Title: Mathematical Physics Lecture

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$\varphi : \mathbb{R} \longrightarrow X$

X a manifold

$S(\varphi) = \int \|\partial_t \varphi\|^2$

Phase space $= T^*X$

coords. $q_i = \varphi_i$

$p_i = \partial_t \varphi_i$

Lagrangian $L \subseteq T^*X$

specifies initial data.

L has to be Lagrangian,

because

$L = \{f_1, \dots, f_n = 0\}$ where

$\{f_i, f_j\} = 0$

$\dim X = n$

$\dim T^*X = 2n$

If we ask that more than n functions vanish, e.g. $p_1 \cdots p_n = 0$

$$q_1 = 0$$

Then, we violate uncertainty

Concretely state will satisfy

$$\hat{p}_i \psi = \frac{\partial \psi}{\partial q_i} = 0$$

$$q_i \psi = 0$$

$\hat{p}_i q_i \psi = 0$

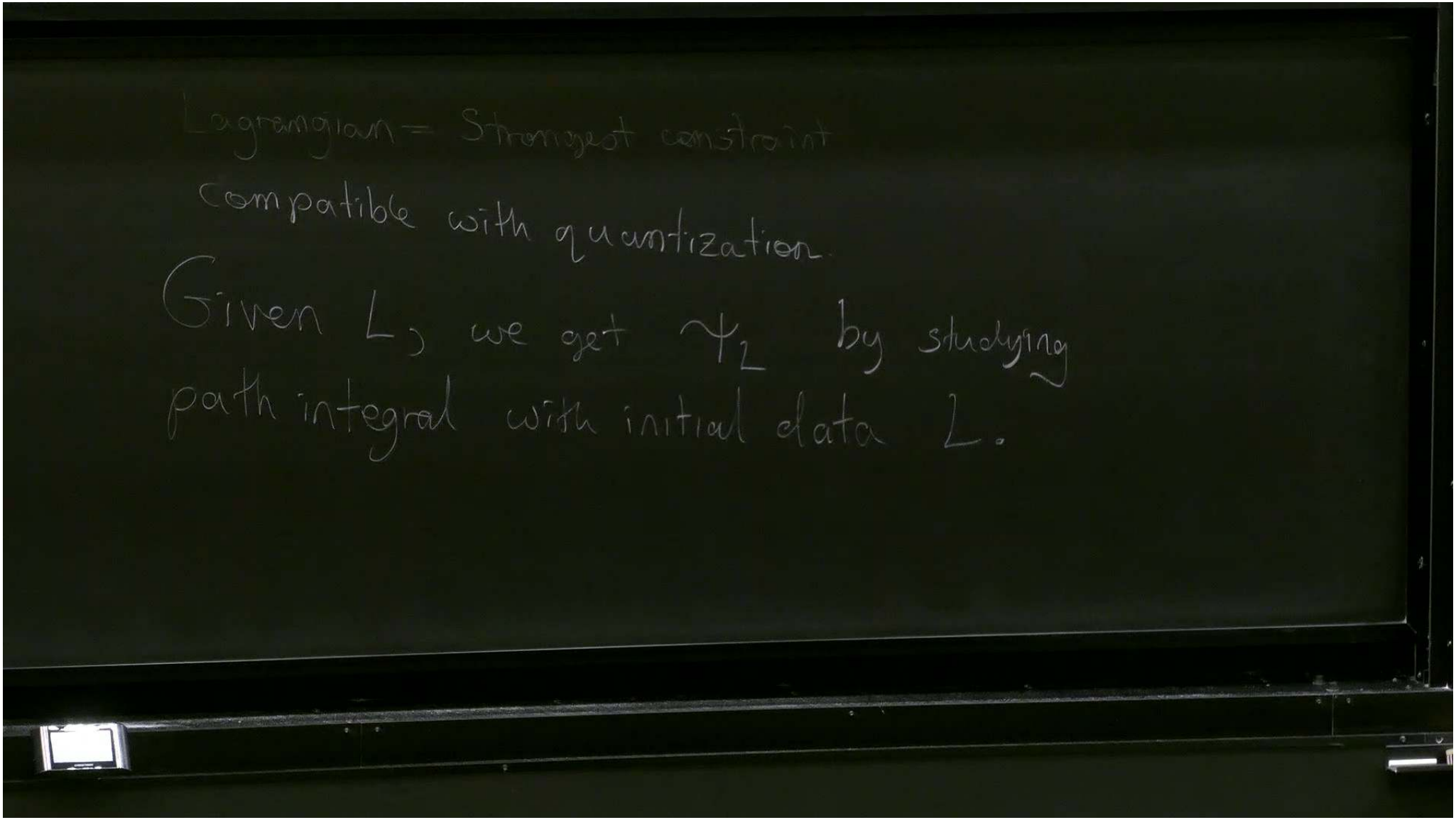
$q_i \hat{p}_i \psi = 0$

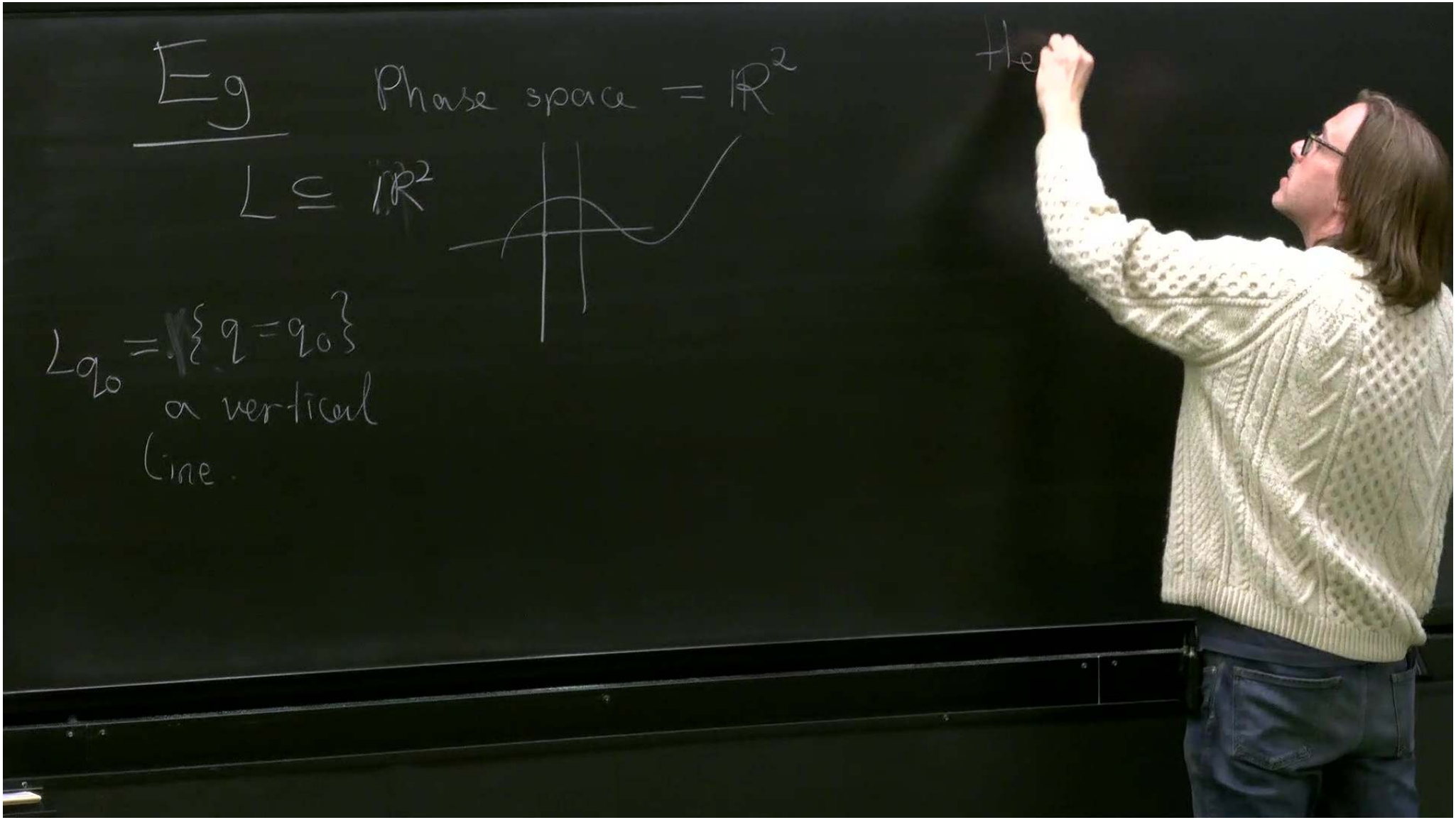
$\Rightarrow [\hat{p}_i, q_i] \psi = 0$

$\hbar \psi = 0$

and $\psi = 0$

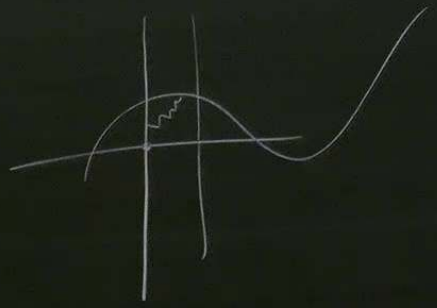
If we impose $< n$ equations, the quantum state $\psi(q)$ is a fn of n variables, imposing $< n$ diff. eqns $\Rightarrow \psi(q)$ undetermined.


Lagrangian = Strongest constraint
compatible with quantization.
Given L, we get ψ_L by studying
path integral with initial data L.


Eg
Phase space $= \mathbb{R}^2$
$L \subseteq \mathbb{R}^2$
$L_{q_0} = \{q = q_0\}$
a vertical line.

Phase space $= \mathbb{R}^2$

$\mathbb{R}^2$



Heuristic

$$\psi_L(q_0) = \int \exp\left(\frac{i}{\hbar}\int_0^1 p(t)\,dq(t)\right)$$

Paths
$p(t), q(t)$
$q(0) = q_0$
$p(1), q(1) \in L$

Vacuum of 2d Scalar Field

$\varphi \quad \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\int \frac{\partial \varphi}{\partial x^\mu} \frac{\partial \varphi}{\partial x^\nu} g_{\mu\nu} (\det g)^{1/2}$$

- Euclidean signature
- g is a metric on $\mathbb{R}^2$

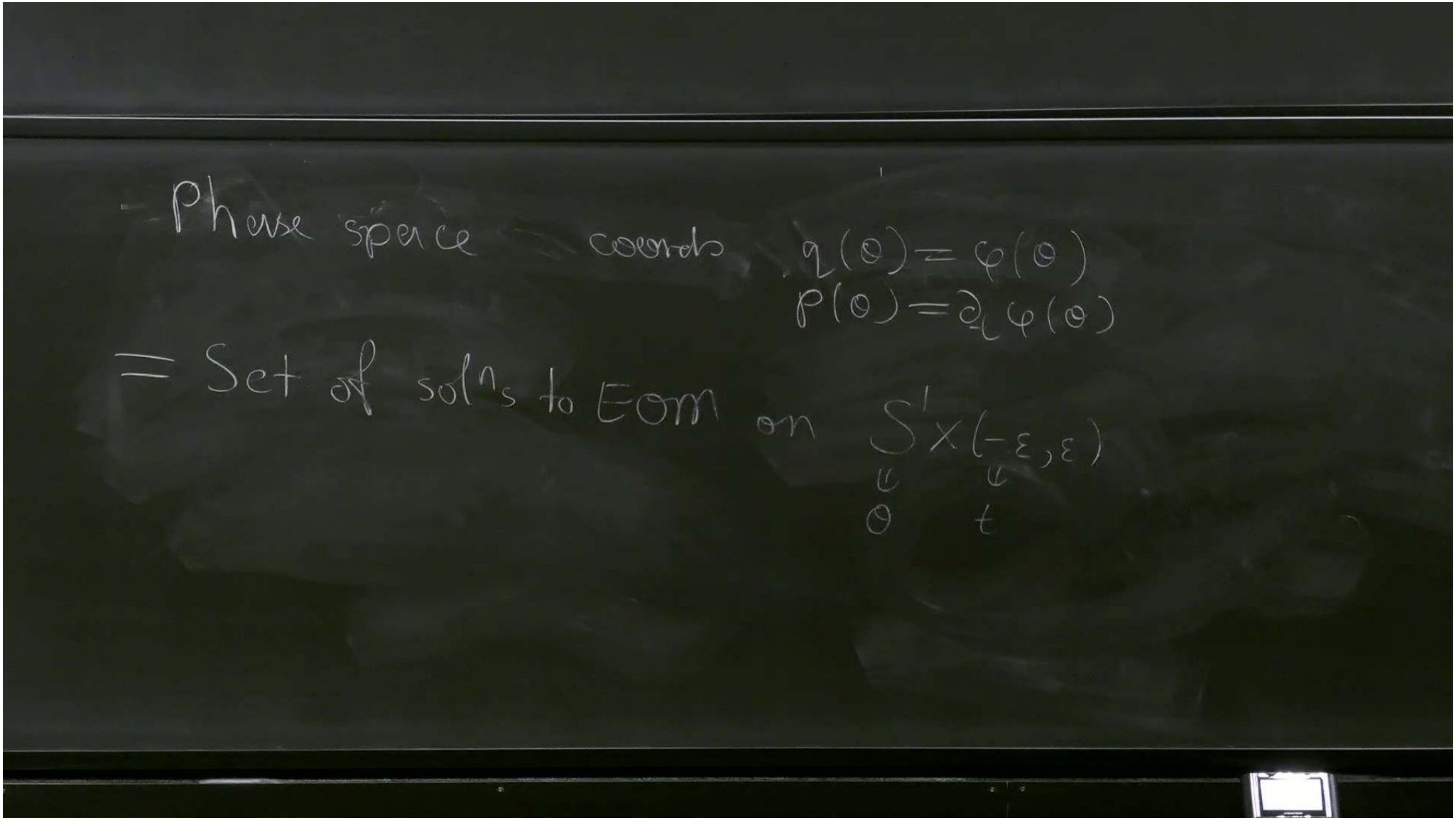
um of 2d Scalar Field

$R^2 \longrightarrow R$

$\frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} g_{\mu\nu} (\det g^{\mu\nu})^{\frac{1}{2}}$

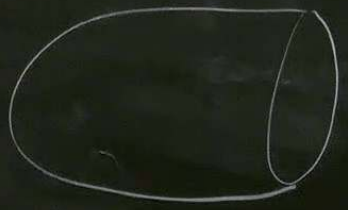
em signature
metric on $\mathbb{R}^2$

$g \longmapsto e^f g$
(Weyl rescaling)
then, S is unchanged
Special to dimension 2


Phase space coords
$q(\theta) = \varphi(\theta)$
$p(\theta) = \partial_t \varphi(\theta)$
= Set of soln's to EOM on $S^1 \times (-\varepsilon, \varepsilon)$
θ
t

Claim

Vacuum state is Lagrangian $L = \{\varphi, \text{which solve EOM on a disc}\}$



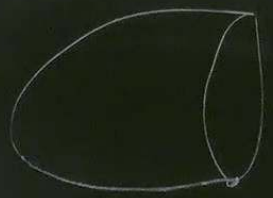
metric on disc doesn't matter, because of conformality.

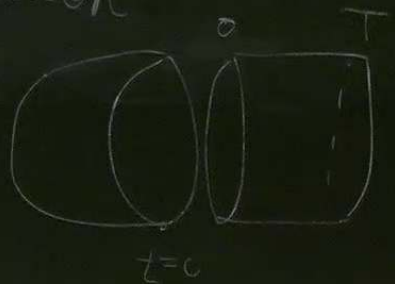
Why is this preserved by Hamiltonian?

Abstractly

Hamiltonian is time evolution

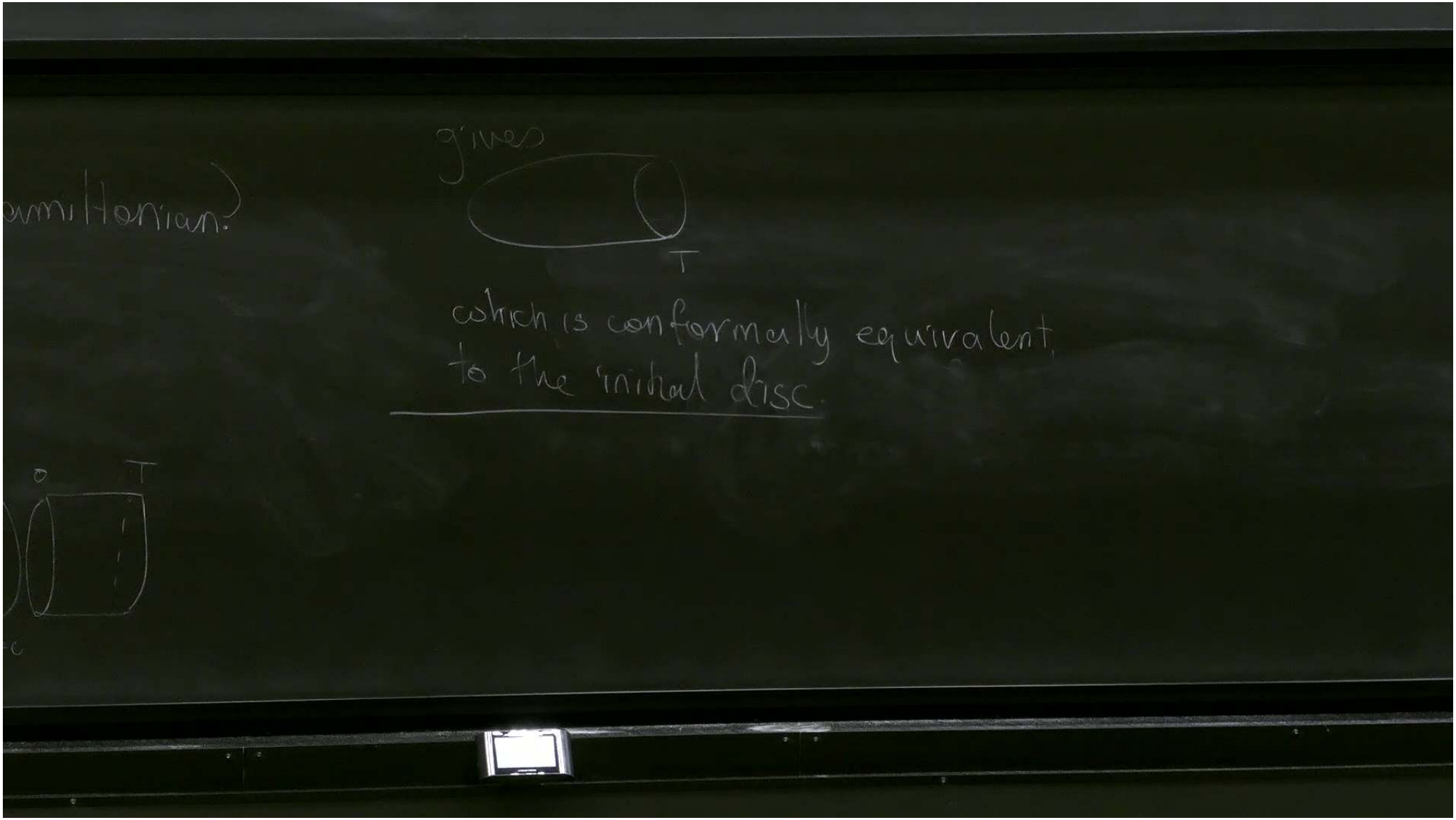
apply e^{-TH} get





gives

which is


gives
T
which is conformally equivalent
to the initial disc
T

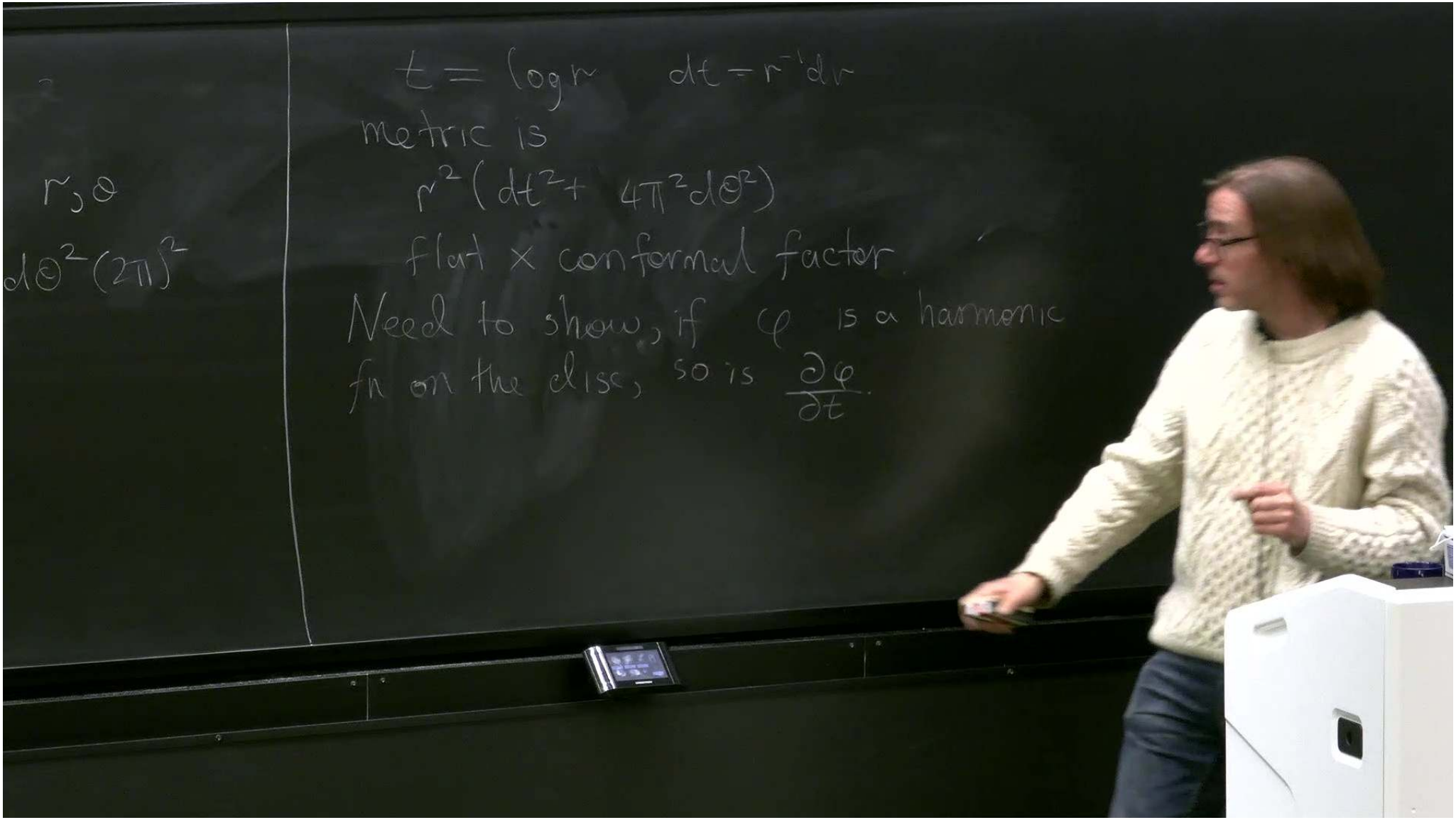
Concretely:

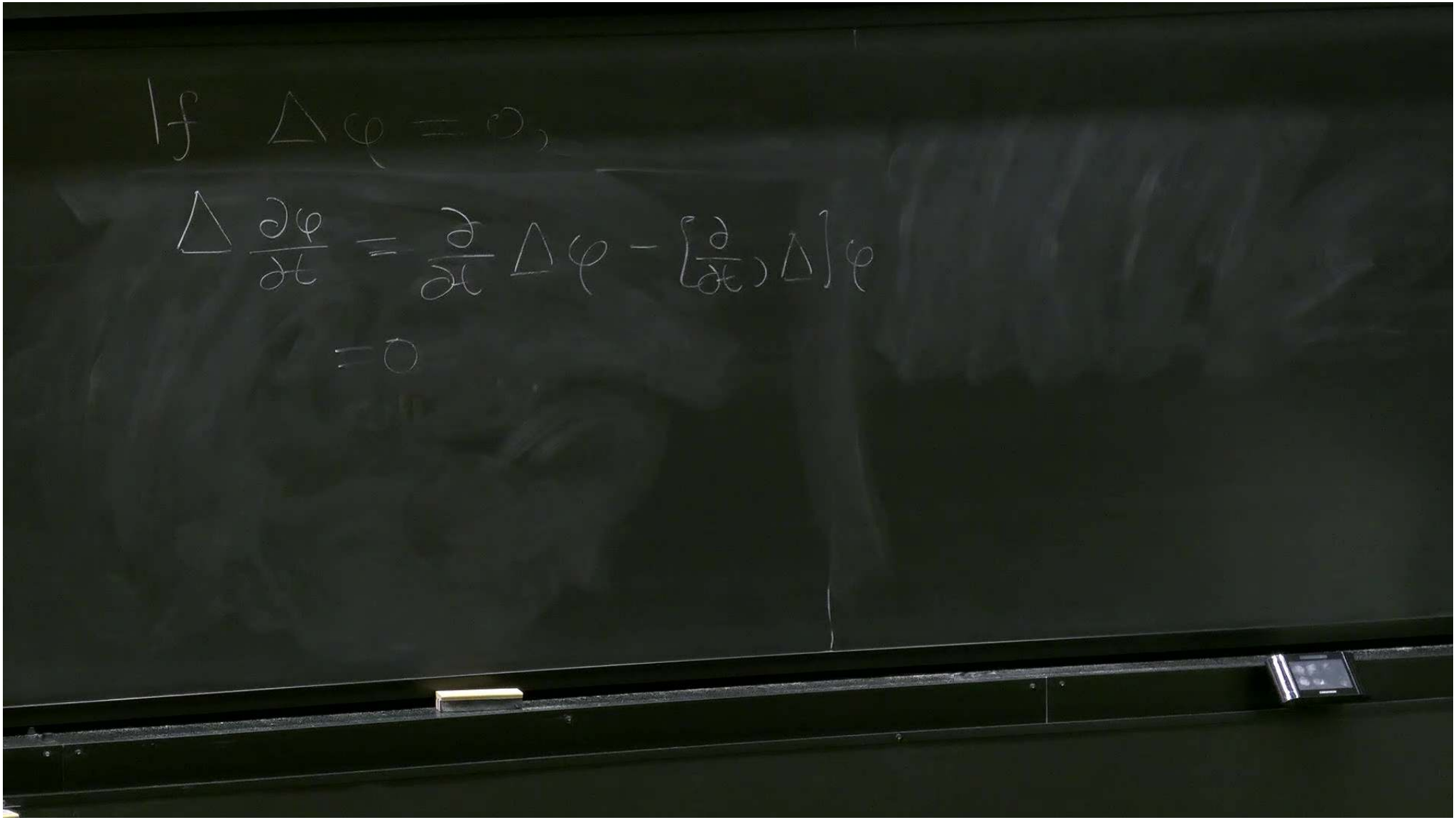
Disc has polar coords r, θ

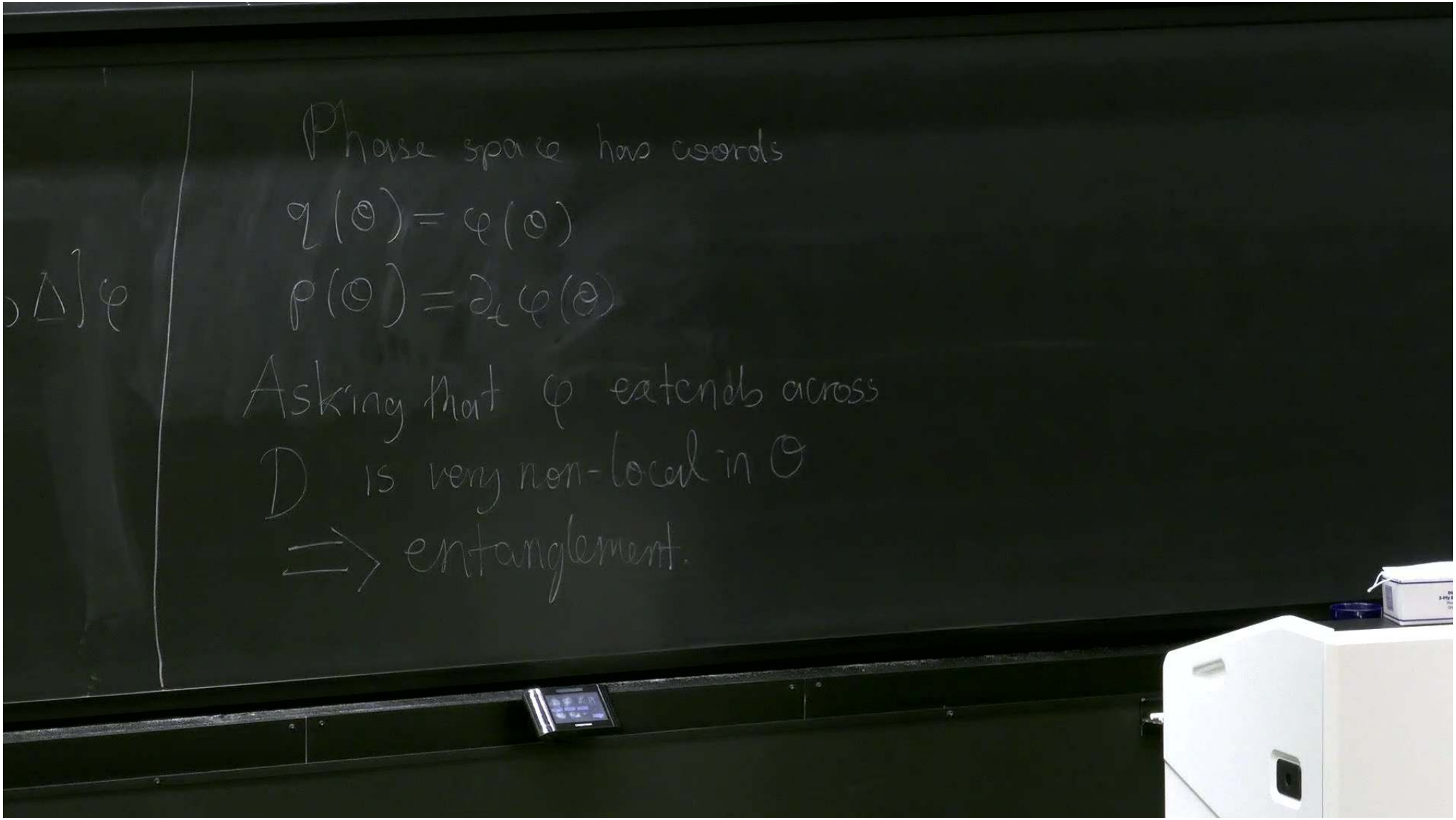
metric $dr^2 + r^2 d\theta^2 (2\pi)^2$

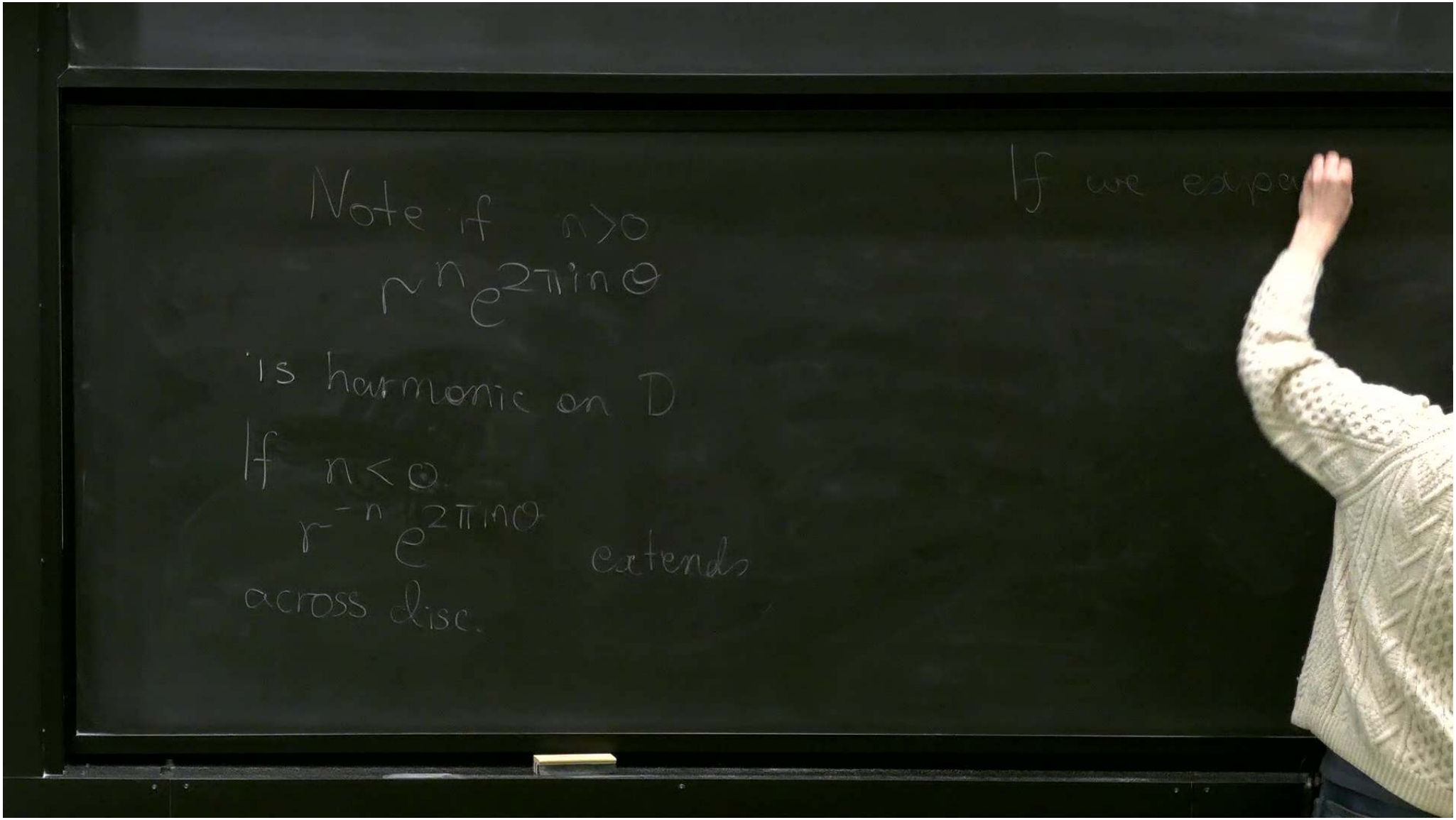
Laplace operator

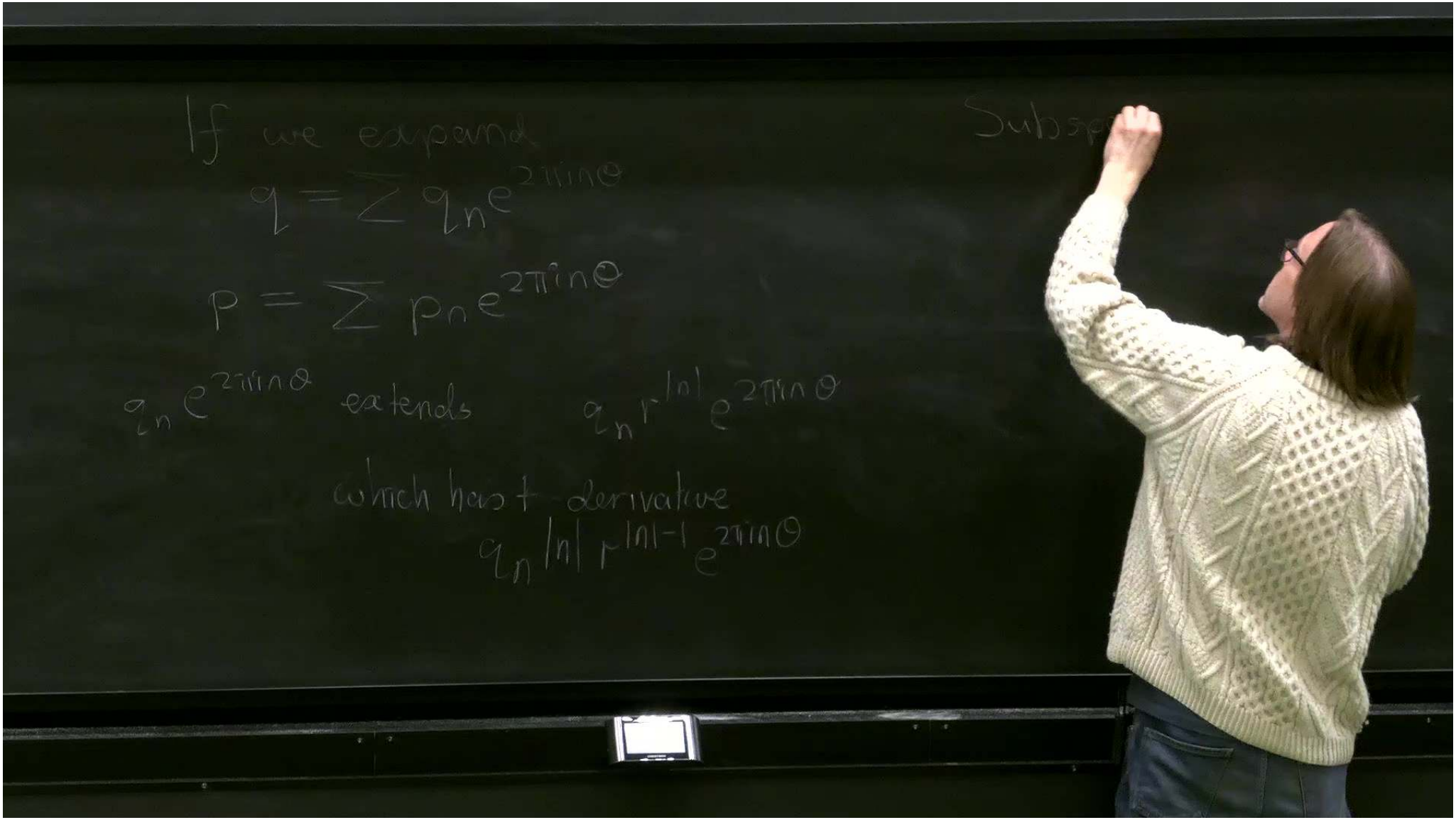
$$\Delta = \frac{\partial}{\partial r^2} - r^{-1}\frac{\partial}{\partial r} + \frac{1}{(2\pi)^2} r^{-2} \partial_\theta^2$$

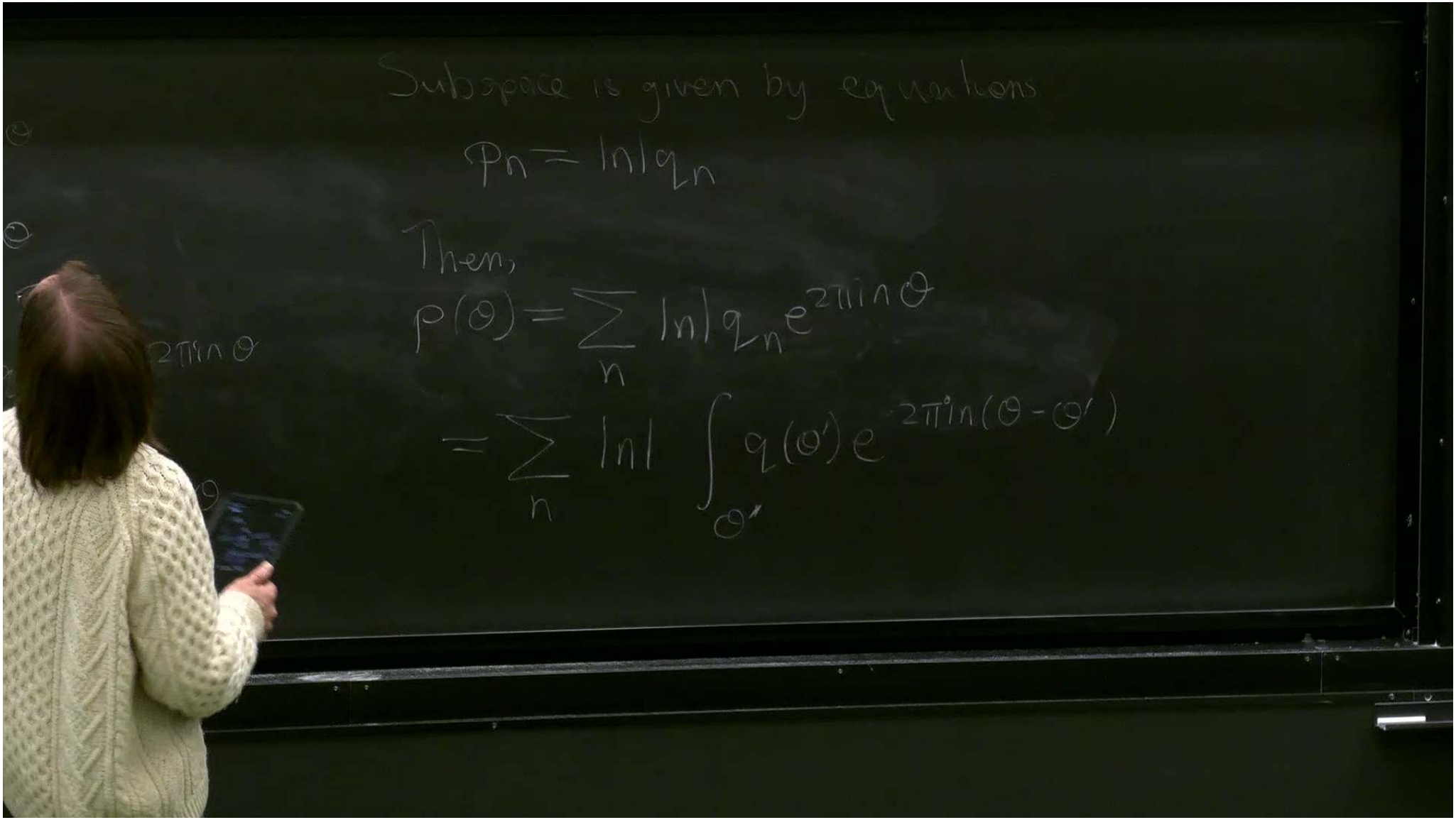

$t = \log r \quad dt = r^{-1} dr$
metric is
$r^2 (dt^2 + 4\pi^2 d\theta^2)$
flat × conformal factor
Need to show, if φ is a harmonic fn on the disc, so is $\frac{\partial \varphi}{\partial t}$.
r, θ
$d\theta^2 (2\pi)^2$


If $\Delta\varphi = 0$,
$\Delta \frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial t}\Delta\varphi - \left[\frac{\partial}{\partial t}, \Delta\right]\varphi$
$= 0$


Phase space has coords
$q(\theta) = \varphi(\theta)$
$p(\theta) = \partial_t \varphi(\theta)$
Asking that φ extends across
D is very non-local in θ
$\Rightarrow$ entanglement.


Note if n>0
$r^n e^{2\pi i n\theta}$
is harmonic on D
If n<0
$r^{-n} e^{2\pi i n\theta}$
extends
across disc.
If we


If we expand
$q = \sum q_n e^{2\pi i n \theta}$
$p = \sum p_n e^{2\pi i n \theta}$
$q_n e^{2\pi i n \theta}$ extends $q_n r^{|n|} e^{2\pi i n \theta}$
which has t derivative
$q_n |n| r^{|n|-1} e^{2\pi i n \theta}$


Subspace is given by equations
$p_n = \ln|q_n$
Then,
$p(\theta) = \sum_n \ln|q_n e^{2\pi i n\theta}$
$= \sum_n \ln|\int_{\theta'} q(\theta') e^{-2\pi i n(\theta-\theta')}$

$$p(\theta) = \sum_{n>0} 2n \int \cos 2\pi n(\theta - \theta') q(\theta')$$

Very non-local

Local/not entangled state might be

$$p(\theta) = \underbrace{\frac{\partial}{\partial\theta} q(\theta) + q(\theta)}_{\text{Value of } q \text{ near } \theta}$$

Non local:

$$p(\theta) = \int q(\theta')$$

depends on all θ'

Gauge Symmetry

If we have a complex scalar field φ, couple to a $U(1)$ gauge field by replacing the action $\int (d\varphi\, d\bar{\varphi})$ by $\int D_A\varphi D_A\bar{\varphi} + \|F_A\|^2$

But - A_t is still dynamical!

- Still must impose gauge invariance.

If χ is generator of gauge trans.

$$\delta A_t = \frac{\partial \chi}{\partial t} \qquad \delta \bar{\phi} = -\chi \bar{\phi}$$

$$\delta \underline{\phi} = \chi \underline{\phi}$$