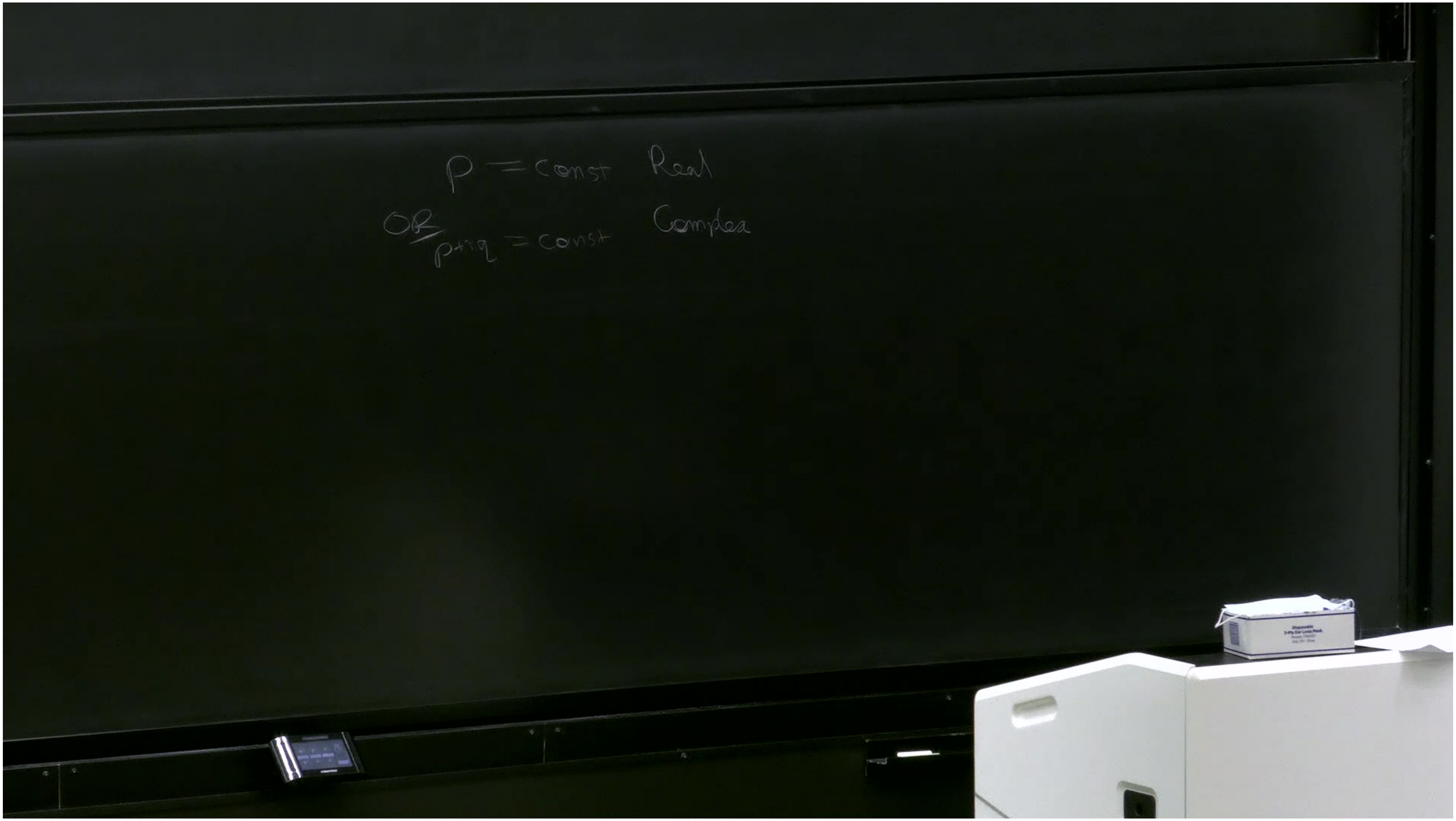

p = const Real
OR p+iq = const Complex

M symp. manifold

$L \subseteq M$ is Lagrangian if

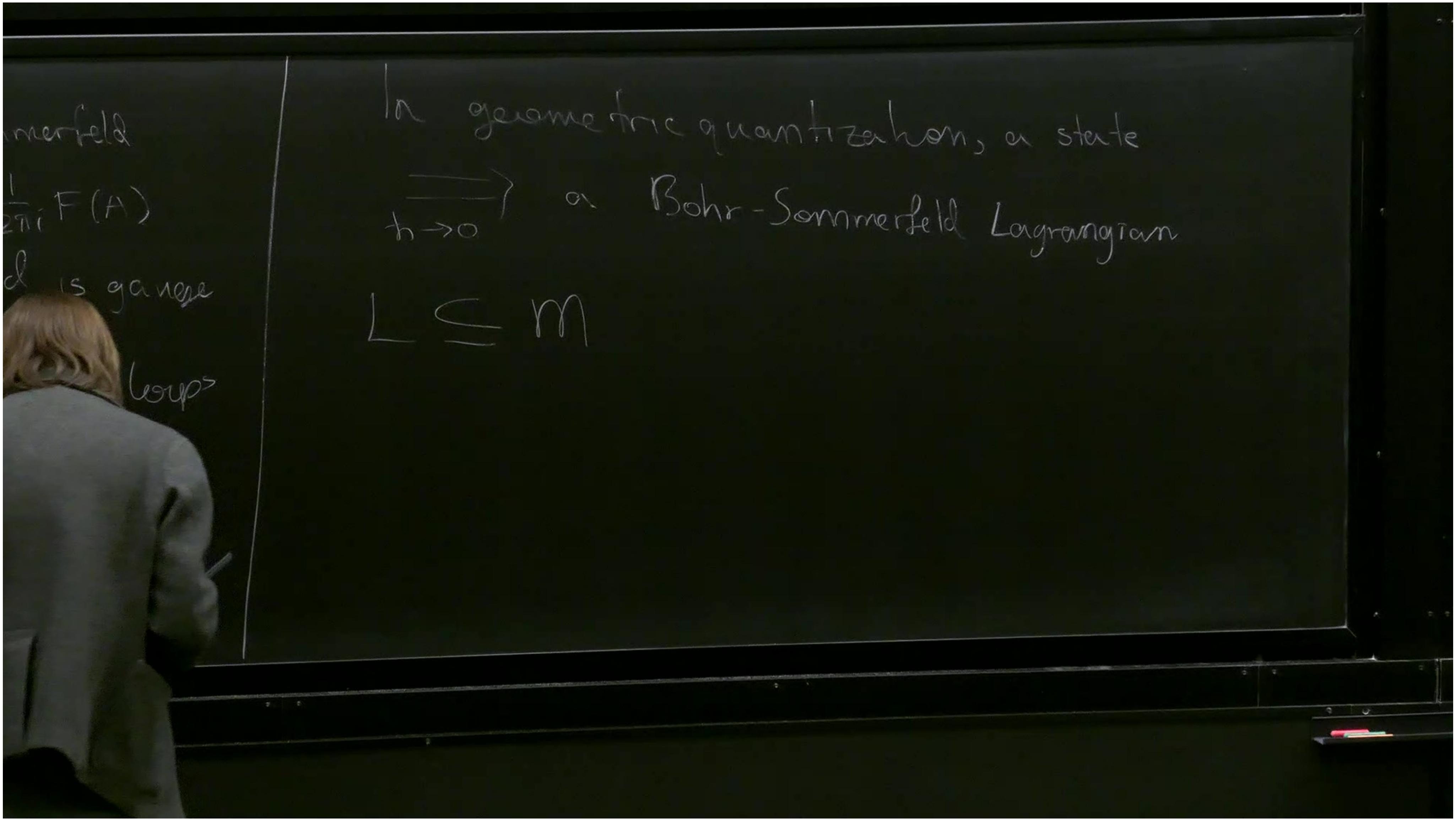
1) $\dim L = \frac{1}{2} \dim M$

2) $\omega\big|_L = 0$

L is Bohr-Sommerfeld if, when $\omega = \frac{1}{2\pi i} F(A)$

$u(1)$ gauge field is gauge trivial on L

($\Longleftrightarrow$ all Wilson loops vanish)


In geometric quantization, a state
$\xrightarrow[\hbar \to 0]{}$ a Bohr-Sommerfeld Lagrangian
$L \subseteq M$

-Sommerfeld

$\omega = \frac{1}{2\pi i} F(A)$

ge field is ganese

n2

ll Wilson loops vanish)

In geometric quantization, a state $\xRightarrow[\hbar \to 0]{}$ a Bohr-Sommerfeld Lagrangian

$L \subseteq M$

Example

$\mathbb{R}^2$, coords p, q

State is $\psi(q)$

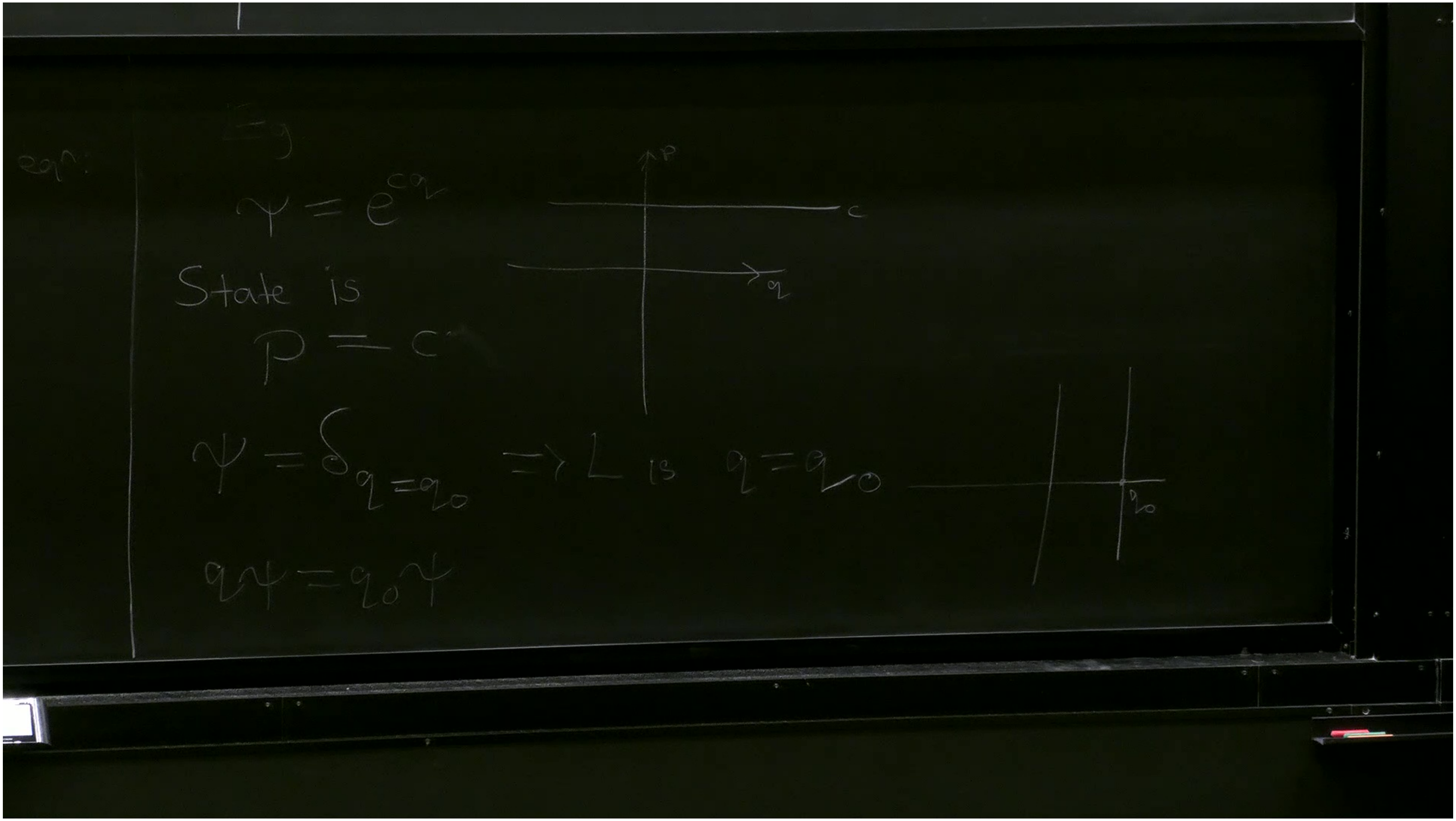
Satisfies equation

$$p \cdot \psi(q) = \hbar i \frac{\partial \psi}{\partial q}$$

Turn this into an algebraic eqn:

$$p = \frac{1}{\psi} \frac{\partial \psi}{\partial q}$$

submanifold of $\mathbb{R}^2$


eqn:
Eg
$\psi = e^{cq}$
State is
$p = c$
$\psi = \delta_{q=q_0} \Rightarrow L$ is $q = q_0$
$q\psi = q_0\psi$
p
c
q
q_0

H is the Hamiltonian

Given any Lagrangian $L \subseteq M$

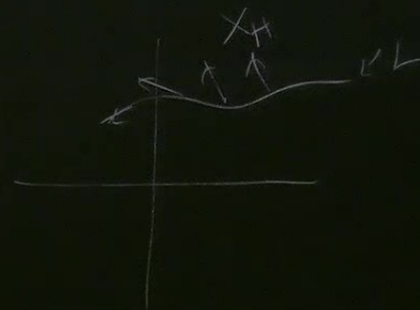
Can find coords. so

$$\omega = dp_i \wedge dq_i$$

$$L = \{p_i = 0\}$$

How does H move L?

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$$



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If we have an eigenstate $\hat{H}\psi = \alpha\psi$

$\hbar \to 0$, the Lagrangian L_ψ is fixed by X_H.

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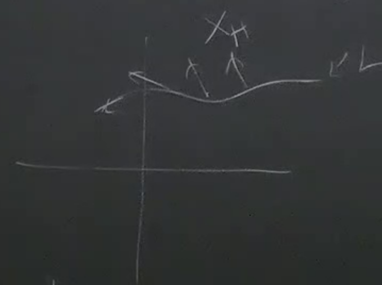


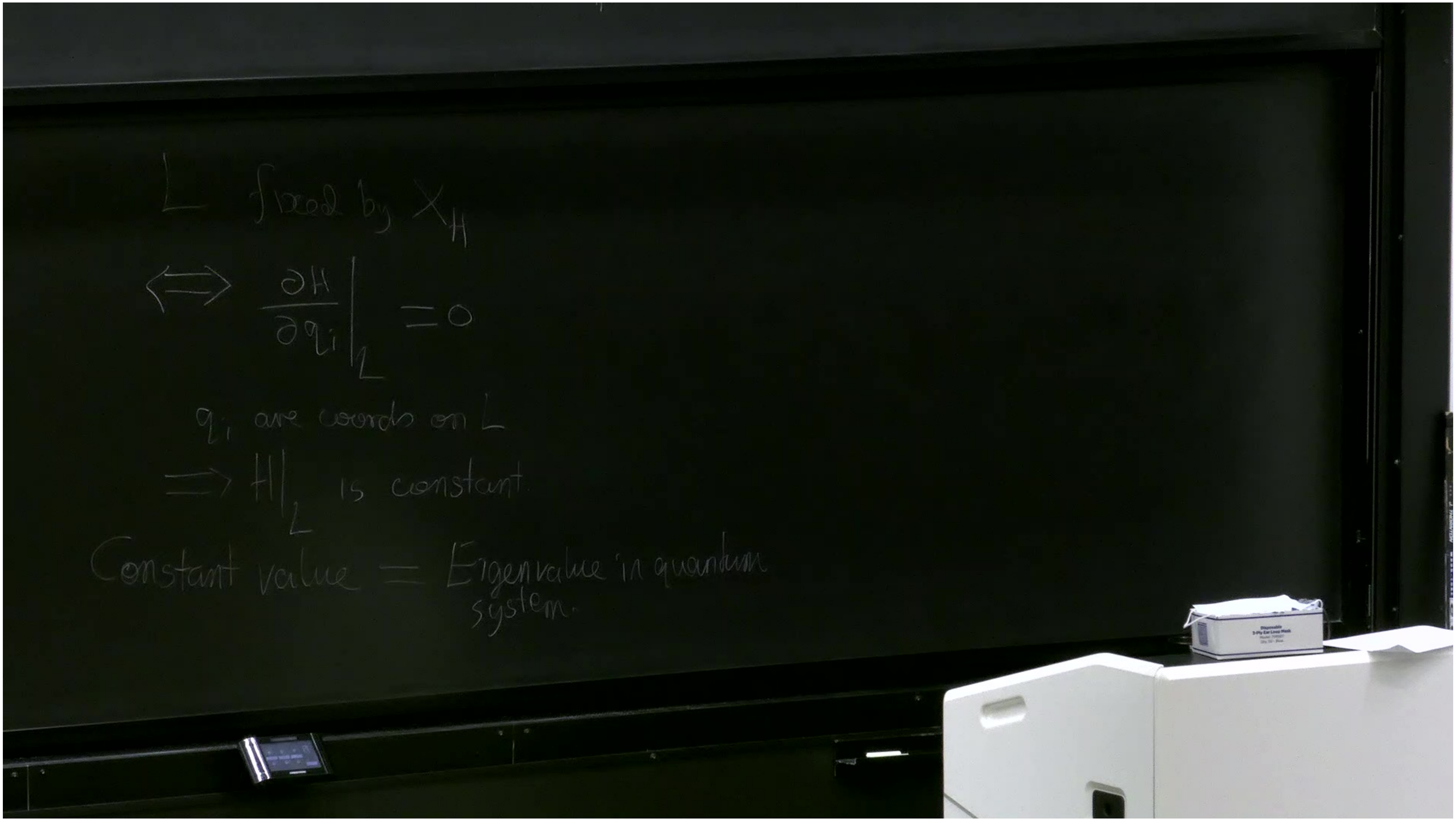
If we have an eigenstate $\hat{H}\psi = \alpha\psi$

$\hbar \to 0$, the Lagrangian L_ψ is fixed by X_H.

L is fixed by $X_H \iff H$ is constant on L

$$L = \{p_i = 0\}$$

$$X_H = \underbrace{\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}}_{\text{Tangent to } L} - \underbrace{\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}}_{\text{point normal to } L}$$



L fixed by X_H
$\Longleftrightarrow \left.\frac{\partial H}{\partial q_i}\right|_L = 0$
q_i are coords on L
$\Longrightarrow H|_L$ is constant.
Constant value = Eigenvalue in quantum system.

Example

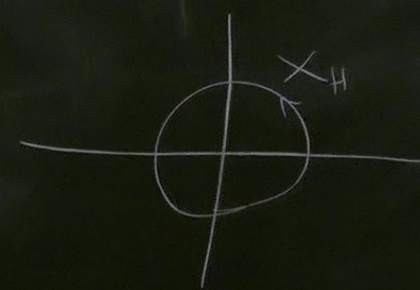
Harmonic oscillator

Phase space is $\mathbb{R}^2$

$\omega = dp\, dq$

$H = \frac{1}{2}(p^2+q^2)$

$X_H = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}$



Polar coords r, θ

$$\omega = r\,dr\,d\theta$$

$$H = \frac{1}{2} r^2$$

$$\alpha = \frac{1}{2} r^2 d\theta$$

$$d\alpha = \omega$$

Polarize by the map

$\mathbb{R}^2 \longrightarrow \mathbb{R}$ given by r

States = fibres which are

BS orbits.

$r = r_0$ and

$\displaystyle\int_{r=r_0} \alpha$ is an integer $\times 2\pi$

$\displaystyle\int \frac{1}{2} r_0^2 d\theta = \pi r_0^2$, so r_0^2 is an integer.

Spectrum: States where r_0^2 is an integer

On this state, H is constant w. integer value!

Spectrum $= 0, 1, 2, \ldots$

If we take 2 decoupled systems, w. Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, a state $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is decomposable if

$$\psi = \psi_1 \otimes \psi_2$$

Otherwise, ψ is entangled.

Suppose my 2 systems have phase space

$\mathbb{R}^2$, coordinates p_1, q_1 p_2, q_2

$$\psi = \psi_1 \otimes \psi_2 = \psi_1(q_1)\psi_2(q_2)$$

Semiclassically,
we get the equations

$$p_1 \left(\psi_1(q_1)\, \psi_2(q_2)\right)$$

$$= \frac{\partial}{\partial q_1} \psi_1(q_1)\, \psi_2(q_2)$$

$$p_1 = \frac{1}{\psi_1}\, \psi_1(q_1)$$

$$p_2 = \frac{1}{\psi_2}\, \psi_2(q_2)$$

$$p_1 \left(\psi_1(q_1)\,\psi_2(q_2)\right)$$
$$= \frac{\partial}{\partial q_1}\psi_1(q_1)\,\psi_2(q_2)$$

$$p_1 = \frac{1}{\psi_1}\,\psi_1(q_1)$$
$$p_2 = \frac{1}{\psi_2}\,\psi_2(q_2)$$

each equation involves one set of variables

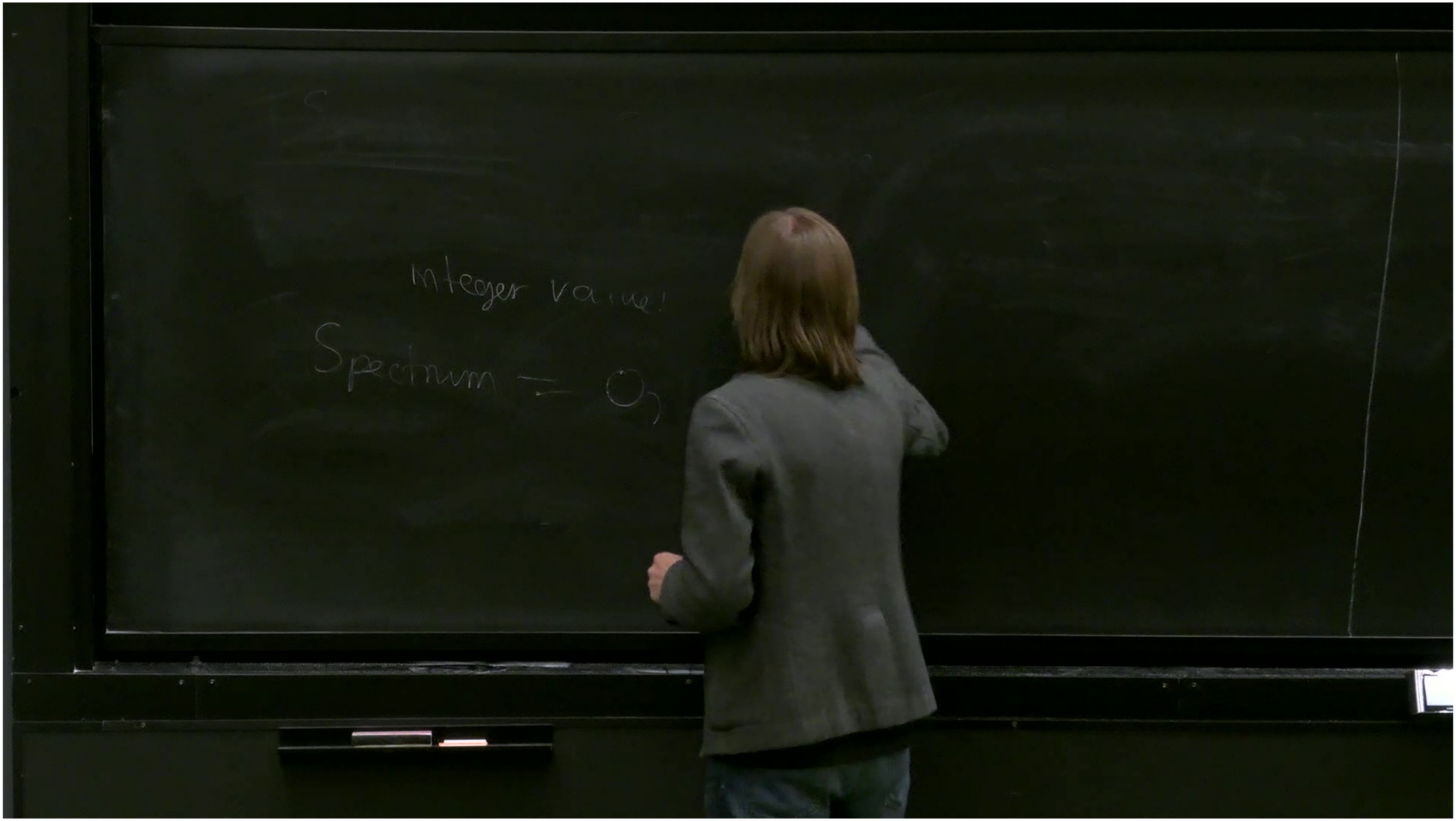
In general, if M_1, M_2 are symplectic

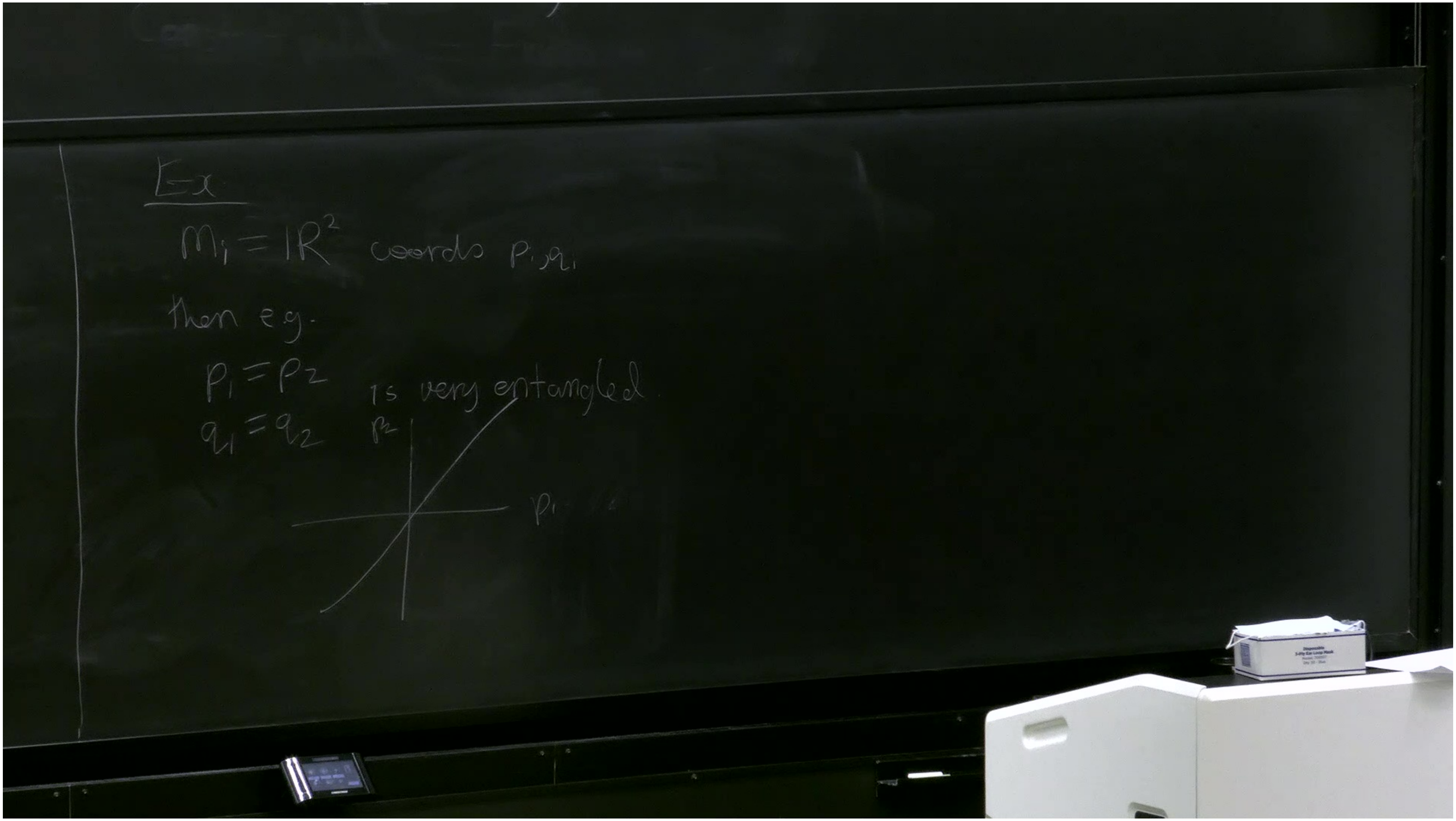
$L \subseteq M_1 \times M_2$

is decomposable if $L = L_1 \times L_2$

$L_i \subseteq M_i$ are Lagrangians.

Otherwise it is entangled.


integer value!
Spectrum = O,

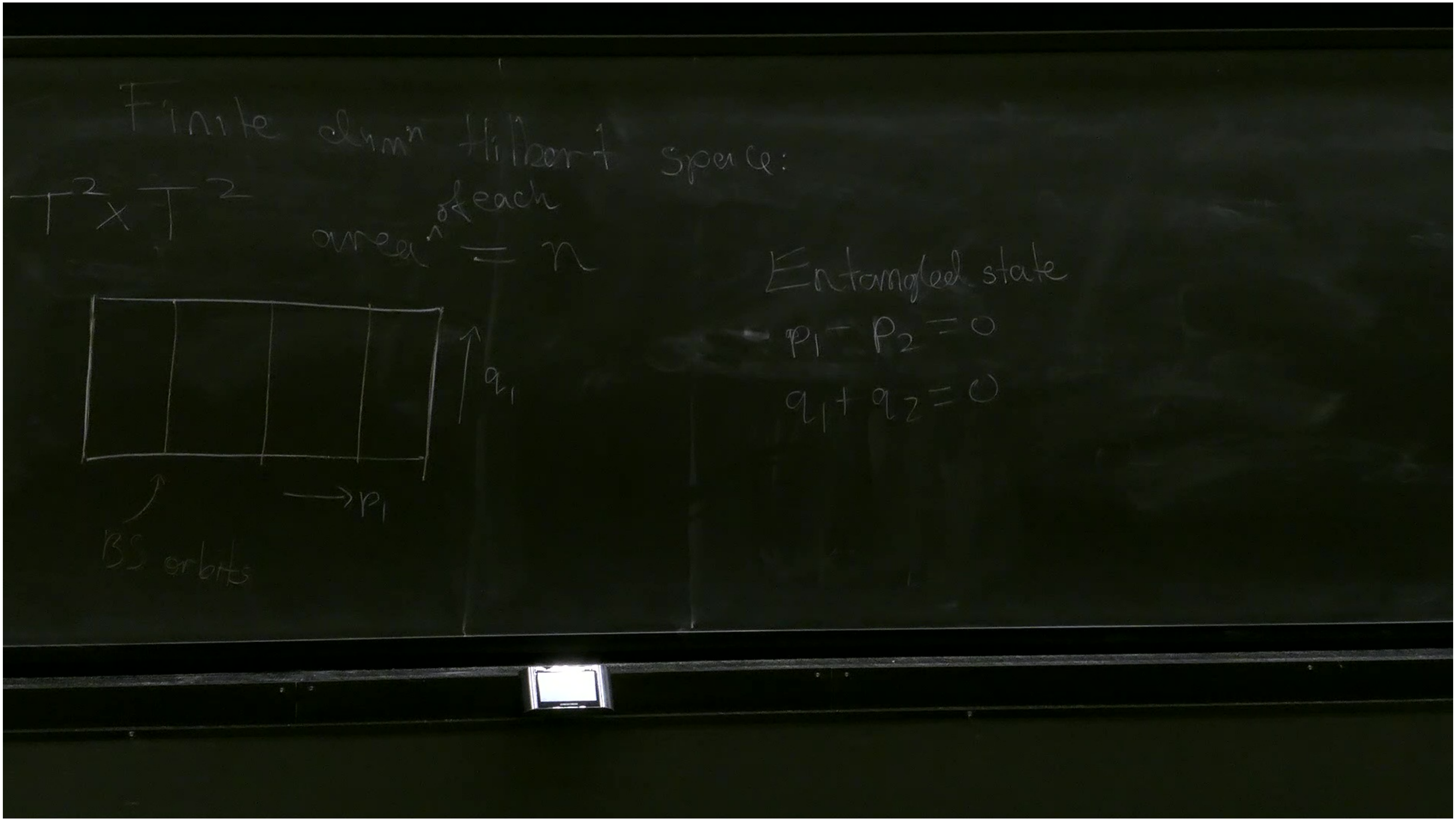

Ex
$M_i = \mathbb{R}^2$ coords p_i, q_i
then e.g.
$p_1 = p_2$ is very entangled
$q_1 = q_2$
p_2
p_1

$$\frac{\partial}{\partial q_1}\psi = \frac{\partial}{\partial q_2}\psi$$

$$q_1\psi = -q_2\psi$$

$$\psi = \delta_{q_1+q_2=0}\, e^{q_1-q_2}$$

$$\int \delta_{q_1=r}\,\delta_{q_2=-r}$$



Finite dim^n Hilbert space:
$T^2 \times T^2$
area of each $= n$
q_1
p_1
BS orbits
Entangled state
$p_1 - p_2 = 0$
$q_1 + q_2 = 0$

ilbert space:
each
$= n$

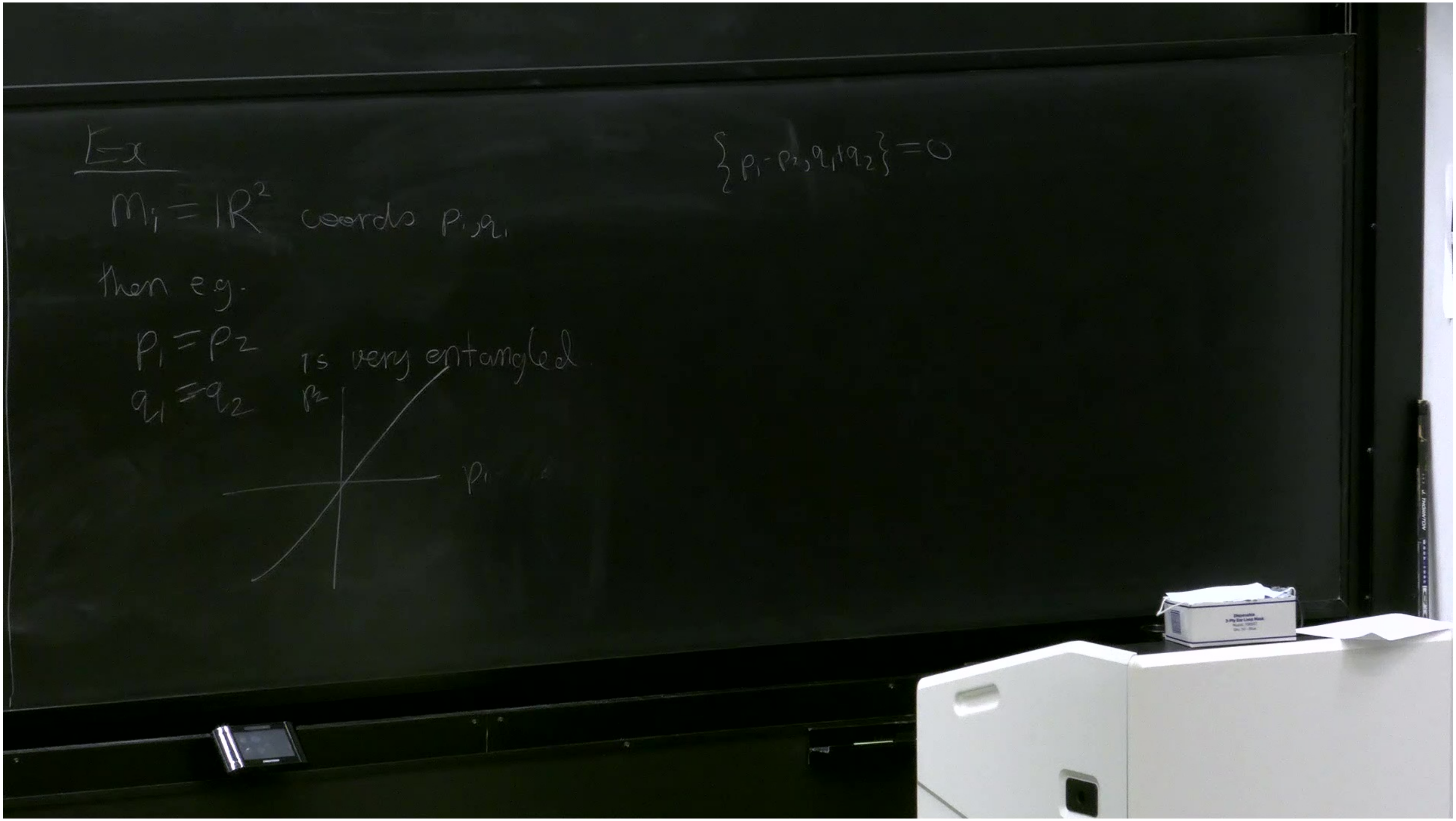
q_1

Entangled state

$p_1 - p_2 = 0$

$q_1 + q_2 = 0$

If you work this out,
$n = 2$, get standard
entangled state in
$\mathbb{C}^2 \otimes \mathbb{C}^2$


Ex
$M_i = \mathbb{R}^2$ coords p_i, q_i
then e.g.
$p_1 = p_2$
$q_1 = q_2$
is very entangled
$\{p_1 - p_2, q_1 + q_2\} = 0$
p_2
p_1