

Title: Mathematical Physics Lecture

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Quantizing Hamiltonian systems.

2 points of view.

1) Deformation (a.k.a. canonical) quantization

2) Geometric quantization

On \mathbb{R}^2 , we can quantize the algebra of functions
gen. by p, q by declaring

$$[p, q] = \hbar$$

Replace $\{p, q\} = 1$ by $[p, q] = \hbar$.

If M is a symplectic manifold
a \star product on M is a way
of multiplying functions like

$$f, g \in C^\infty(M)$$

the star product is

$$f \star g = fg + \hbar \{f, g\} + \hbar^2 f \star_2 g + \dots$$

Such that:

1) \star is associative:

$$f \star (g \star h) = (f \star g) \star h$$

2) At each order in \hbar , $f \star_n g$ involves $\leq 2n$ derivatives of f, g .

A def-quantization of M = choice of a star product

$(C^\infty(M), \star) =$ operators in the quantum system

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$(C^\infty(M), \star) =$ operators in the quantum system

$$f \star g - g \star f = \hbar \{f, g\} + O(\hbar^2)$$

$$\sum_{n=0}^{\infty} \frac{(\hbar)^n}{n!} \left(\sum_{i_1, \dots, i_n} \partial_{i_1} \dots \partial_{i_n} f \right) \left(\sum_{j_1, \dots, j_n} \partial_{j_1} \dots \partial_{j_n} g \right)$$

On \mathbb{R}^2 there is a formula for
the $*$ product called the Moyal
product:

$$f(p, q) * g(p, q) = \left(e^{\hbar(\partial_{p_1} \partial_{q_2} - \partial_{p_2} \partial_{q_1})} f(p_1, q_1) g(p_2, q_2) \right) \Big|_{\substack{p_1 = p_2 \\ q_1 = q_2}}$$

$$= fg + \left(\frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} \right) \hbar$$

$$+ \frac{\hbar^2}{2} \left(\frac{\partial^2 f}{\partial p^2} \frac{\partial^2 g}{\partial q^2} - \dots \right)$$

$$p_1 = p_2$$
$$q_1 = q_2$$



Theorem (Fedosov)

\exists a \star product on
any symplectic manifold.

Other examples of symplectic
manifolds:

$\mathbb{R} \times S^1$ coords p, q $q \sim q+1$

Then, the classical alg has generators

$$p, Q = e^{2\pi i q}$$

If we quantize so that

$$[p, q] = 1$$

then, we find

$$[p, Q] = e^{2\pi i q} Q$$

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If we quantize so that

$$[p, q] = \hbar$$

then, we find

$$[p, Q] = 2\pi i \hbar Q$$

$$[p, -] = \hbar \frac{\partial}{\partial q}$$

$$\hbar \frac{\partial}{\partial q} e^{2\pi i q} = \hbar 2\pi i e^{2\pi i q}$$

Example



$S' \times S'$

$p, q \quad p \sim p+1$

$q \sim q+1$

$\omega = A dp dq$

$A = \text{Area of torus}$

$$\{p, q\} = \frac{1}{A}$$

$$[p, q] = \frac{h}{A}$$

$$P = e^{2\pi i p}$$

$$Q = e^{2\pi i q}$$

$$PQ = QP e^{\frac{i\hbar 2\pi i}{A}}$$

Often convenient to absorb \hbar into A

$$PQ = QP e^{\frac{2\pi i}{A}}$$

Large A = classical limit.

Geometric Quantization

On \mathbb{R}^2 $[p, q] = \hbar$

Usual Hilbert space is functions of q , where p acts by $\hbar \frac{\partial}{\partial q}$

Or we could take functions of p

= where q acts by $-\hbar \frac{\partial}{\partial p}$

$$f(q) \xleftrightarrow[\text{trans}]{\text{Fourier}} \hat{f}(p)$$

Fourier trans intertwines action of the algebra.

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If M is a symplectic manifold of $\dim^n 2n$

a Lagrangian submanifold

$L \subseteq M$ is a submanifold
of $\dim^n n$

satisfying the following equivalent
conditions:

1) $\omega|_L = 0$

If M is a symplectic manifold
a \star product on M is a way
of multiplying functions like

$$f, g \in C^\infty(M)$$

the star product is

$$f \star g = fg + \frac{\hbar}{i} \{f, g\} + \frac{\hbar^2}{2} f \star_2 g + \dots$$

Such

1)

2)

A

(C^∞)

$f \star g$

ch that:

1) \star is associative:
 $f \star (g \star h) = (f \star g) \star h$

$$\frac{\partial^2}{\partial x_1 \partial x_2} \left(\frac{\partial}{\partial x_3} f \right) \frac{\partial^2}{\partial x_1 \partial x_2} g$$

2) At each order in \hbar , $f \star_n g$ involves $\leq 2n$ derivatives of f, g .

def-quantization of $M =$ choice of a star product

$\mathcal{O}^\hbar(M, \star) =$ operators in the quantum system

$$g - g \star f = \hbar \{f, g\} + O(\hbar^2)$$

Defⁿ

A Lagrangian foliation
is a map $M \xrightarrow{\pi} N$

M is symplectic, $\dim^n 2n$

N is of $\dim. n$

π is a submersion
($\frac{\partial \pi_i}{\partial x_k}$ is of rank n)

And

Each fibre is
Lagrangian.

ersion
rank n)

$$\text{E.g. } \mathbb{R}^2 \rightarrow \mathbb{R}$$
$$(p, q) \rightarrow q$$

is a Lagrangian foliation.

Defⁿ (1st approx - require corrections!)

The Hilbert space of the quantum system is $C^\infty(N)$, when we have a Lagrangian foliation.

$$PQ = QP e^{\frac{\hbar 2\pi i}{A}}$$

Often convenient to absorb \hbar into A

$$PQ = QP e^{\frac{2\pi i}{A}}$$

Large A = classical limit.

Ex: $M = \mathbb{R} \times S^1$

p, q coords

$$\omega = dp \wedge dq$$

The map $\mathbb{R} \times S^1 \rightarrow S^1$
 $(p, q) \rightarrow q$
is a Lagrangian foliation.

Hilbert space = functions of q
(periodic)

Basis is $Q^n = e^{2\pi i n q}$

$$p \rightsquigarrow \hbar \frac{\partial}{\partial q} = \hbar 2\pi i Q \frac{\partial}{\partial Q}$$

Q^n is an eigenvector of
 p

$H = p^2$, then Q^n is an

= functions of q

dir)

$$Q^n = e^{2\pi i n q}$$

$$= \hbar 2\pi i n \frac{\partial}{\partial q}$$

eigenvector of

$H = p^2$, then Q^n is an eigenvector
eigenvalue $-\hbar^2 (2\pi n)^2$

as these are Fourier dual to \mathbb{Q}^n

Soln Bohr-Sommerfeld orbits.

Assume that $\omega = d\alpha$ $\alpha \in \Omega^1(M)$

$\pi: M \rightarrow N$ is a Lagrangian foliation.

$\pi \in N$ is a Bohr-Sommerfeld orbit, if

* maps $\gamma: S^1 \rightarrow \pi^{-1}(h)$



$$e^{2\pi i \int_{\gamma} \alpha} = 1$$



h

\mathbb{Z}

Or, $[\alpha] \in H^1(\pi^{-1}(h), \mathbb{Z})$

"Correct" Hilbert space is

then functions on $\{n \in \mathbb{N}, n \text{ is a BS orbit}\}$

Ex:

$$\mathbb{R} \times S^1 \rightarrow \mathbb{R}$$

$$p, q \rightarrow p$$

$$\omega = dpdq$$

$$\alpha = pdq$$

For $p \in \mathbb{R}$,

$$\int_{S^1} p dq = p$$

BS condition: $p \in \text{an integer}$

✓

= functions of q

(ic)

$$Q^n = e^{2\pi i n q}$$

$$= \hbar 2\pi i n \frac{\partial}{\partial q}$$

eigenvector of

$H = p^2$, then Q^n is an eigenvector
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