Title: Mathematical Physics Lecture

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Given a Lagrangian system,

the <u>phase space</u> is a classical Hamiltonian system

that's equivalent.

i.e. Phase Space is a symplectic manifold M, with a Hamiltonian i.e. Phase Space is a symplectic manifold M, with a Hamiltonian $H \in C^\infty(M)$

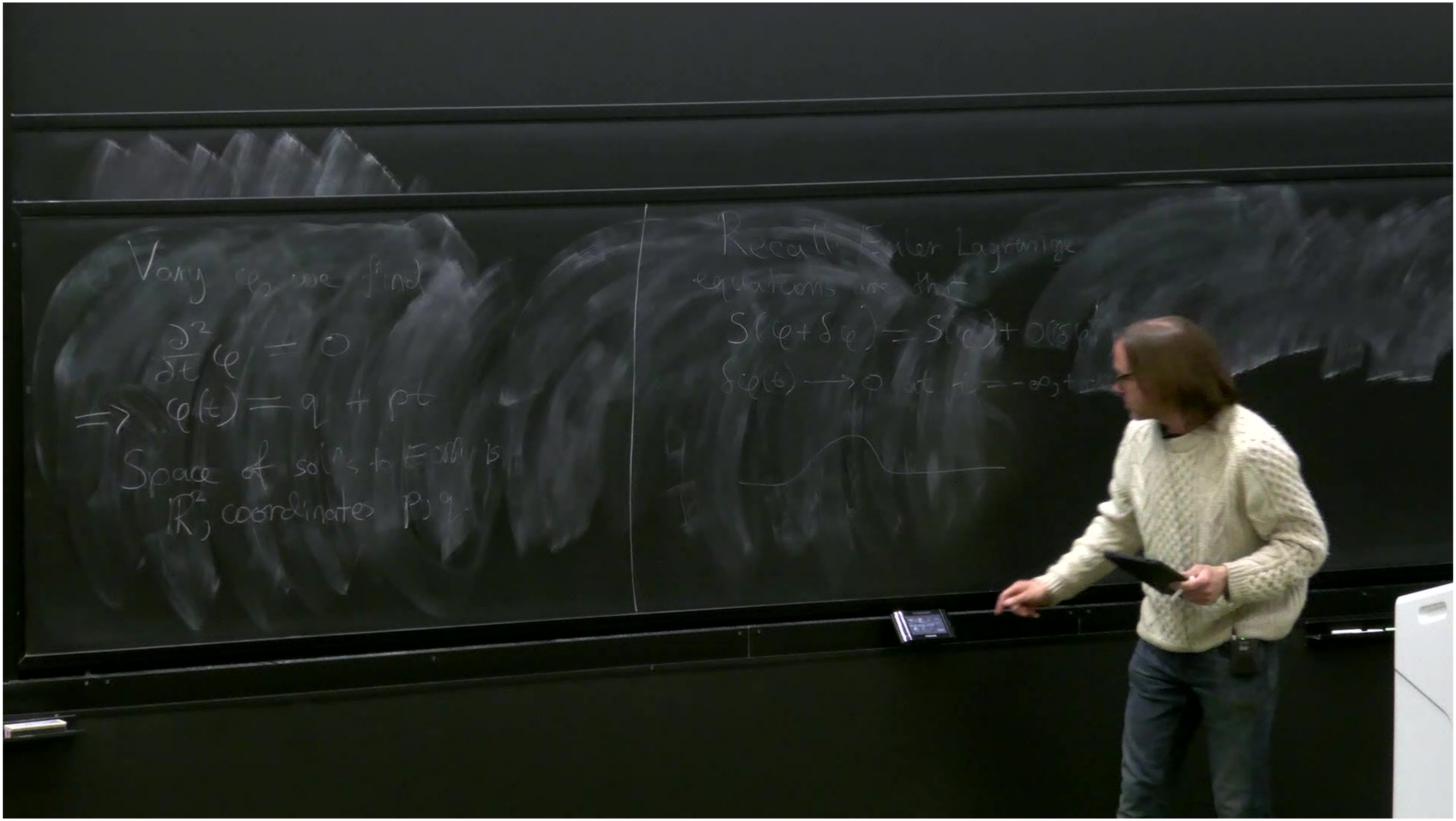
Phase Space is ALWAYS the space of soln's to EOM on Space $\times \mathbb{R}$

- Why is this symplectic?
- What is Hamiltonian?

Example:

$\varphi(t)$ a field depends only on t

$-\frac{1}{2}\int \varphi(t)\, \partial_t^2 \varphi(t)$


Vary φ we find
∂²/∂t² φ = 0
=> φ(t) = q + pt
Space of sol'ns to EOM is
ℝ², coordinates p, q.
Recall: Euler Lagrange
equations are that
S(φ + δφ) = S(φ) + O((δφ)²)
δφ(t) → 0 at t = -∞, +

$\mathbb{R}$, coordinates p, q.

If $\varphi(t)$ satisfies EOM, can consider

$$S_{[0,1]}(\varphi) = \int_0^1 \mathcal{L}(\varphi) - \frac{1}{2}\int_0^1 \varphi\, \partial_t^2 \varphi$$

$$\delta S_{[0,1]} = S_{[0,1]}(\varphi + \delta\varphi) - S_{[0,1]}(\varphi)$$

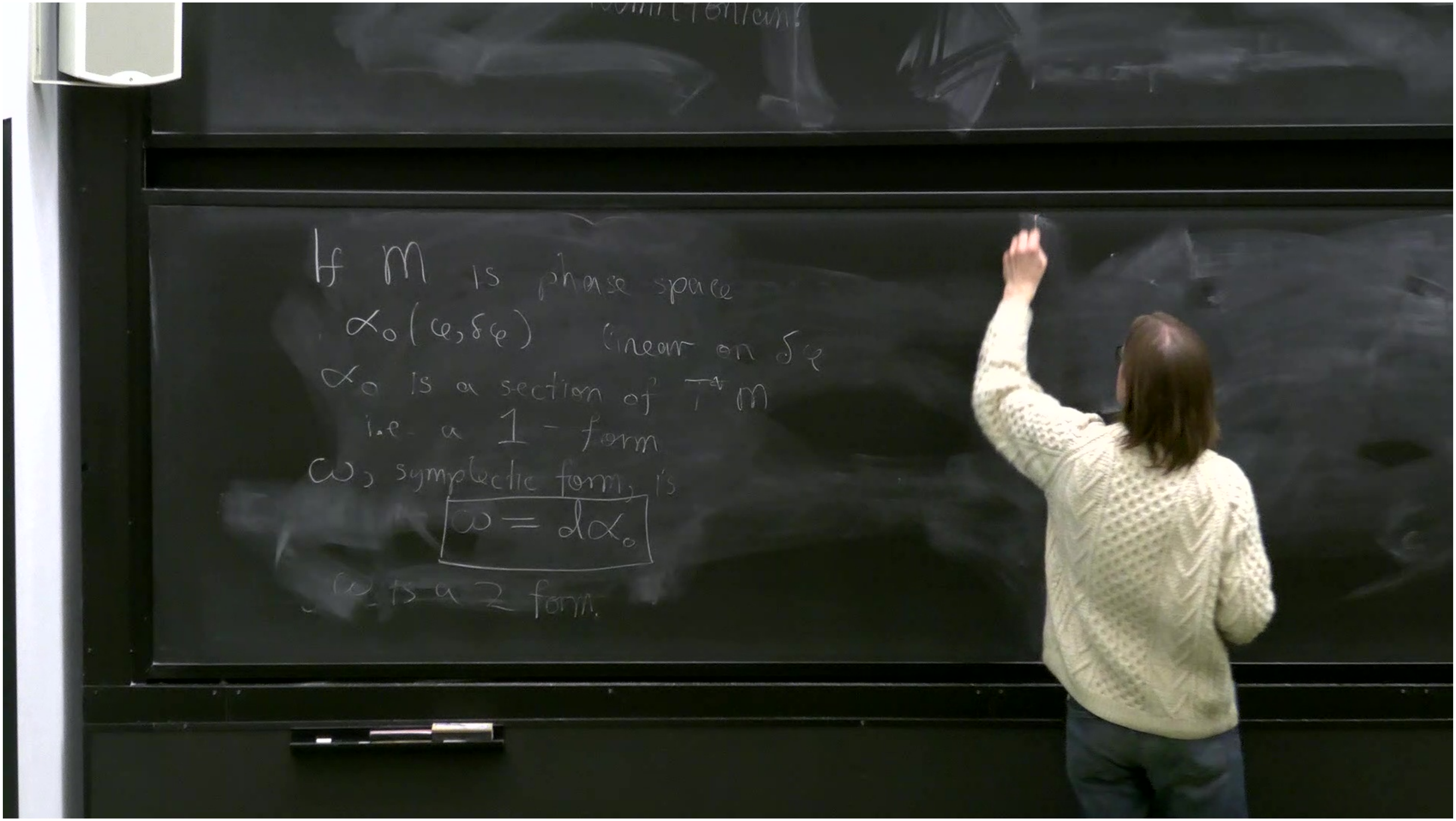
$\delta\varphi$ satisfies EOM

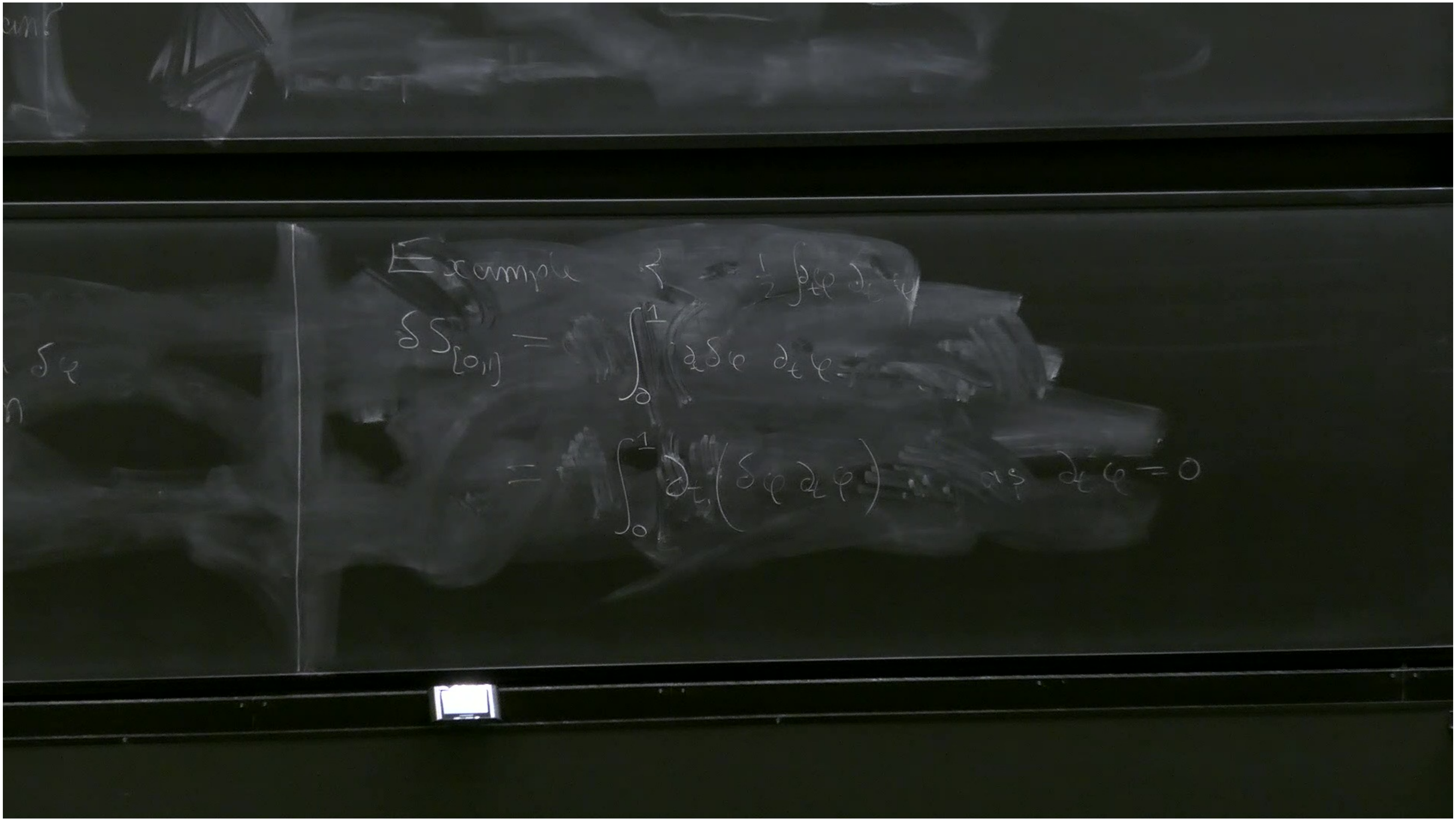
Because φ satisfies EOM, bulk term vanished

$$\delta S_{[0,1]} = \alpha_0 - \alpha_1$$

α_0: contribution from 0

α_1: " from 1


If M is phase space
α₀(φ, δφ) linear on δφ
α₀ is a section of T*M
i.e. a 1-form
ω, symplectic form, is
ω = dα₀
ω is a 2 form.


Example $S = \frac{1}{2}\int \partial_t\varphi \, \partial_t\varphi$
$\delta S_{[0,1]} = \int_0^1 \partial_t\delta\varphi \, \partial_t\varphi$
$= \int_0^1 \partial_t(\delta\varphi \, \partial_t\varphi)$ as $\partial_t^2\varphi = 0$

$\boxed{\omega = d\alpha_0}$

a 2 form.

$\alpha_0 = -\delta\varphi\,\partial_t\varphi \qquad \varphi(0)=q \quad \varphi'(0)=p$

$- dq\, p$

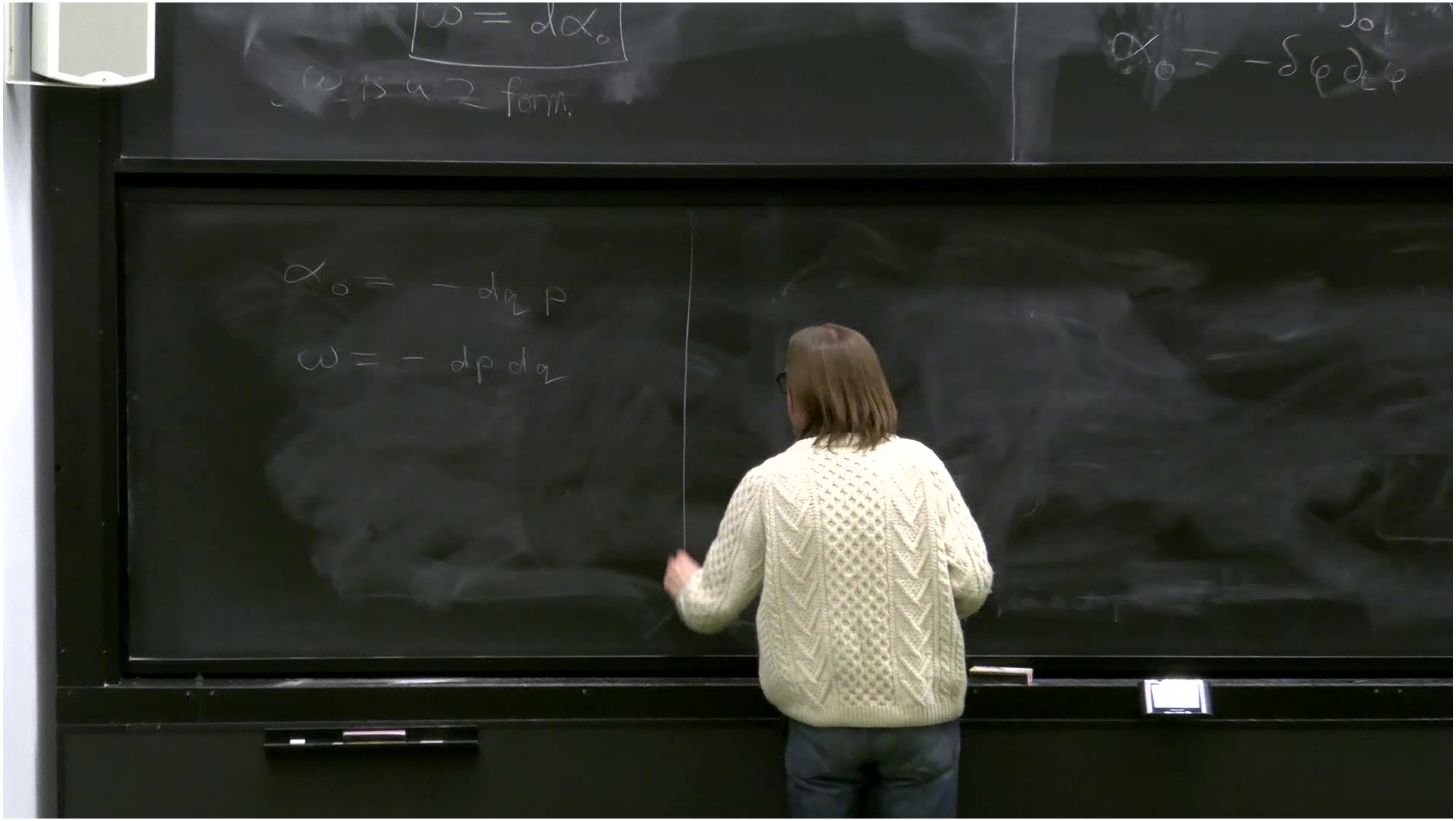
$- dp\, dq$

What about H?

Space of sol'ns to EOM has a time translation symmetry, ∂_t

H is the function generating this symmetry

$\{H, f\} = \frac{\partial f}{\partial t}$


ω = dα₀
ω is a 2 form
∫₀
α₀ = −δφ ∂ₜφ
α₀ = − dq p
ω = − dp dq

In our example,

$$q = \varphi(0)$$

$$p = \varphi'(0)$$

$$p = \frac{\partial q}{\partial t}$$

$$\frac{\partial p}{\partial t} = 0$$

$$\{H, q\} = p$$

$$\{H, p\} = 0$$

And, $\{p, q\} = -1$

$$H = -\frac{1}{2} p^2$$

Alternative way to derive $\{\ \}$ and hence ω:

Coordinates on phase space are operators in the system.

Eg, here, $\varphi(0)$, $\varphi'(0)$

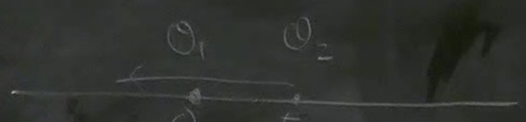
Gauge theory, measure $\mathrm{tr}\, F^2$, ...

Gravity, use probe lines going to spatial ∞

We need to give a Poisson bracket on operators

$\mathcal{O}_1, \mathcal{O}_2$ operators

$$\{\mathcal{O}_1, \mathcal{O}_2\} = \lim_{t \to 0_+} \mathcal{O}_1(0)\,\mathcal{O}_2(t) - \lim_{t \to 0_-} \mathcal{O}_1(0)\,\mathcal{O}_2(t)$$



Caution work "semiclassically"
(leading non-trivial term)

Example $\frac{1}{2}\int \partial_t \varphi \, \partial_t \varphi$

$$\delta S_{[0,1]} = \int_0^1 \partial_t \delta\varphi \; \partial_t \varphi$$

$$= \int_0^1 \partial_t \left(\delta\varphi \, \partial_t \varphi \right) \quad \text{as} \quad \partial_t^2 \varphi = 0$$

$$\alpha_0 = -\delta\varphi \, \partial_t \varphi \qquad \varphi(0) = q \quad \varphi'(0) = p$$

$P(t)$ satisfies

$$\frac{1}{2}\partial_t^2 P(t) = \delta_{t=0}$$

P is what we get by solving EOM with a source $\varphi(0)$

$$\frac{1}{2}\int (\partial_t\varphi)^2 + \varphi(0)$$

Vary φ, IBP,

$$\Rightarrow \quad \frac{1}{2}\partial_t^2 P(t) = \delta_{t=0}$$

) satisfies

t) $= \delta_{t=0}$

at we get by

m with a source

$\frac{1}{2}\int (\partial_t \varphi)^2 + \varphi(0)$

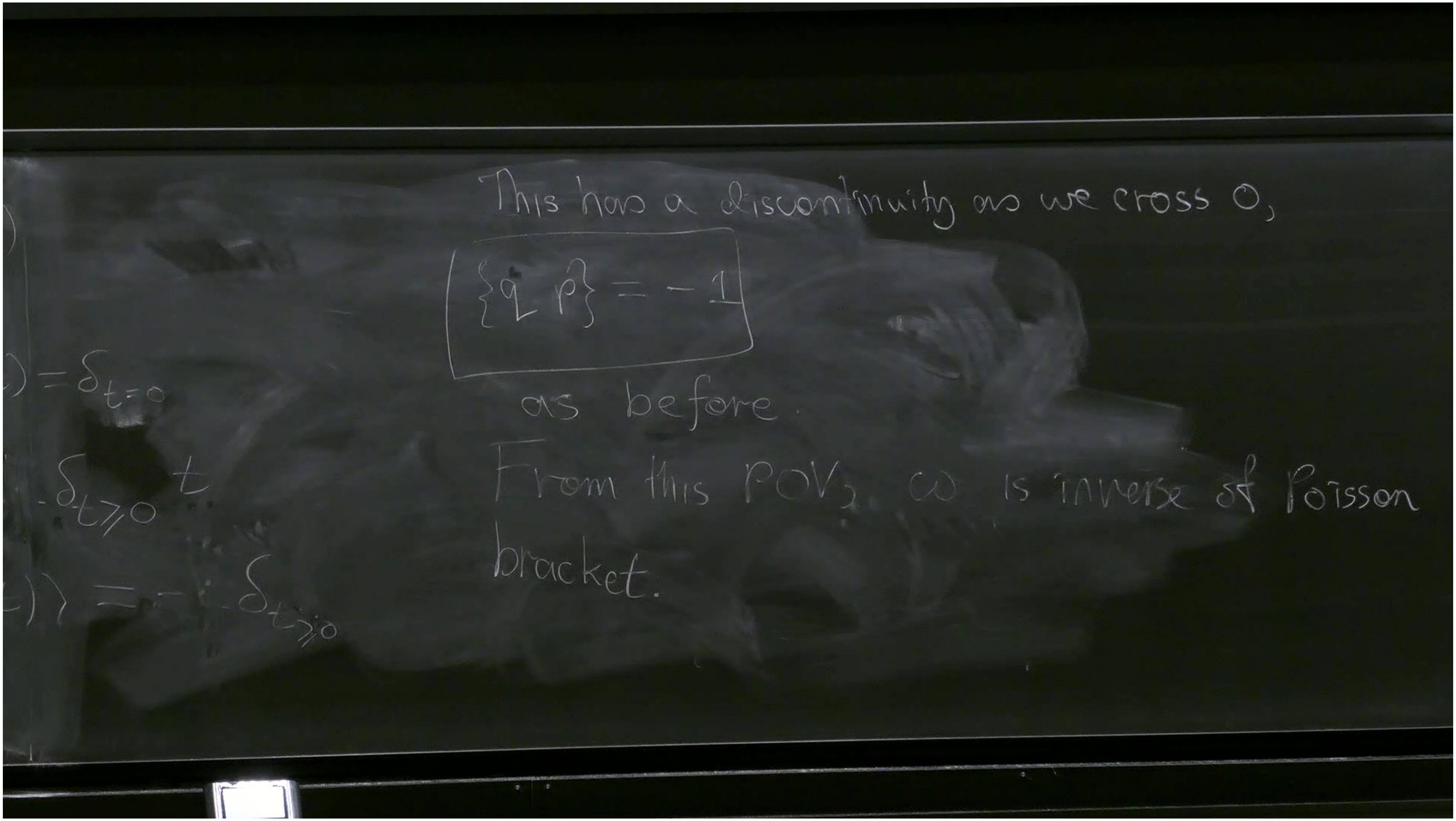
Vary φ, IBP,

$\Rightarrow \frac{1}{2}\partial_t^2 P(t) = \delta_{t=0}$

$P(t) = -2\,\delta_{t\geq 0}\, t$

$\langle \varphi(0)\, \varphi'(t) \rangle = -2\,\delta_{t\geq 0}$

This how


This has a discontinuity as we cross 0,
$\{q, p\} = -1$
as before.
From this POV, ω is inverse of Poisson bracket.

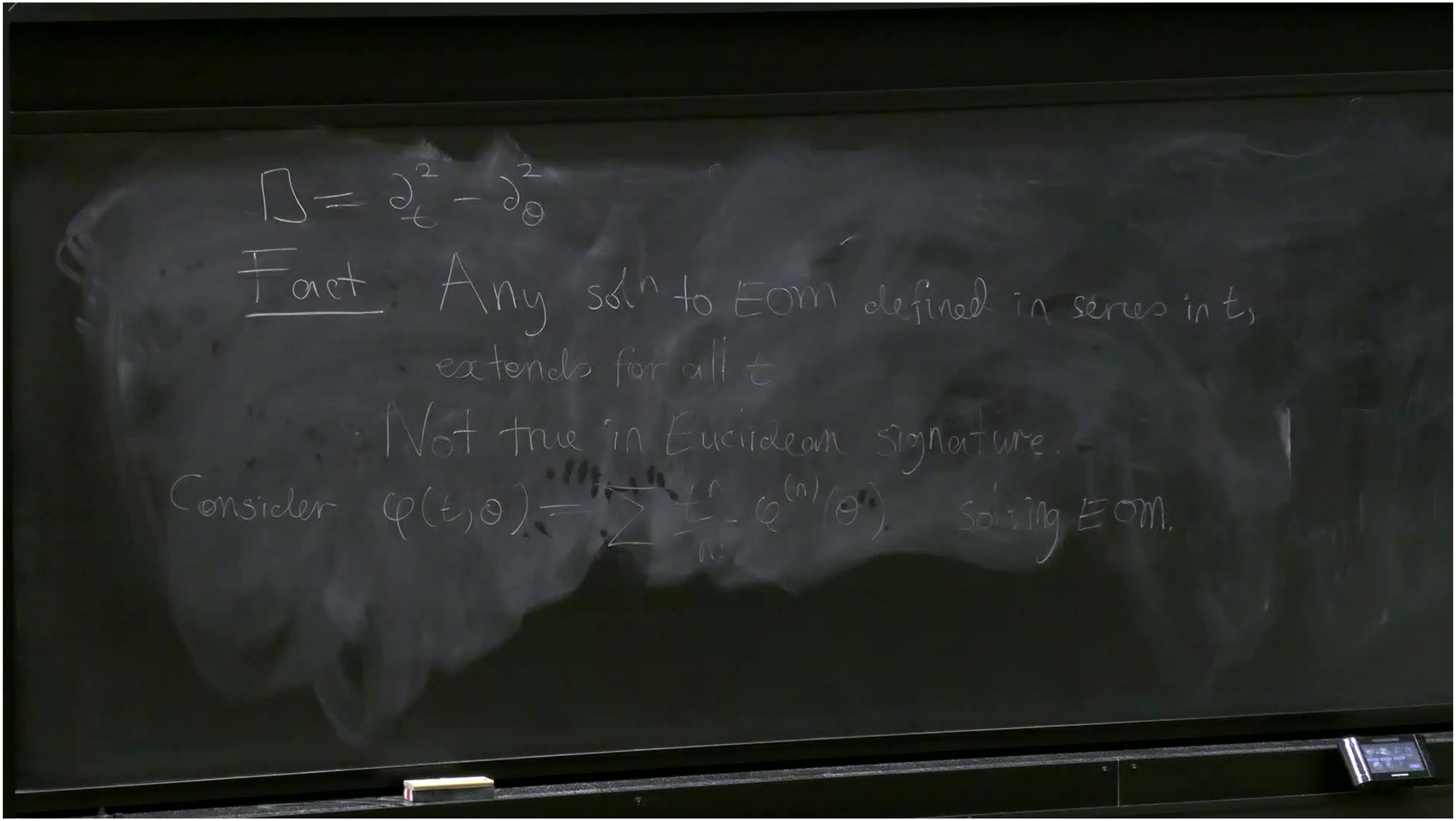
What is the phase space of a classical field theory?

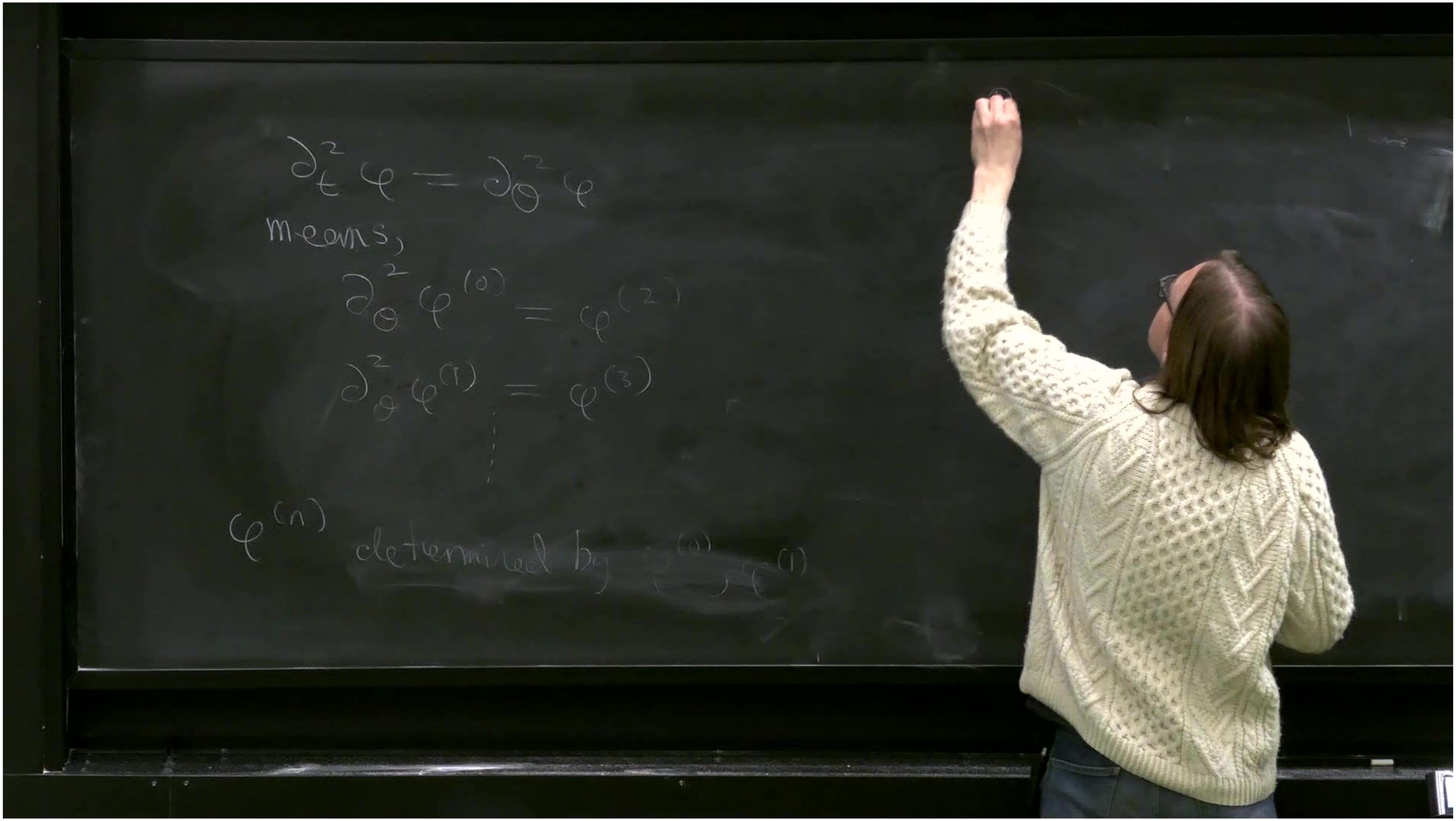
2d scalar field

$\varphi(t, \theta)$ Lagrangian is

$\theta \sim \theta + 1$

$$\int_{S^1 \times \mathbb{R}} \varphi \, \Box \, \varphi$$


$\Box = \partial_t^2 - \partial_\theta^2$
Fact: Any sol^n to EOM defined in series in t,
extends for all t
Not true in Euclidean signature.
Consider $\varphi(t,\theta) = \sum \frac{t^n}{n!} \varphi^{(n)}(\theta)$. solving EOM.


$\partial_t^2 \varphi = \partial_\theta^2 \varphi$
means,
$\partial_\theta^2 \varphi^{(0)} = \varphi^{(2)}$
$\partial_\theta^2 \varphi^{(1)} = \varphi^{(3)}$
$\varphi^{(n)}$ determined by $\varphi^{(0)}, \varphi^{(1)}$

$\partial_\theta^2 \varphi$

$= \varphi^{(2)}$

$= \varphi^{(3)}$

ned by $\varphi^{(0)}, \varphi^{(1)}$

Phase space has coordinates

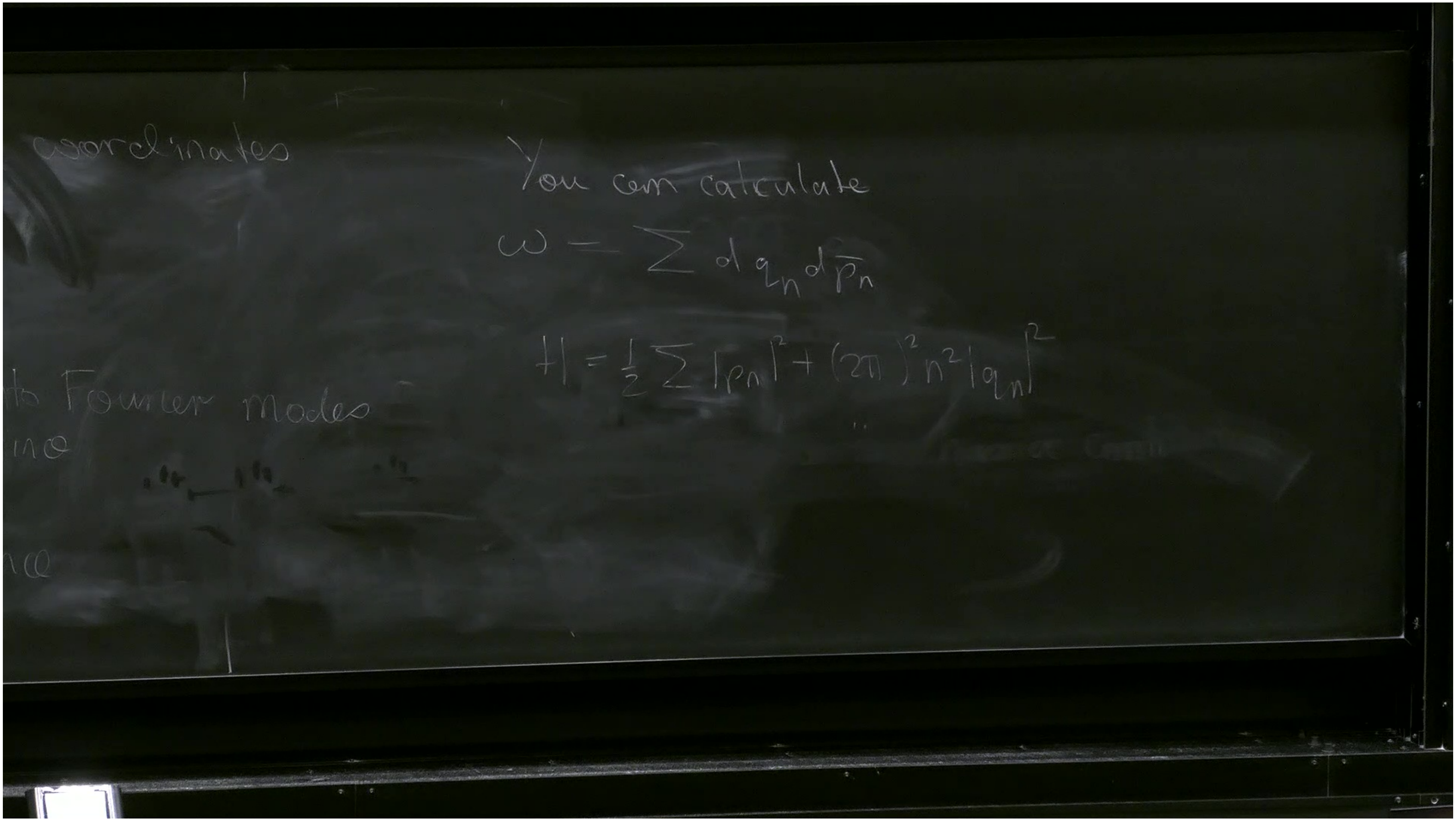
$\varphi_{\parallel}^{(0)}, \varphi^{(1)}$

$q(\theta) \quad p(\theta)$

Expand p, q into Fourier modes

$$q = \sum_n q_n e^{2\pi i n\theta}$$

$$p = \sum p_n e^{2\pi i n\theta}$$


coordinates
You can calculate
$\omega = \sum dq_n d\bar{p}_n$
$H = \frac{1}{2} \sum |p_n|^2 + (2\pi)^2 n^2 |q_n|^2$
Fourier modes