Title: Mathematical Physics Lecture

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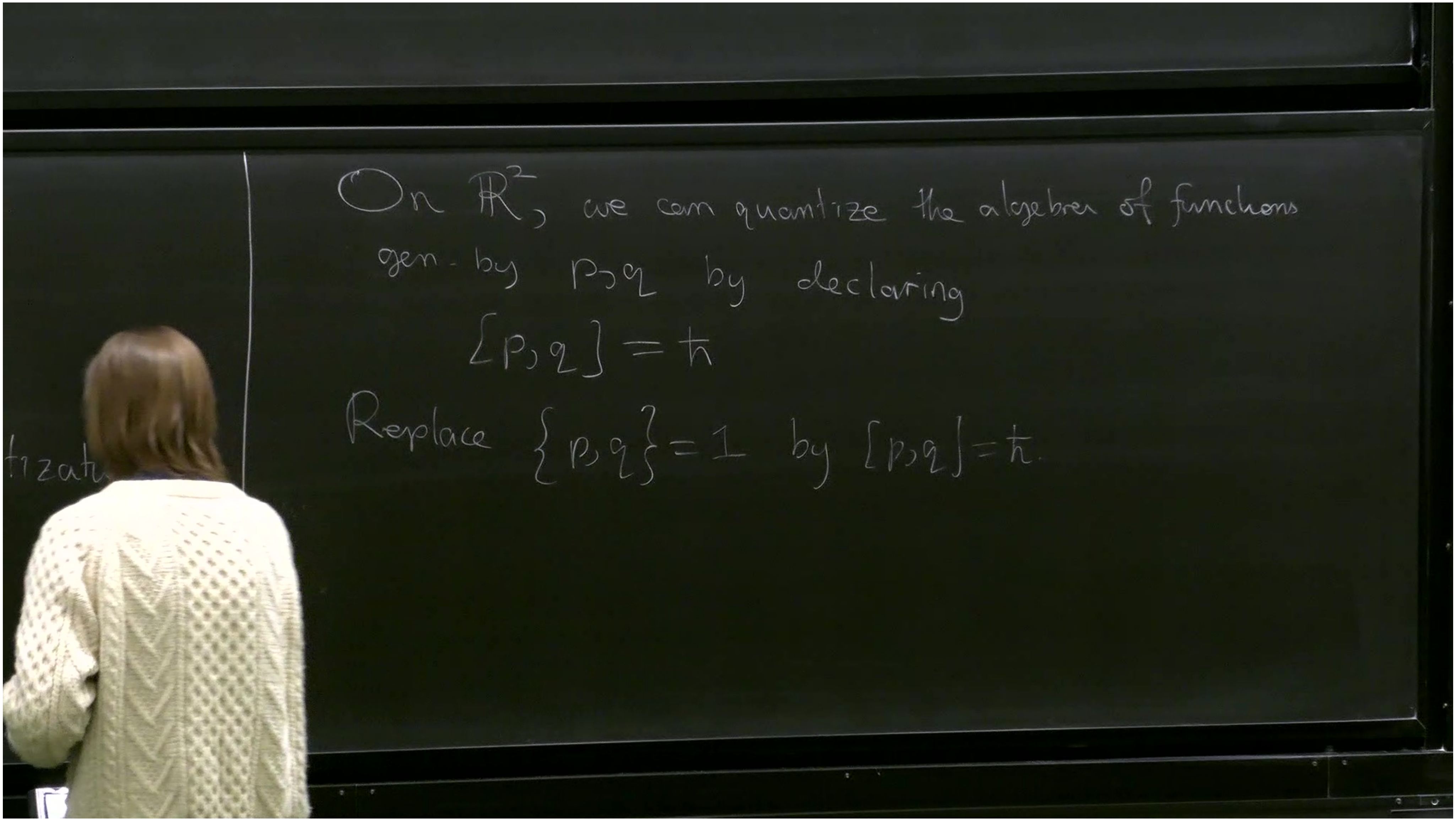
Quantizing Hamiltonian systems.

2 points of view:

1) Deformation (a.k.a. canonical) quantization

2) Geometric quantization


On $\mathbb{R}^2$, we can quantize the algebra of functions
gen. by p, q by declaring
$[p,q] = \hbar$
Replace $\{p,q\} = 1$ by $[p,q] = \hbar$.

If M is a symplectic manifold
a $\star$ product on M is a way
of multiplying functions like
$f, g \in C^\infty(M)$
the star product is

$$f \star g = fg + \hbar\{f,g\} + \hbar^2 f \star_2 g + \ldots$$

Such that:

1) $*$ is associative

$$f*(g*h) = (f*g)*h$$

2) At each order in $\hbar$, $f *_n g$ involves $\leq 2n$ derivatives of f, g.

A def. quantization of M = choice of a star product

$(C^\infty(M), *)$ = operators in the quantum system

Such that:

1) $*$ is associative

$$f*(g*h) = (f*g)*h$$

$$\left(T^{i_1 i_2 i_3 j_1 j_2 j_3} \partial_{i_1}\partial_{i_2}\partial_{i_3} f \, \partial_{j_1}\partial_{j_2}\partial_{j_3} g \right)$$

2) At each order in $\hbar$, $f *_n g$ involves $\leq 2n$ derivatives of f, g.

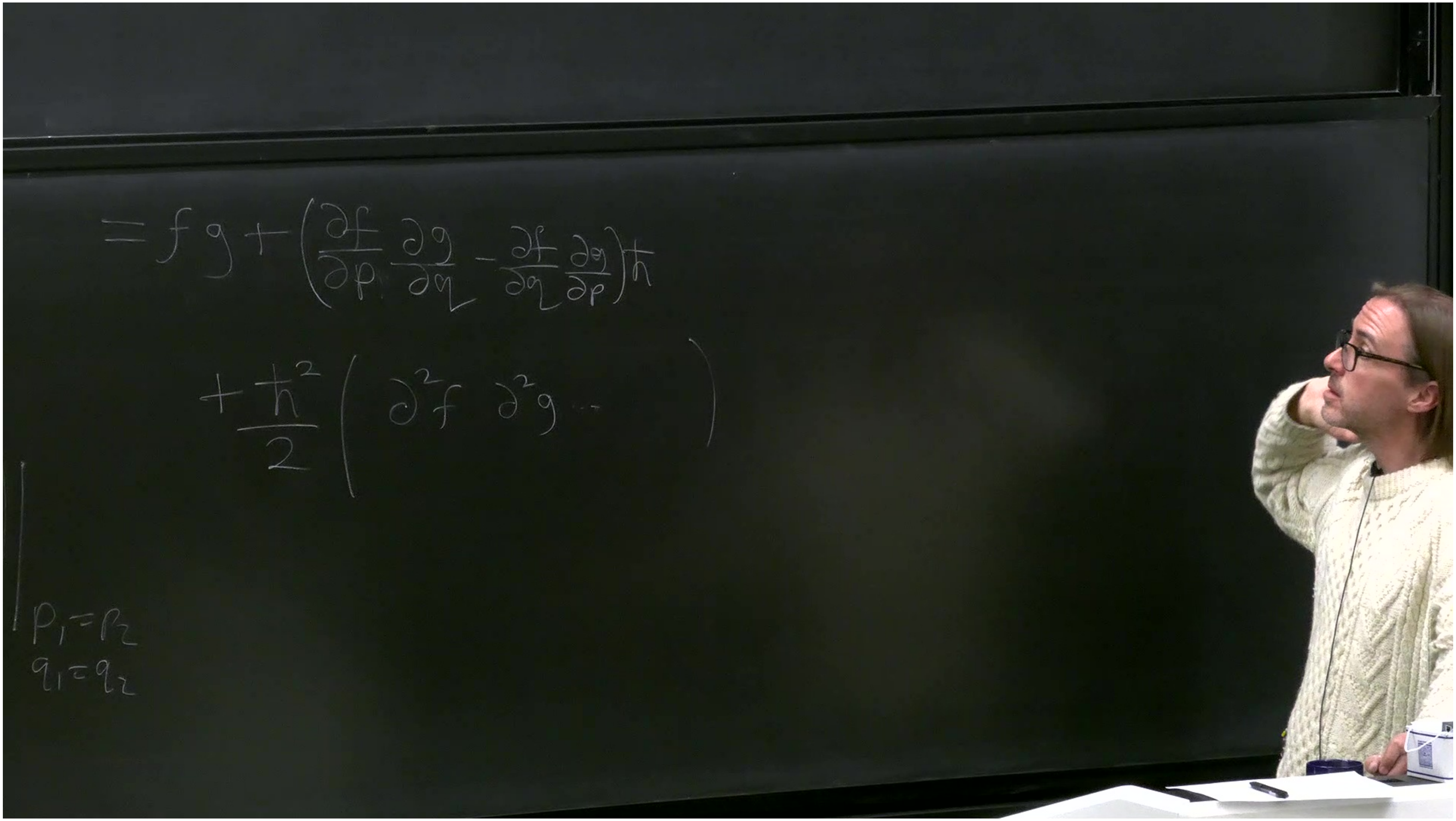
A def. quantization of M = choice of a star product

$(C^\infty(M), *)$ = operators in the quantum system

$$f*g - g*f = \hbar\{f,g\} + O(\hbar^2)$$

On $\mathbb{R}^2$ there is a formula for the $*$ product called the Moyal product:

$$f(p,q) * g(p,q) = \left(e^{\hbar(\partial_{p_1}\partial_{q_2} + \partial_{p_2}\partial_{q_1})} f(p_1,q_1)\, g(p_2,q_2) \right)\Big|_{\substack{p_1=p_2 \\ q_1=q_2}}$$


$= fg + \left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\right)\hbar$
$+ \frac{\hbar^2}{2}\left(\partial^2 f \, \partial^2 g \cdots \right)$
$p_1 = p_2$
$q_1 = q_2$

Theorem (Fedosov)

$\exists$ a $\star$ product on any symplectic manifold.

Other examples of symplectic manifolds:

$\mathbb{R}\times S^1$ coords p, q $q \sim q+1$

Then, the classical alg. has generators

$p, \; Q = e^{2\pi i q}$

If we quantize so that

$[p, q] = 1$

then, we find

$[p, Q] = e^{2\pi i \hbar} Q$

Theorem (Fedosov)

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Then, the classical alg has generators

$p, Q = e^{2\pi i q}$

If we quantize so that

$[p, q] = \hbar$

then, we find

$[p, Q] = 2\pi i \hbar Q$

$[p, -] = \hbar \frac{\partial}{\partial q}$

$\hbar \frac{\partial}{\partial q} e^{2\pi i q} = \hbar 2\pi i e^{2\pi i q}$

Example

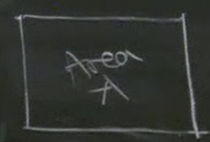
$S^1 \times S^1$

$p, q \qquad p \sim p+1$

$\qquad\qquad q \sim q+1$

$\omega = A\, dp\, dq$

$A = $ Area of torus



$\{p, q\} = \frac{1}{A}$

$[p, q] = \frac{i\hbar}{A}$

$P = e^{2\pi i p}$

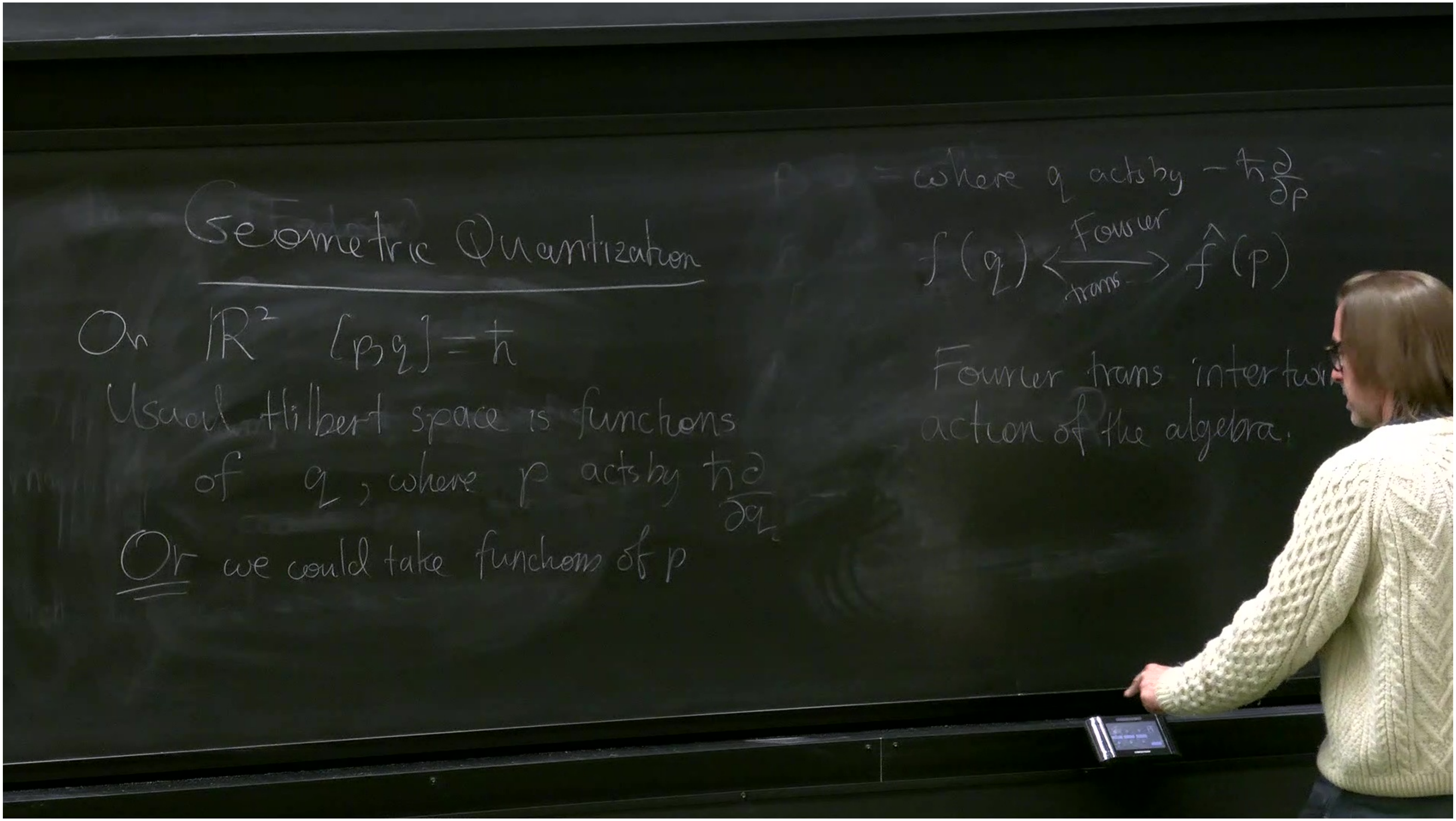
$Q = e^{2\pi i q}$

$$PQ = QP\, e^{\frac{\hbar 2\pi i}{A}}$$

Often convenient to absorb $\hbar$ into A

$$PQ = QP\, e^{\frac{2\pi i}{A}}$$

Large A = classical limit.


Geometric Quantization
On $\mathbb{R}^2$ $[p,q] = \hbar$
Usual Hilbert space is functions of q, where p acts by $\hbar \frac{\partial}{\partial q}$
Or we could take functions of p
where q acts by $-\hbar \frac{\partial}{\partial p}$
$f(q) \xleftrightarrow[\text{trans}]{\text{Fourier}} \hat{f}(p)$
Fourier trans intertwin
action of the algebra.

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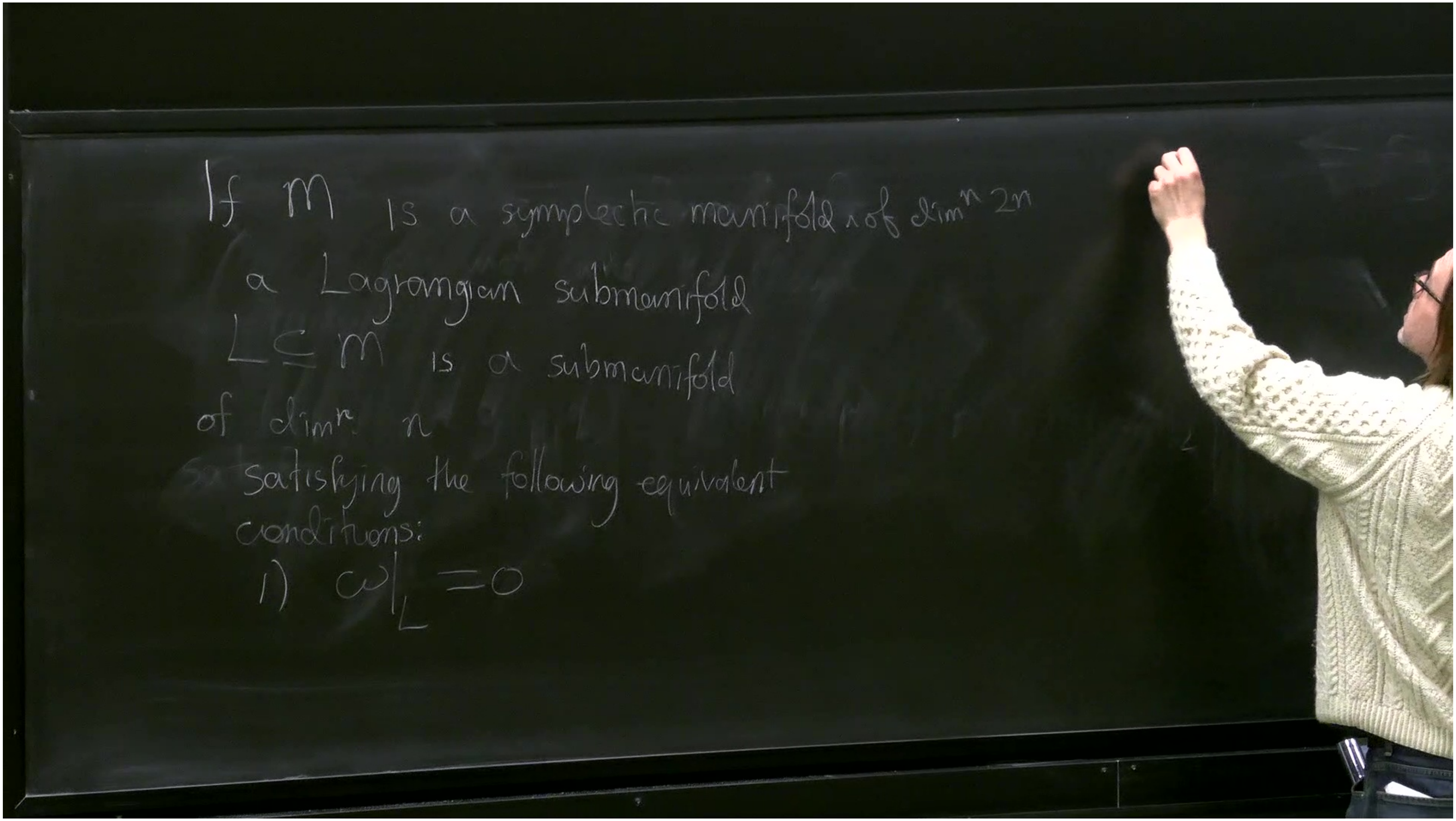
If M is a symplectic manifold of dimⁿ 2n
a Lagrangian submanifold
L ⊆ M is a submanifold
of dimⁿ n
satisfying the following equivalent
conditions:
1) ω|L = 0

If M is a symplectic manifold

a $\star$ product on M is a way of multiplying functions like

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the star product is

$$f \star g = f.g + \hbar\{f, g\} + \hbar^2 f \star_2 g + \ldots$$

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1) $*$ is associative

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$\to T^{i_1 i_2 i_3 j_1 j_2 j_3} \partial_{i_1}\partial_{i_2}\partial_{i_3} f \; \partial_{j_1}\partial_{j_2}\partial_{j_3} g)$

2) At each order in $\hbar$, $f *_n g$ involves $\leq 2n$ derivatives of f, g.

def. quantization of M = choice of a star product

$(C^\infty(M), *)$ = operators in the quantum system

$f*g - g*f = \hbar\{f,g\} + O(\hbar^2)$

Def^n

A Lagrangian foliation is a map $M \xrightarrow{\pi} N$

M is symplectic, dim^n $2n$

N is of dim. n

π is a submersion

$\left(\frac{\partial \pi_i}{\partial x_k} \text{ is of rank } n \right)$

And

Each fibre is Lagrangian.

rsion

rank n)

E.g. $\mathbb{R}^2 \longrightarrow \mathbb{R}$

$(p,q) \longrightarrow q$

is a Lagrangian foliation.

<u>Def^n</u> (1st approx - require corrections!)

The Hilbert space of the quantum system is $C^\infty(N)$, when we have a Lagrangian foliation.

$$PQ = QP\, e^{\frac{\hbar 2\pi i}{A}}$$

Often convenient to absorb $\hbar$ into A

$$PQ = QP\, e^{\frac{2\pi i}{A}}$$

Large A = classical limit.

Ex: $M = \mathbb{R} \times S^1$

p, q coords

$\omega = dp \wedge dq$

The map $\mathbb{R} \times S^1 \longrightarrow S^1$

$(p, q) \longrightarrow q$

is a Lagrangian foliation.

H

Hilbert space = functions of q
(periodic)

Basis is $Q^n = e^{2\pi i n q}$

$p \rightsquigarrow \hbar \frac{\partial}{\partial q} = \hbar 2\pi i Q \frac{\partial}{\partial Q}$

Q^n is an eigenvector of
p

$H = p^2$, then Q^n is an

= functions of q

dic)

$$Q^n = e^{2\pi i n q}$$

$$= \hbar 2\pi i Q \frac{\partial}{\partial Q}$$

envector of

$H = p^2$, then Q^n is an eigenvector

eigenvalue $-\hbar^2 (2\pi n)^2$

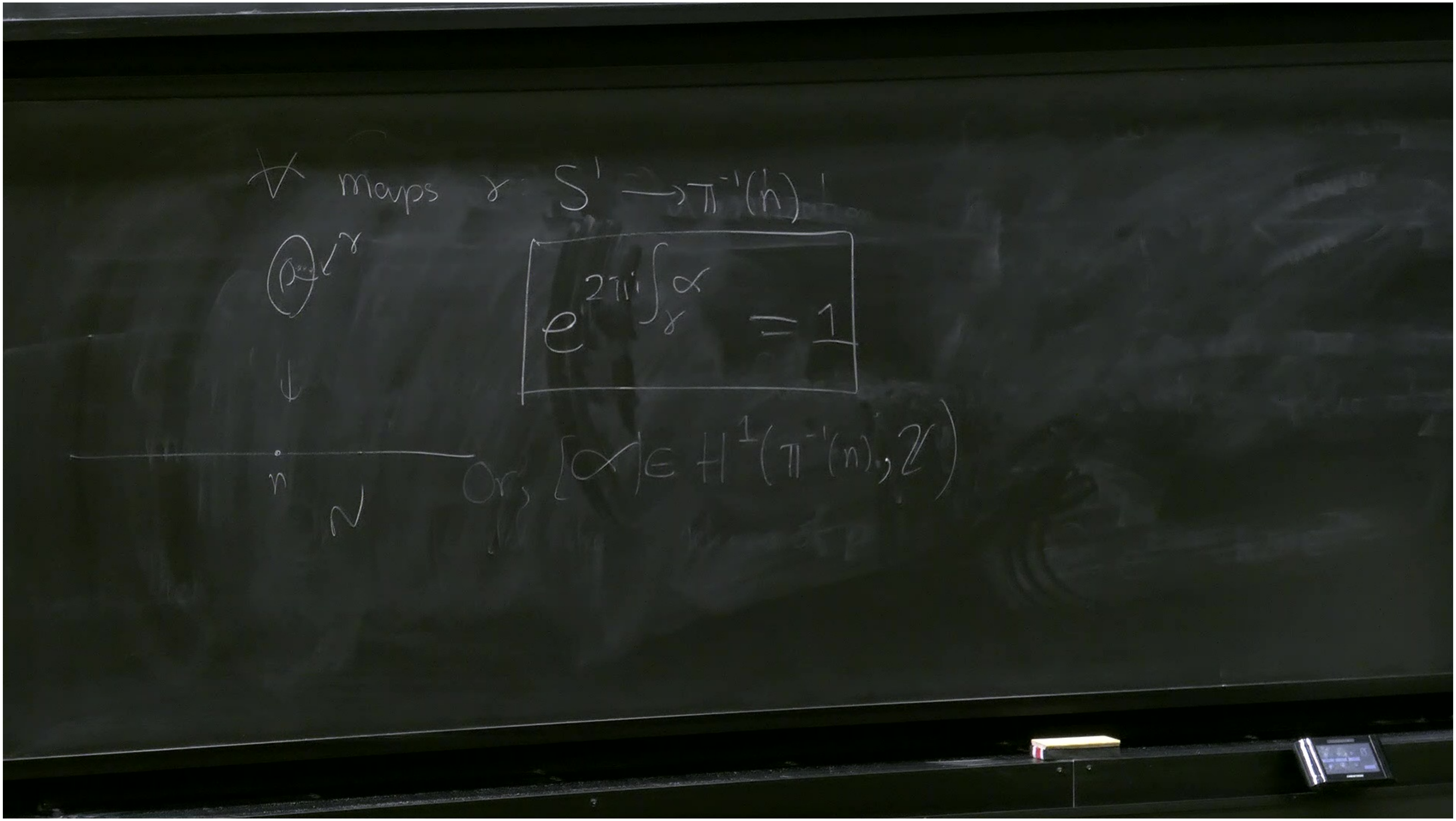
as these are 'Fourier dual to $\mathbb{Q}^n$

<u>Sol^n</u> Bohr–Sommerfeld orbits

Assume that $\omega = d\alpha$ $\alpha \in \Omega^1(M)$

$\pi : M \longrightarrow N$ is a Lagrangian foliation.

$n \in N$ is a Bohr-Sommerfeld orbit, if


∀ maps γ: S¹ → π⁻¹(n) :
e^{2πi ∫_γ α} = 1
Or, [α] ∈ H¹(π⁻¹(n), ℤ)
n
N

"Correct" Hilbert space is

the functions on $\{n \in \mathbb{N},\ n \text{ is a BS orbit}\}$

<u>Ex:</u>

$\mathbb{R} \times S^1 \longrightarrow \mathbb{R}$

$p, q \longrightarrow p$

$\omega = dp \wedge dq$

$\alpha = p\,dq$

For $p \in \mathbb{R}$,

$$\int_{S^1} p\,dq = p$$

BS condition: $p \in$ an integer

$= $ functions of q

dic)

$$Q^n = e^{2\pi i n q}$$

$$= \hbar 2\pi i Q \frac{\partial}{\partial Q}$$

envector of

$H = p^2$, then Q^n is an eigenvector

eigenvalue $-\hbar^2(2\pi n)^2$