Title: Mathematical Physics Lecture

Speakers: Kevin Costello

Collection: Mathematical Physics 2023/24

Date: April 18, 2024 - 1:00 PM

URL: https://pirsa.org/24040062

$\varphi : \mathbb{R} \longrightarrow X$

X a manifold

$S(\varphi) = \int \|\partial_t \varphi\|^2$

Phase space $= T^*X$

coords. $q_i = \varphi_i$

$p_i = \partial_t \varphi_i$

Lagrangian $L \subseteq T^*X$

specifies initial data.

L has to be Lagrangian,

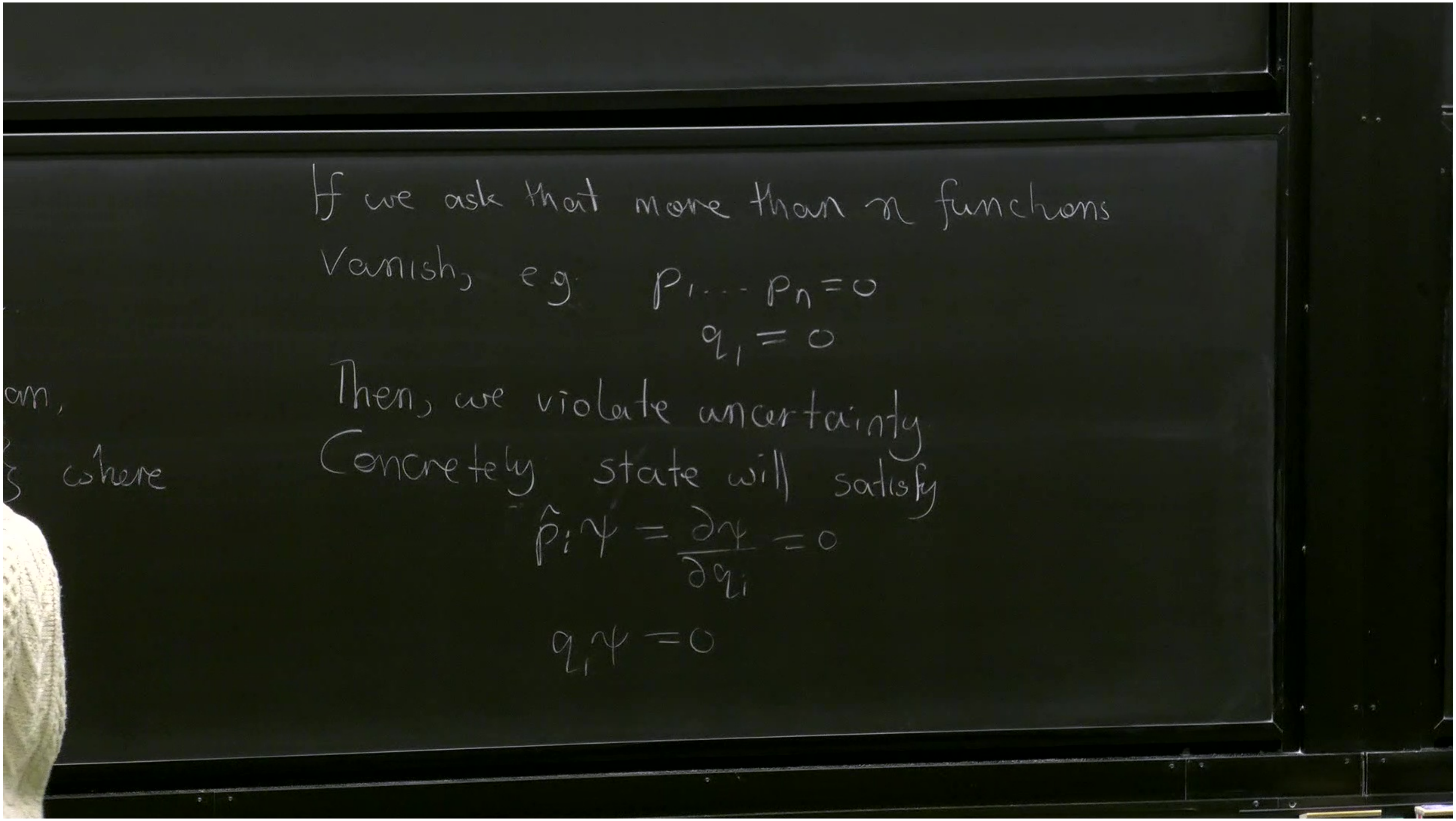
because

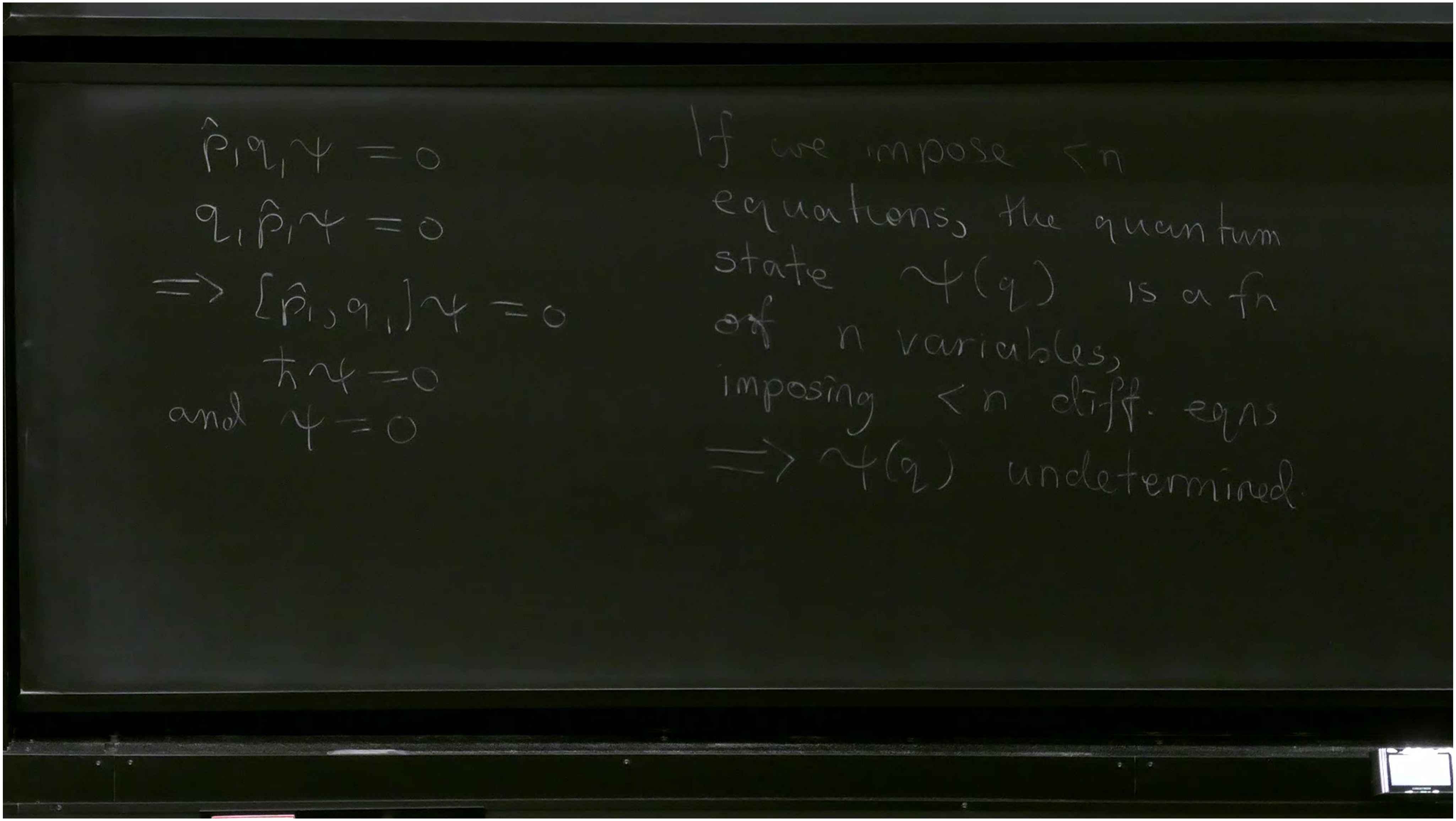
$L = \{f_1, \ldots, f_n = 0\}$ where

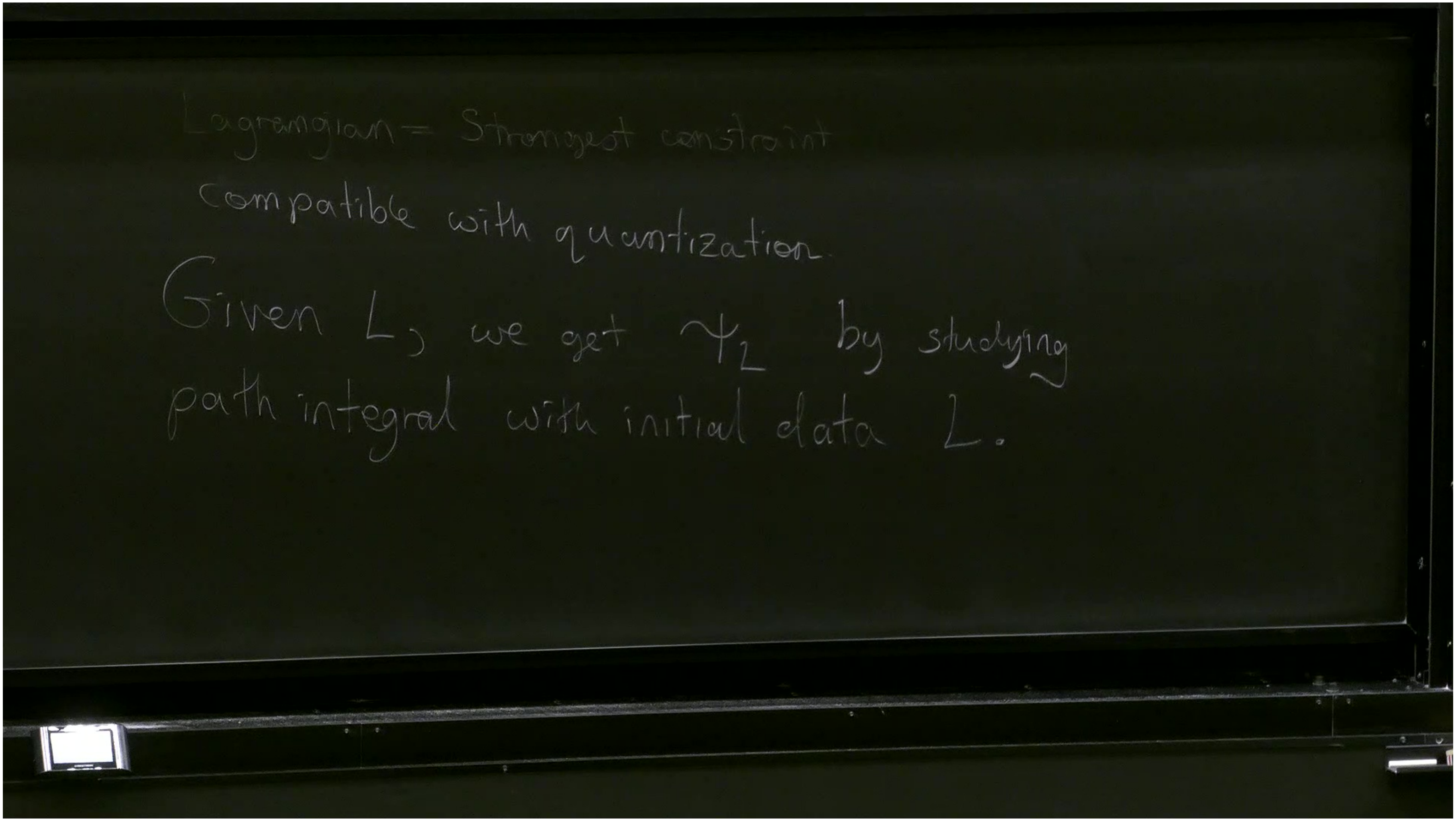
$\{f_i, f_j\} = 0$

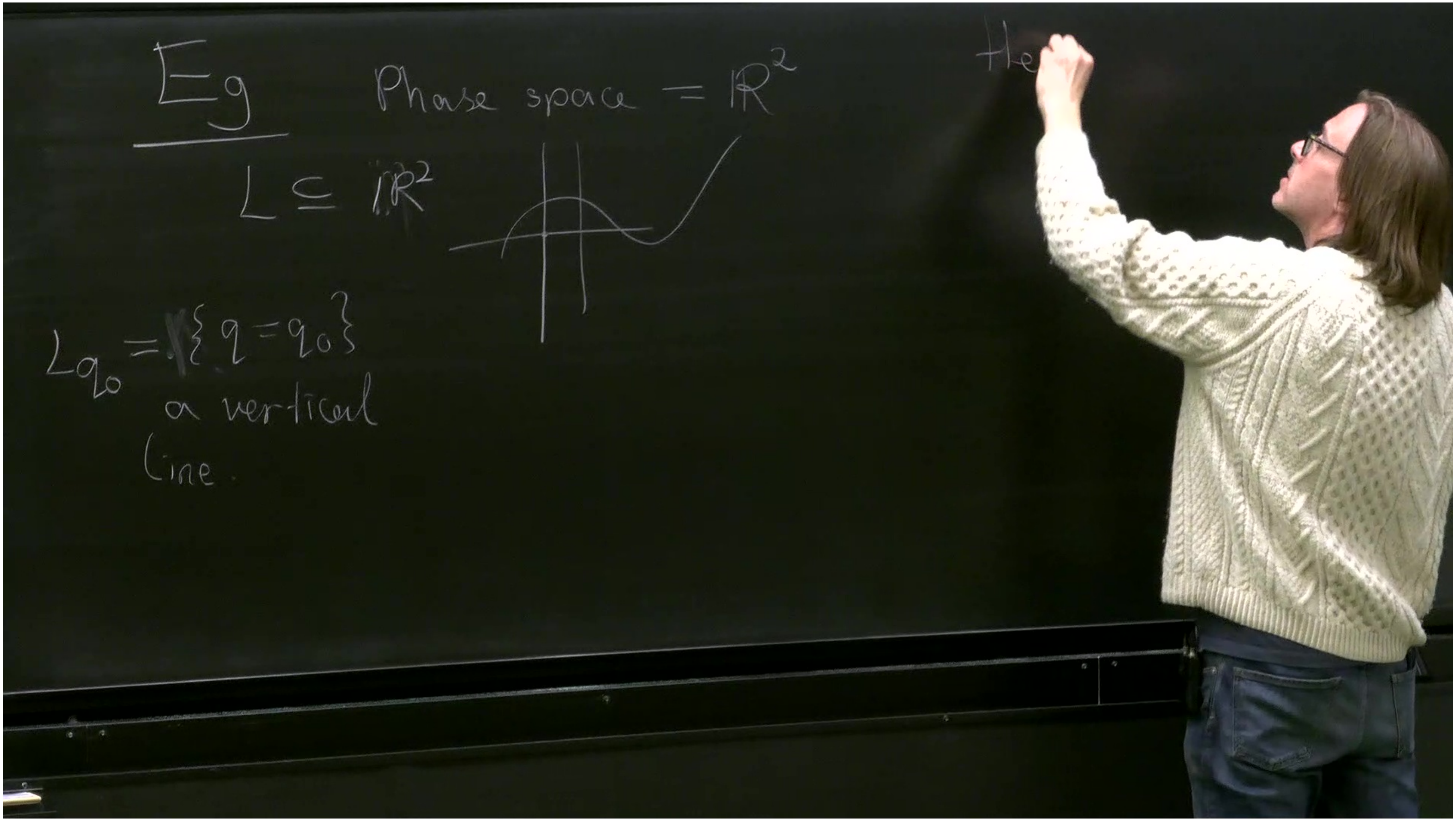
$\operatorname{Dim} X = n$

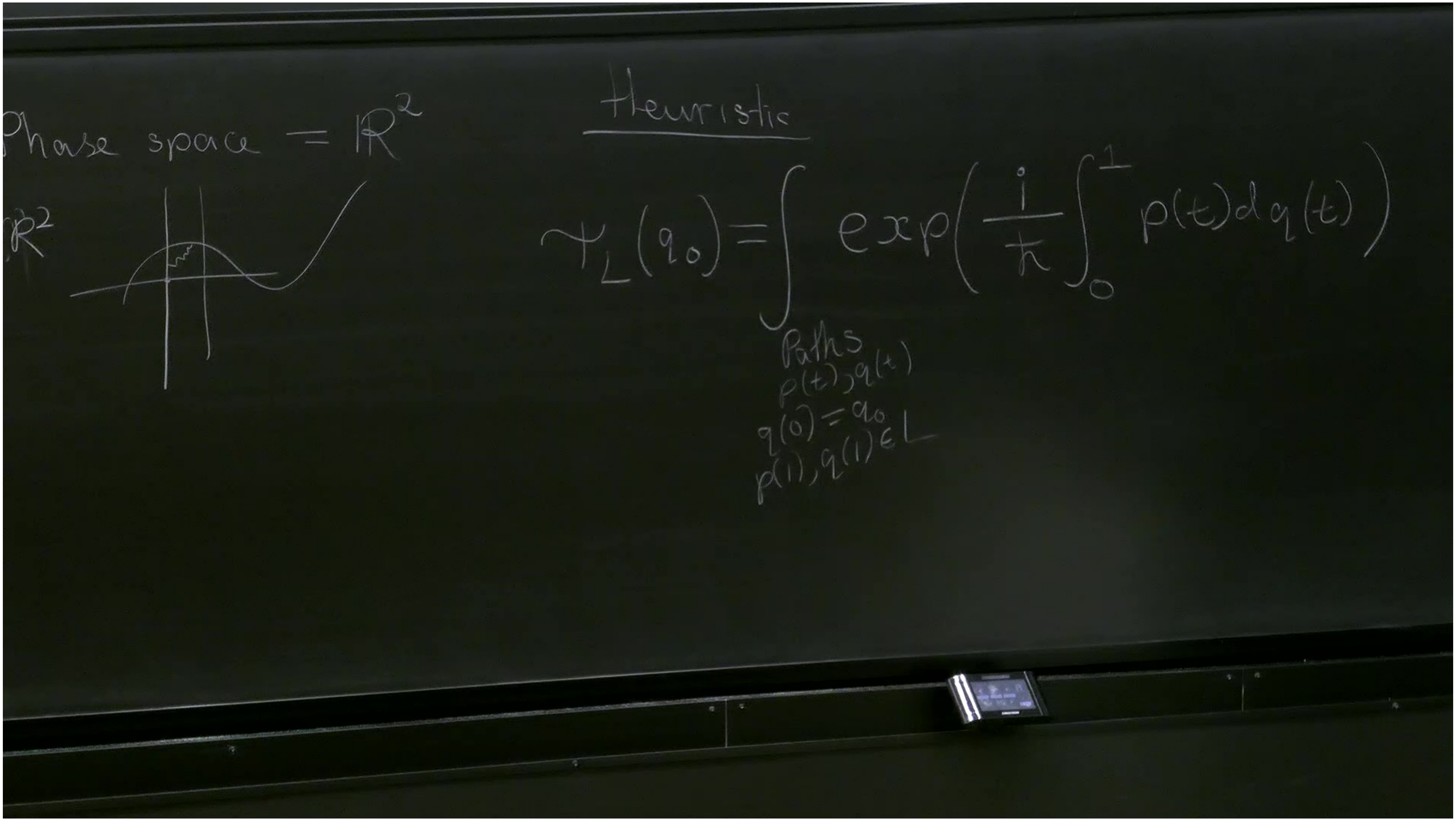
$\operatorname{Dim} T^*X = 2n$


am,
where
If we ask that more than n functions
vanish, e.g. $p_1 \dots p_n = 0$
$q_1 = 0$
Then, we violate uncertainty
Concretely state will satisfy
$\hat{p}_i \psi = \frac{\partial \psi}{\partial q_i} = 0$
$q_1 \psi = 0$


$\hat{p}_1 q_1 \psi = 0$
$q_1 \hat{p}_1 \psi = 0$
$\Rightarrow [\hat{p}_1, q_1] \psi = 0$
$\hbar \psi = 0$
and $\psi = 0$
If we impose $< n$
equations, the quantum
state $\psi(q)$ is a fn
of n variables,
imposing $< n$ diff. eqns
$\Longrightarrow \psi(q)$ undetermined.


Lagrangian = Strongest constraint
compatible with quantization.
Given L, we get ψ_L by studying
path integral with initial data L.


Eg
Phase space = ℝ²
L ⊆ ℝ²
L_q0 = {q = q0}
a vertical line.

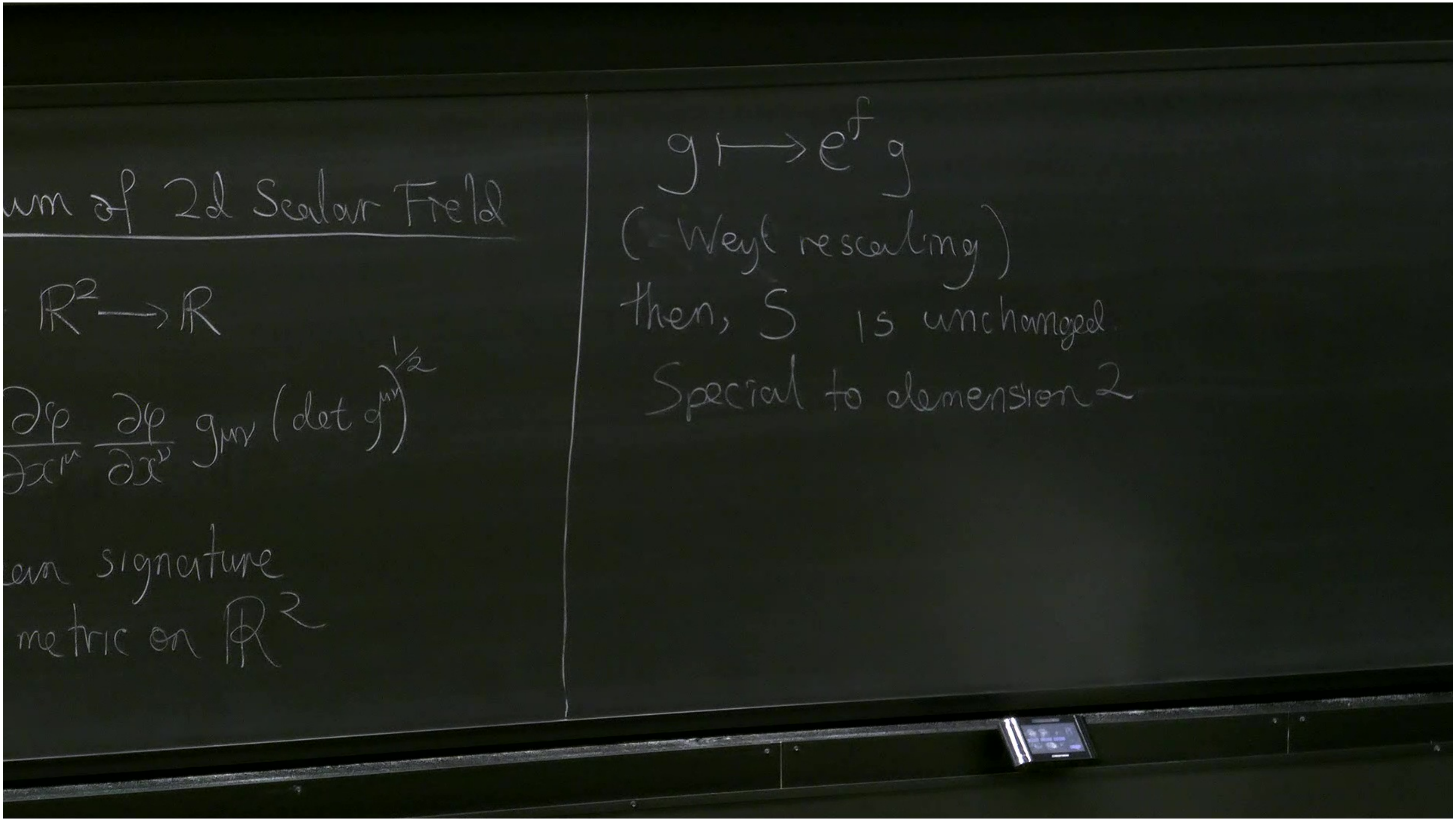

Phase space = ℝ²
ℝ²
Heuristic
ψ_L(q₀) = ∫ exp(i/ħ ∫₀¹ p(t) dq(t))
Paths
p(t), q(t)
q(0) = q₀
p(1), q(1) ∈ L

Vacuum of 2d Scalar Field

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\int \frac{\partial \varphi}{\partial x^\mu} \frac{\partial \varphi}{\partial x^\nu} g_{\mu\nu} (\det g)^{1/2}$$

- Euclidean signature
- g is a metric on $\mathbb{R}^2$

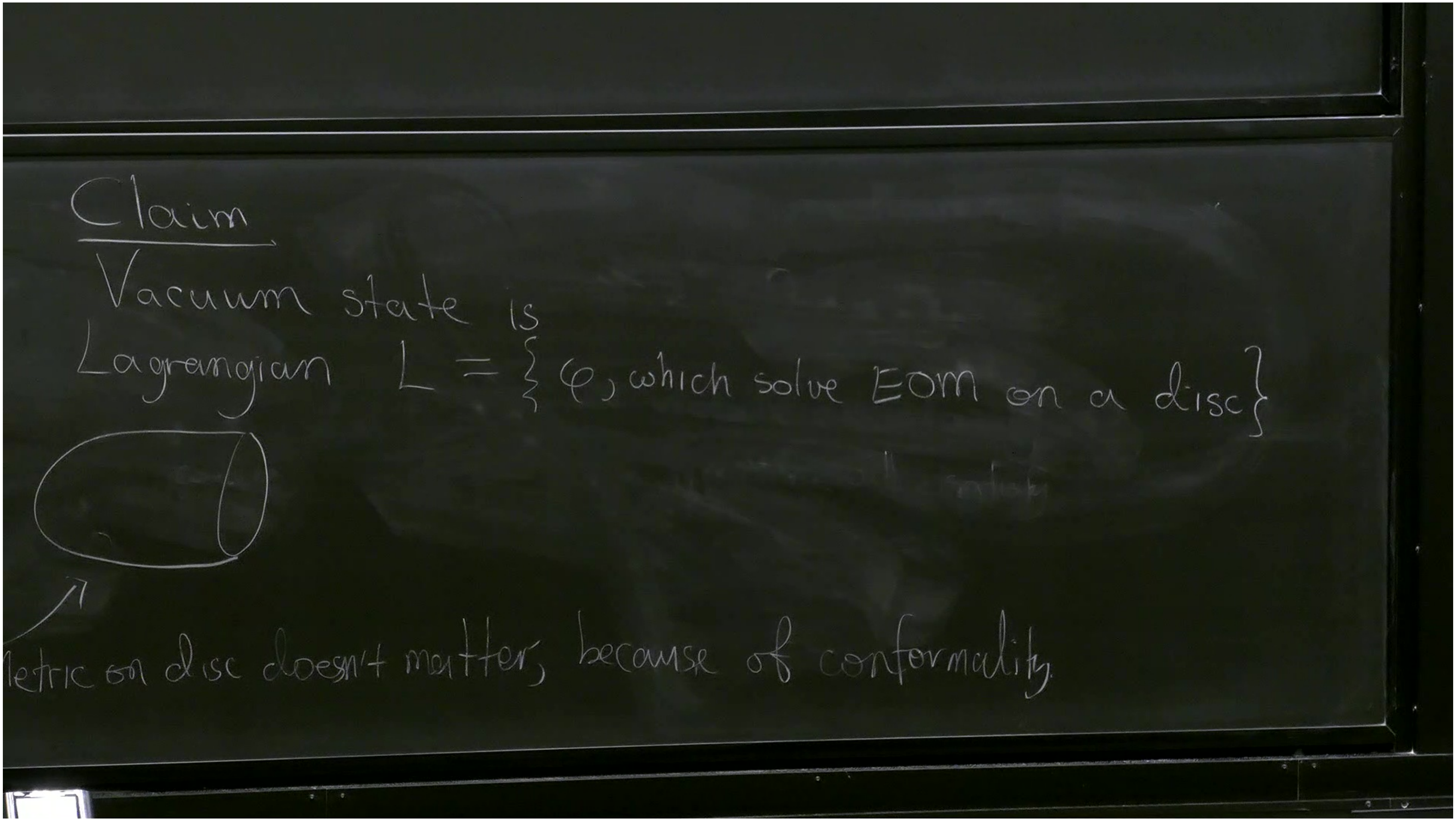

um of 2d Scalar Field
R^2 → R
∂φ/∂x^μ ∂φ/∂x^ν g_μν (det g^μν)^1/2
ean signature
metric on R^2
g ↦ e^f g
(Weyl rescaling)
then, S is unchanged.
Special to dimension 2

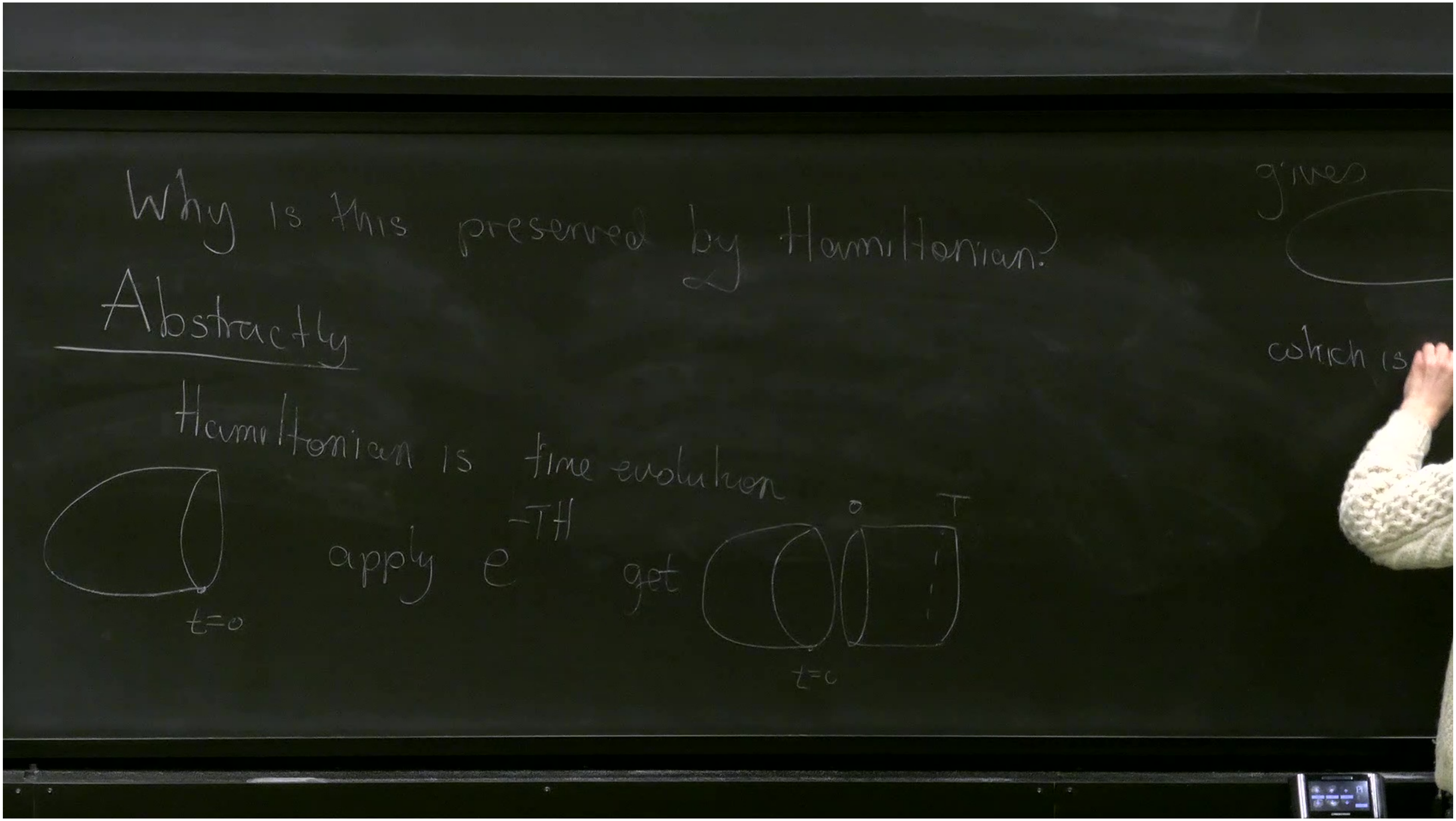
- Phase space — coords $q(\theta) = \varphi(\theta)$

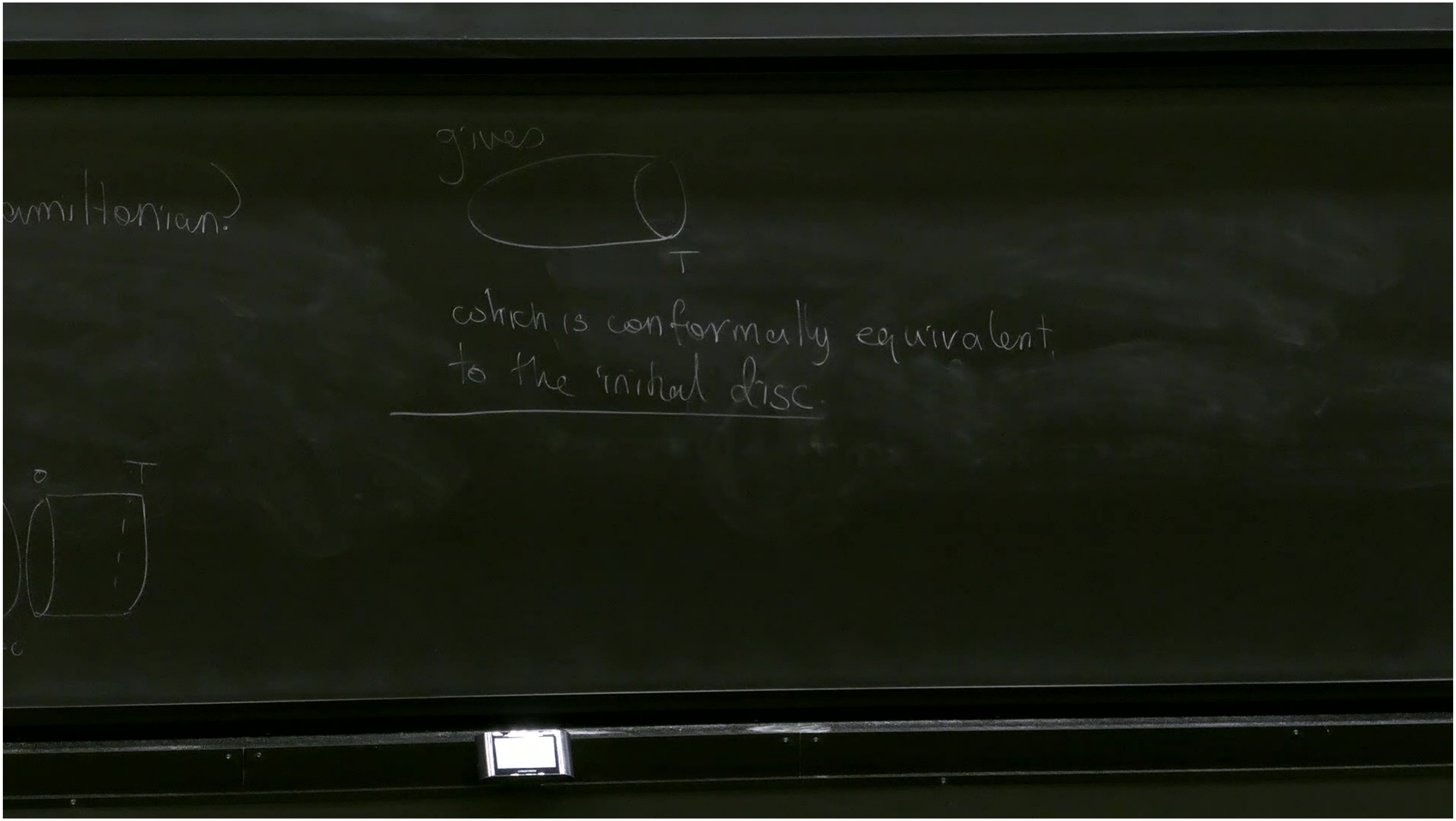
$$p(\theta) = \partial_t \varphi(\theta)$$

= Set of sol^{n}s to EOM on $S^1 \times (-\varepsilon, \varepsilon)$

$$\underset{\theta}{S^1} \ni \theta, \quad t \in (-\varepsilon, \varepsilon)$$


Claim
Vacuum state is
Lagrangian L = { φ, which solve EOM on a disc }
etric on disc doesn't matter, because of conformality


gives
Why is this preserved by Hamiltonian?
Abstractly
which is
Hamiltonian is time evolution
apply e^{-TH} get
0
T
t=0
t=0


amiltonian?
gives
T
which is conformally equivalent
to the initial disc.
0
T

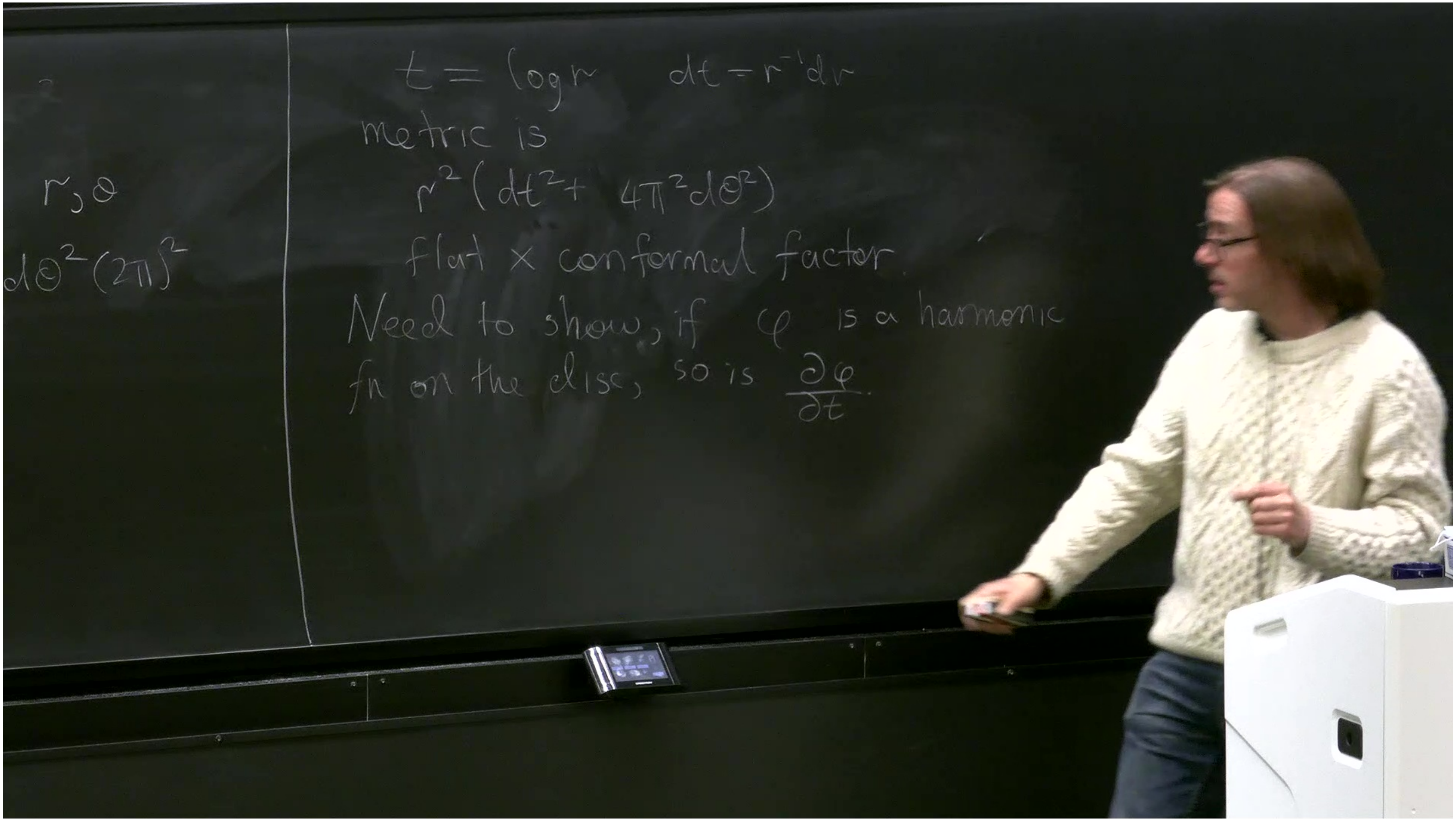
Concretely:

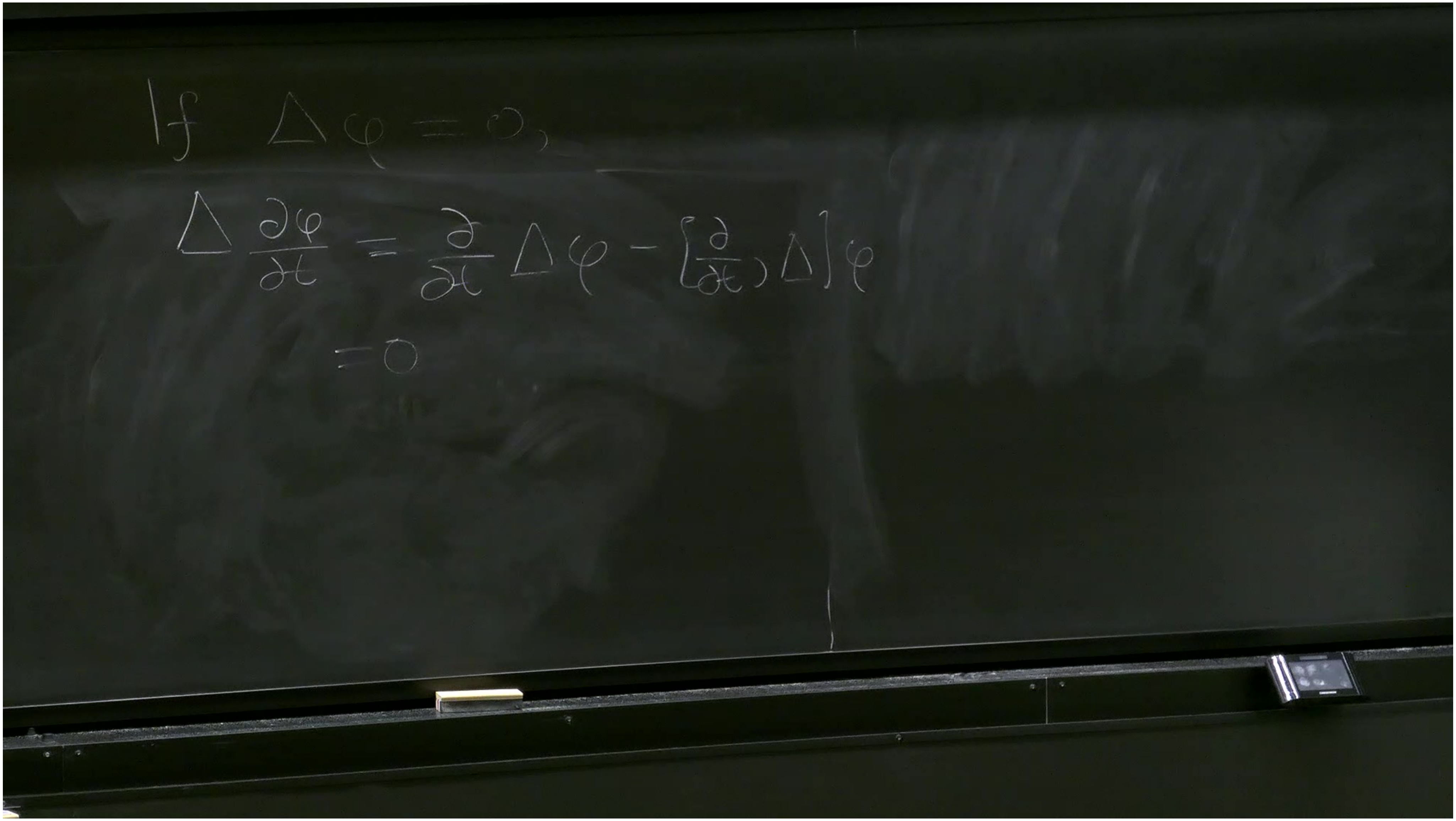
Disc has polar coords r, θ

metric $dr^2 + r^2 \, d\theta^2 (2\pi)^2$

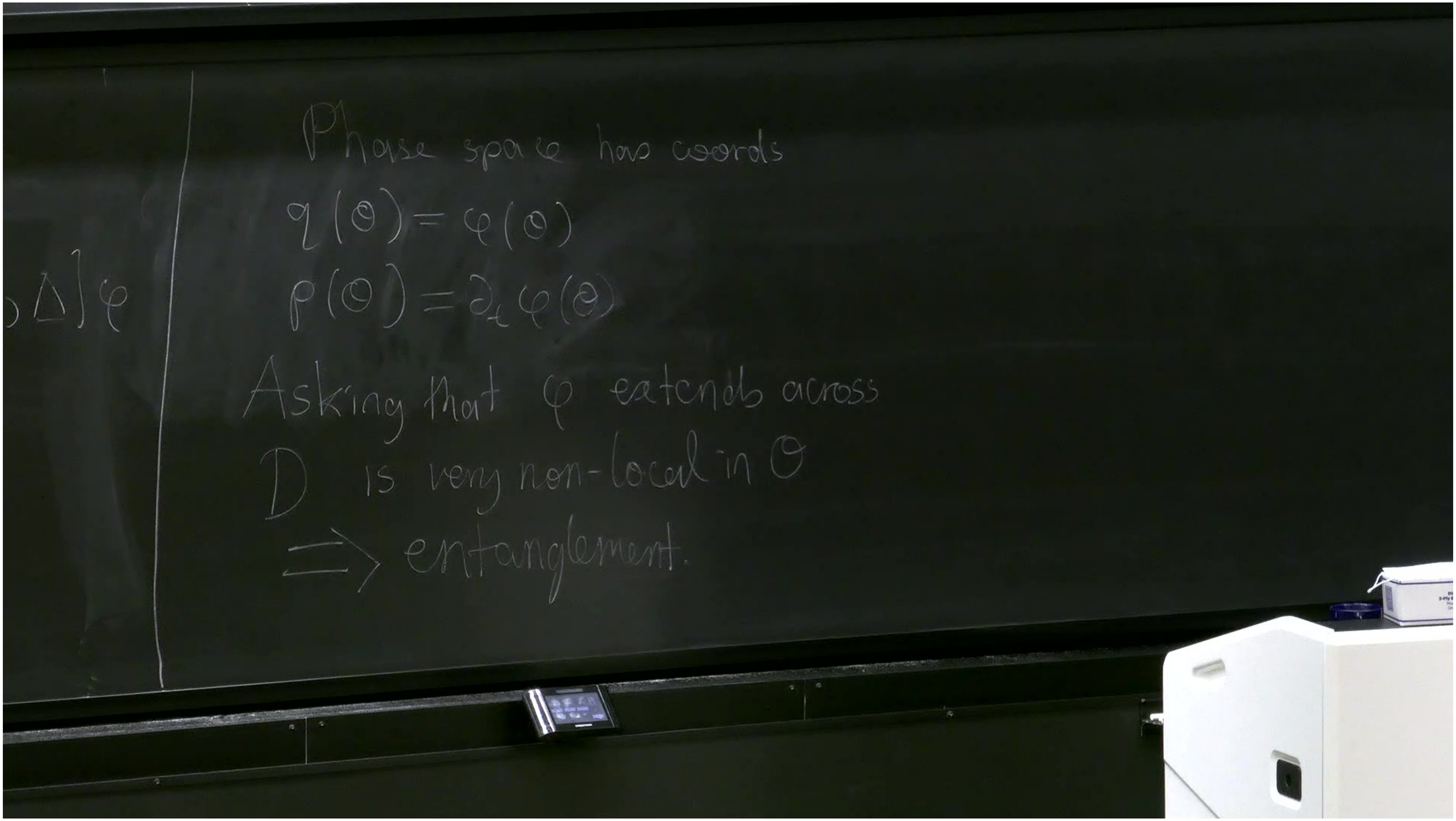
Laplace operator

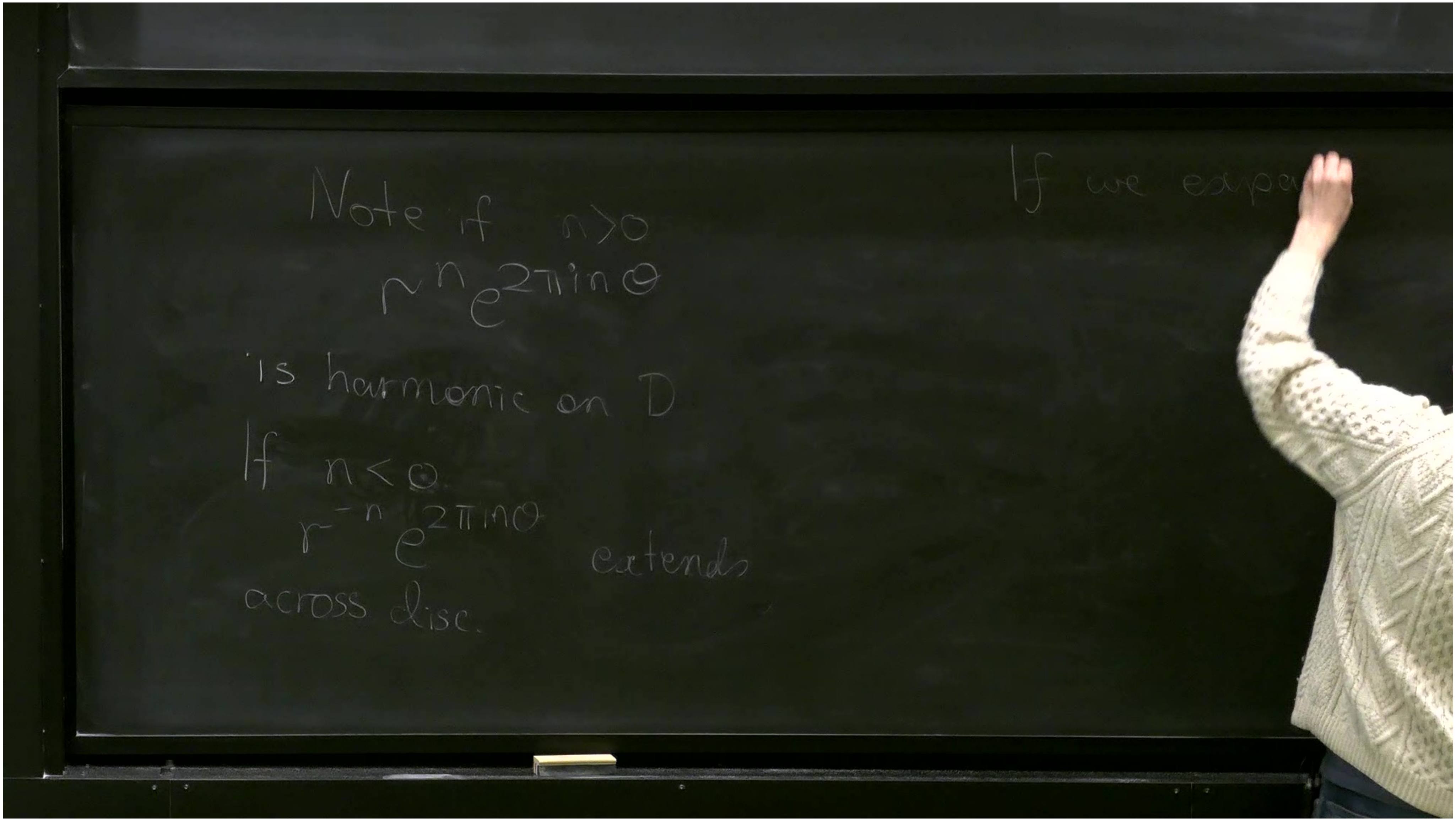
$$\Delta = \frac{\partial}{\partial r^2} - r^{-1}\frac{\partial}{\partial r} + \frac{1}{(2\pi)^2} r^{-2} \partial_\theta^2$$

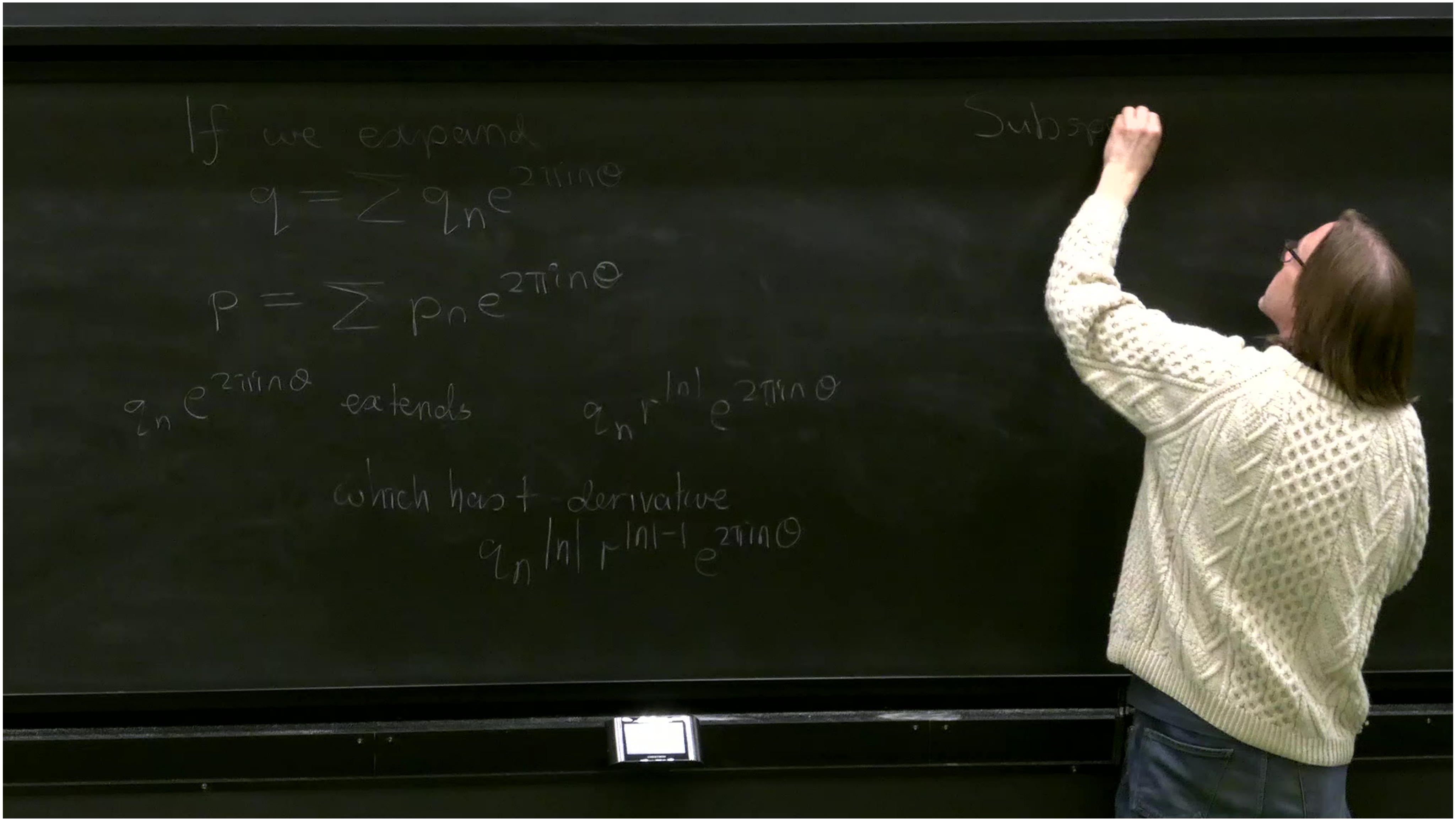

r, θ
dθ² (2π)²
t = log r   dt = r⁻¹dr
metric is
r² (dt² + 4π² dθ²)
flat × conformal factor
Need to show, if φ is a harmonic
fn on the disc, so is ∂φ/∂t

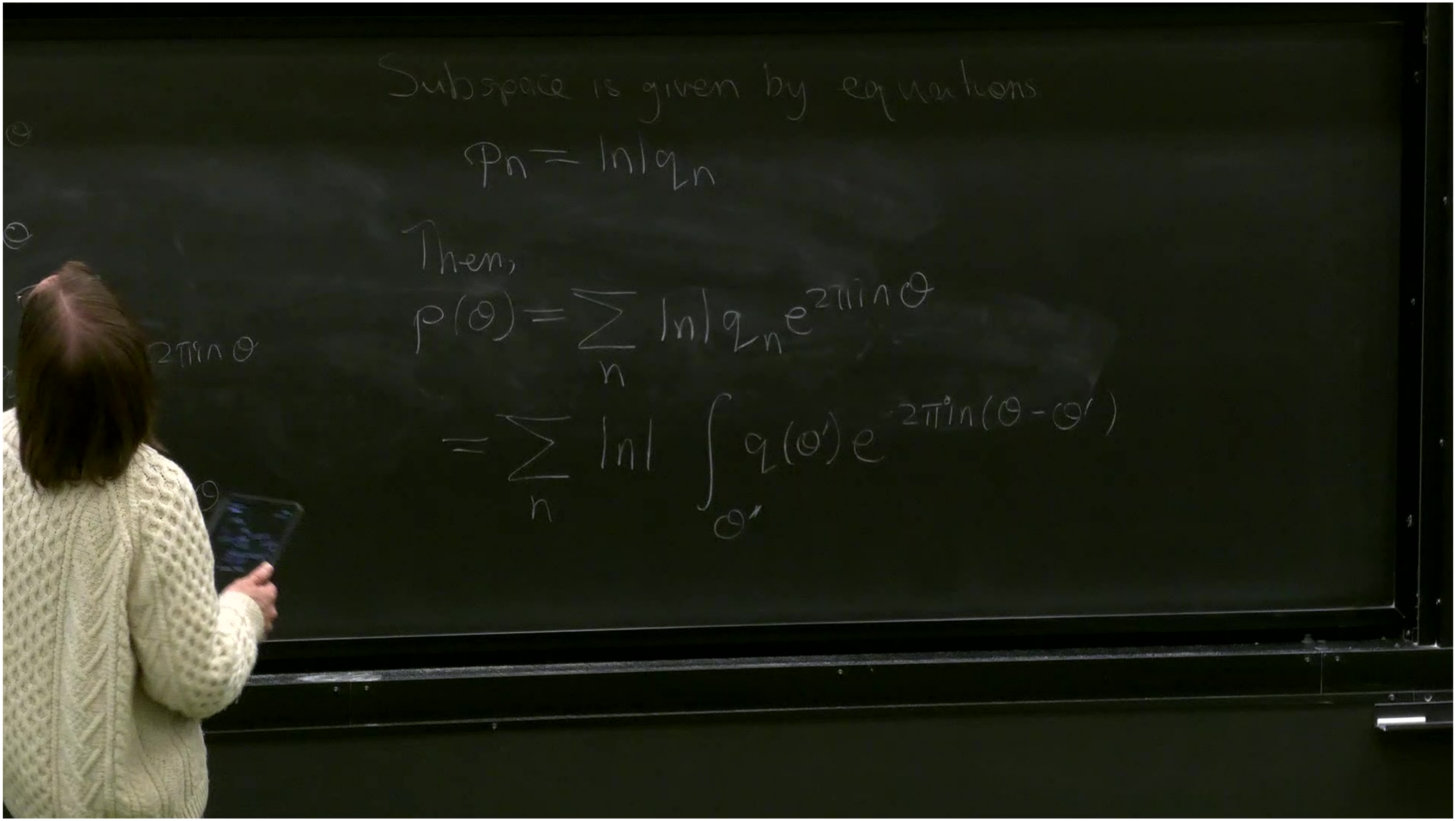


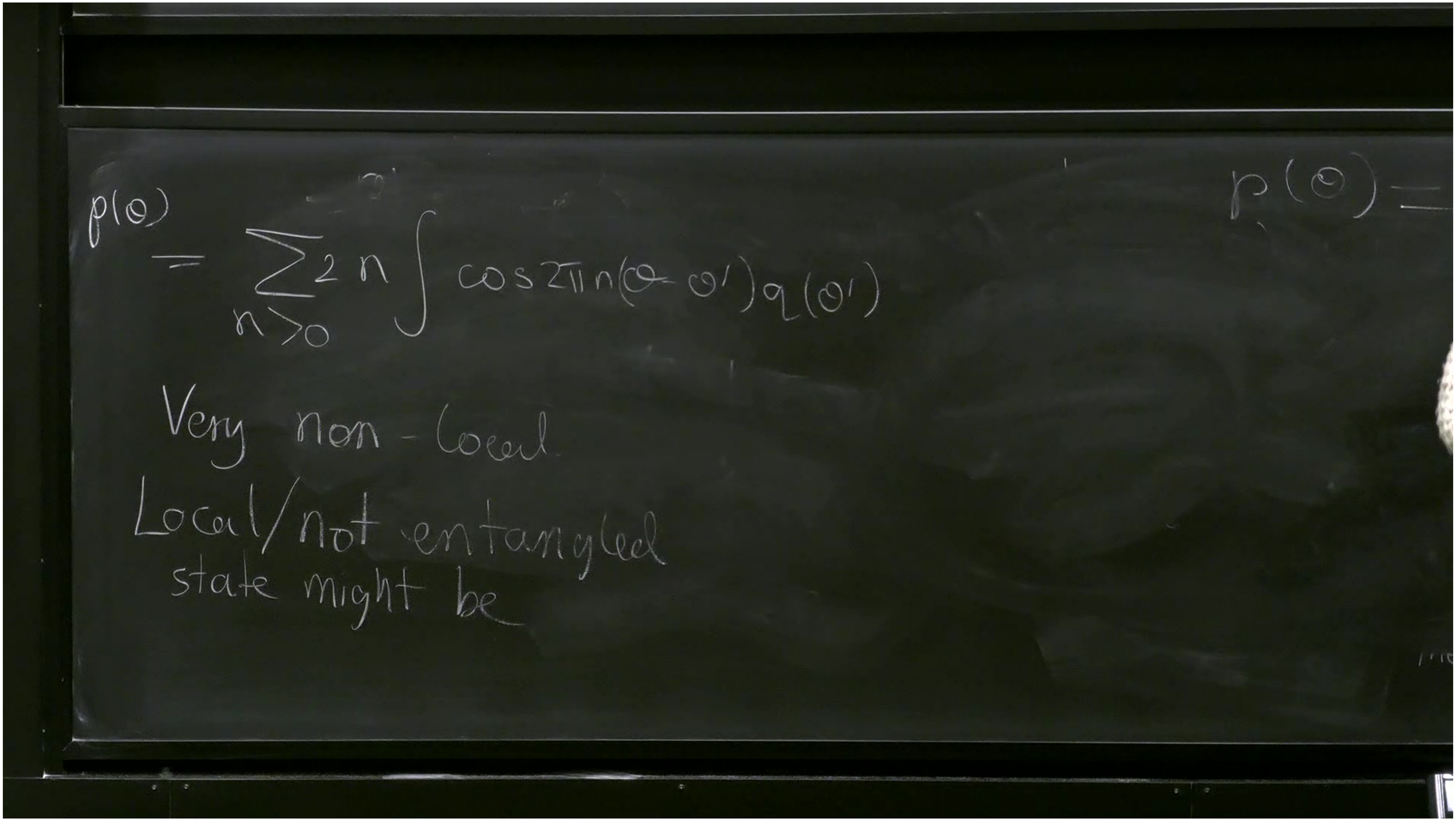
If $\Delta \varphi = 0$,
$\Delta \frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial t} \Delta \varphi - [\frac{\partial}{\partial t}, \Delta] \varphi$
$= 0$

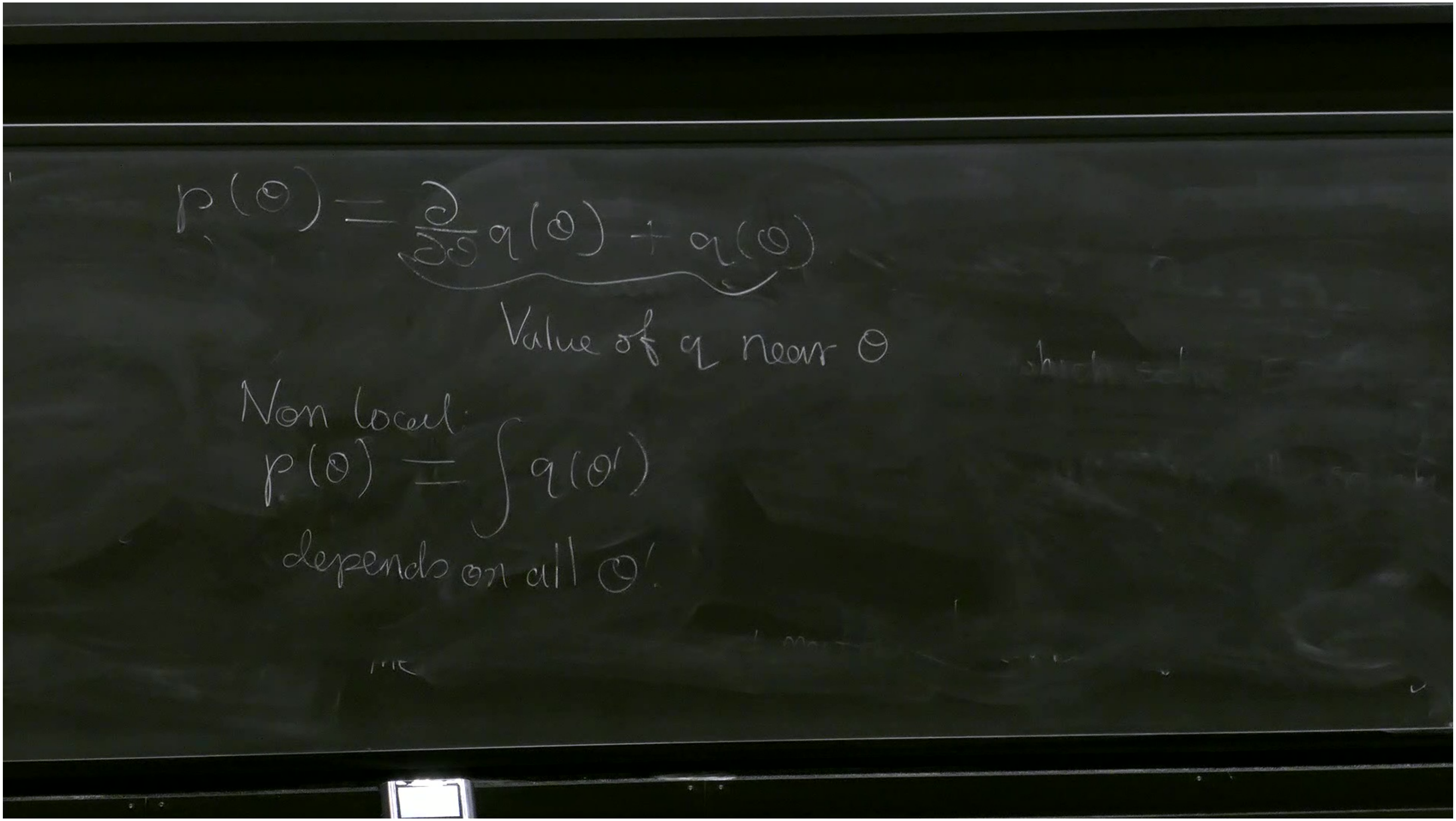

Phase space has coords
q(Θ) = φ(Θ)
p(Θ) = ∂_t φ(Θ)
Asking that φ extends across
D is very non-local in Θ
⇒ entanglement.


Note if n>0
r^n e^2πinθ
is harmonic on D
If n<0
r^-n e^2πinθ extends
across disc.
If we expa


If we expand
$q = \sum q_n e^{2\pi i n\theta}$
$p = \sum p_n e^{2\pi i n\theta}$
$q_n e^{2\pi i n\theta}$ extends $q_n r^{|n|} e^{2\pi i n\theta}$
which has r derivative
$q_n |n| r^{|n|-1} e^{2\pi i n\theta}$
Subsp


Subspace is given by equations
$p_n = |n| q_n$
Then,
$\rho(\theta) = \sum_n |n| q_n e^{2\pi i n \theta}$
$= \sum_n |n| \int_{\theta'} q(\theta') e^{2\pi i n(\theta - \theta')}$

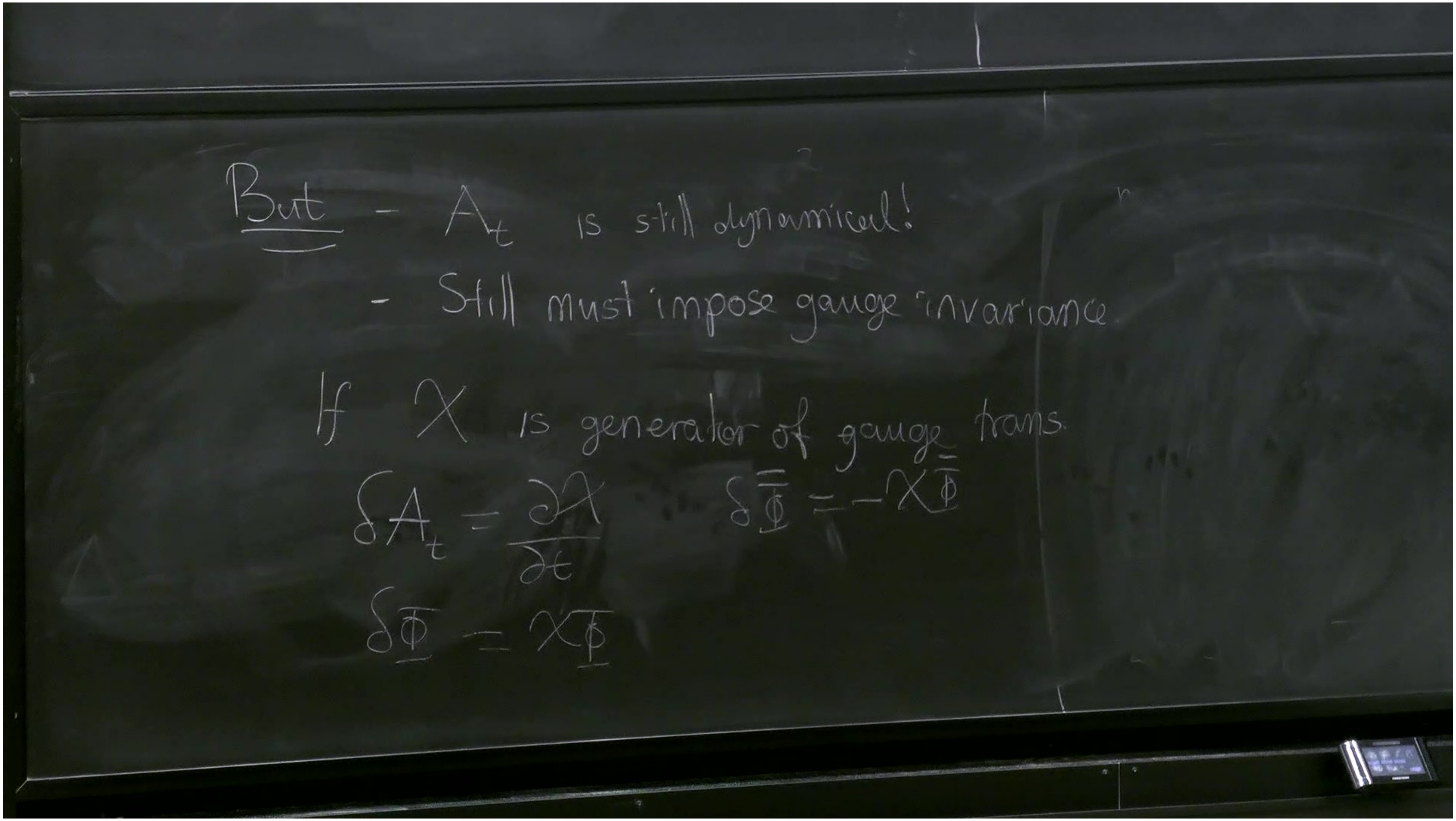

$\rho(\theta)$
$= \sum_{n>0} 2n \int \cos 2\pi n(\theta - \theta')\, q(\theta')$
Very non-local
Local/not entangled
state might be
$\rho(\theta) =$


$p(\theta) = \frac{\partial}{\partial\theta} q(\theta) + q(\theta)$
Value of q near θ
Non local:
$p(\theta) = \int q(\theta')$
depends on all θ'!

Gauge Symmetry

If we have a complex scalar φ, couple to a $U(1)$ gauge field by replacing the action

$$\int (d\varphi \, d\bar{\varphi}) \quad \text{by} \quad \int D_A\varphi \, D_A\bar{\varphi} + \|F_A\|^2$$


But - A_t is still dynamical!
- Still must impose gauge invariance
If χ is generator of gauge trans.
$\delta A_t = \frac{\partial \chi}{\partial t}$
$\delta \bar{\Phi} = -\chi \bar{\Phi}$
$\delta \Phi = \chi \Phi$