Title: Mathematical Physics Lecture

Speakers: Kevin Costello

Collection: Mathematical Physics 2023/24

Date: April 05, 2024 - 11:30 AM

URL: https://pirsa.org/24040059

Geometric quantization to

$\mathbb{R}\times S^1 \longrightarrow \mathbb{R}$

$p\,,\,q$

$q \sim q+1$

$\omega = dp\,dq$

$\alpha = p\,dq$

We saw that the Hilber

...ntization to

We saw that the Hilbert space has a basis given by those $p \in \mathbb{R}$ where

$$e^{2\pi i \frac{1}{\hbar}\int_{q=0}^{1} p\,dq} = 1$$

$$\int_0^1 p\,dq = n\hbar \quad n \text{ an integer}$$

$$p = n\hbar$$

We saw that the Hilbert space has a basis given by those $p \in \mathbb{R}$ where

$$e^{2\pi i \frac{1}{\hbar}\int_{q=0}^{1} p\,dq} = 1$$

$$\int_0^1 p\,dq = n\hbar \quad n \text{ an integer}$$

$$p = n\hbar$$

$$Q = e^{2\pi i q}$$

$$p \Rightarrow \frac{\hbar}{2\pi i}\frac{\partial}{\partial q} = \frac{\hbar}{2\pi i} Q \frac{\partial}{\partial Q}$$

$$p \cdot Q^n = \hbar n Q^n$$

Q^n eigenvalues of p

State where $p = n\hbar$ is also an eigenvalue of p!

$$Q^n \Longleftrightarrow \delta_{p=n\hbar}$$

under Fourier transform

$$Q\, Q^n = Q^{n+1}$$

What is $Q\, \delta_{p=n\hbar}$?

$$Q = e^{2\pi i q}$$

$$[p, q] = \frac{\hbar}{2\pi i}$$

$$q = -\frac{\hbar}{2\pi i}\frac{\partial}{\partial p}$$

$$Q = e^{-\hbar \frac{\partial}{\partial p}}$$

$$Q \cdot p = p - \hbar$$

$$Q \cdot \delta_{p=n\hbar} = \delta_{p-\hbar=n\hbar} = \delta_{p=(n+1}$$

$= \delta_{p=(n+1)\hbar}$

If $M = S^1 \times S^1$ $\quad p \sim p+1 \quad \omega = c\, dp\, dq$

$\qquad\qquad\qquad\qquad q \sim q+1$

what happens?

$c = \text{Area}$

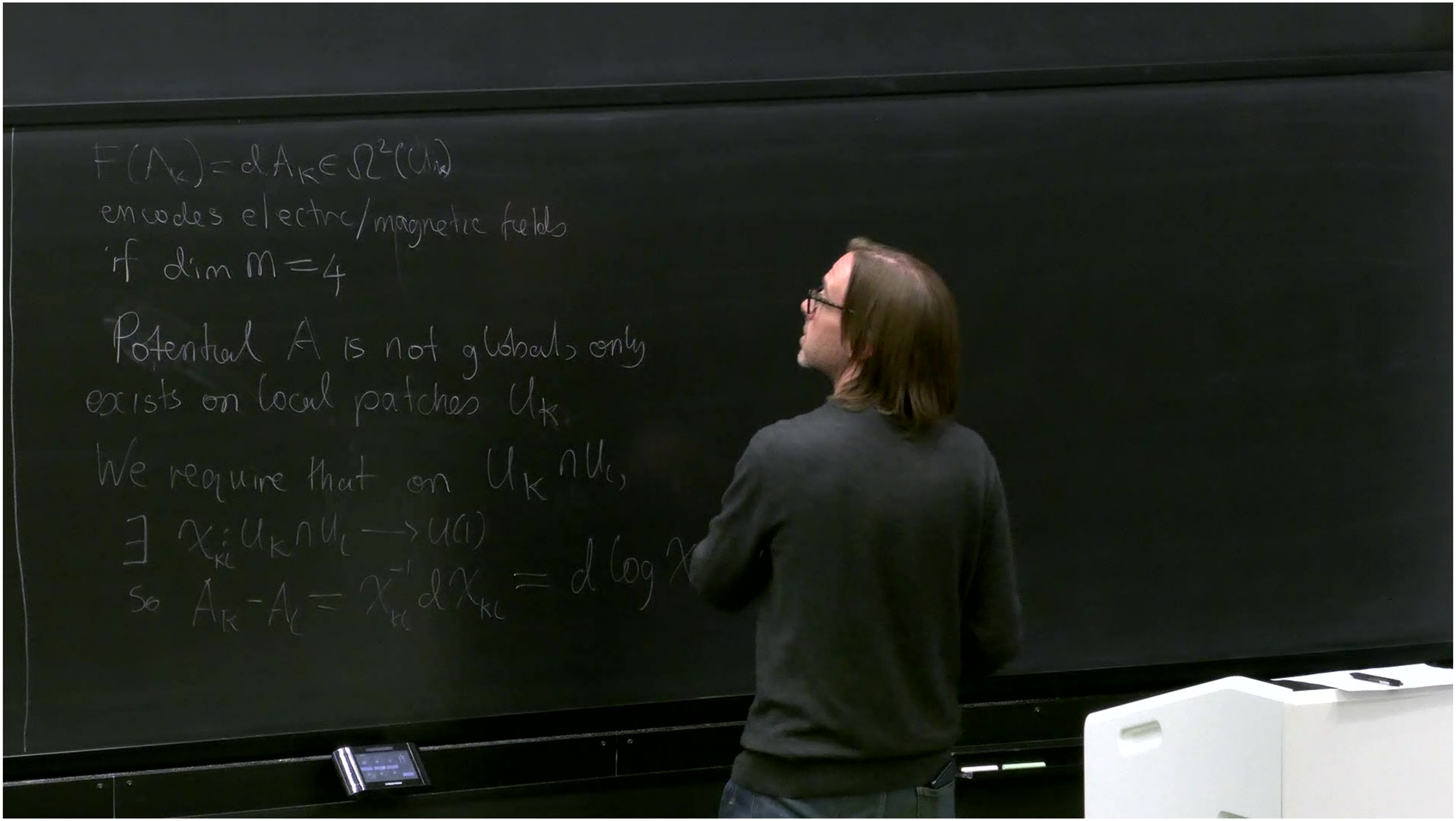
<u>Problem</u> $c \int p\, dq$ is not globally defined - it has a branch cut as p goes from 1 to 0.

<u>Sol^n</u> Ask that $\omega = \dfrac{F(A)}{2\pi}$ where A is a $U(1)$ gauge field. Locally, $\omega = \dfrac{dA}{2\pi}$ but not globally

What is a $U(1)$ gauge field?

M any manifold, covered by opens U_i, to give a $U(1)$ gauge field is to give on each open, a 1-form $A \in \sqrt{-1}\,\Omega^1(U_k)$

Electromagnetic potential in Maxwell theory.


$F(A_k) = dA_k \in \Omega^2(U_k)$
encodes electric/magnetic fields
if dim $M = 4$
Potential A is not global, only
exists on local patches U_k
We require that on $U_k \cap U_l$,
$\exists\ \chi_{kl} : U_k \cap U_l \longrightarrow U(1)$
So $A_k - A_l = \chi_{kl}^{-1} d\chi_{kl} = d \log \chi$

χ_{kl} = gauge transformation, gluing gauge fields together

This implies $F(A_k) = F(A_l)$

as $d(d \log \chi_{kl}) = 0$

Also require $\chi_{lk} = \chi_{kl}^{-1}$

$$\chi_{kl}\,\chi_{lm} = \chi_{km}$$

<u>Example</u> (mo

$$M = S^2$$

Example (magnetic monopole)

$M = \mathbb{R}^3 \setminus 0$

Let $F = \sqrt{-1}\, \dfrac{\varepsilon^{ijk} x_i\, dx_j\, dx_k}{\|x\|^3}$

Then, it turns out F satisfies Maxwells eq^{n}s.

But, $\displaystyle\int_{\|x\|=1} F \neq 0$. So, $F \neq dA$ globally.

Useful fact

On any M, if $F \in \Omega^2(M)$

so that $dF = 0$

and $\int_\Sigma F = 2\pi i \times \text{integer}$

Σ is a closed surface in M

$\Rightarrow F$ is the field strength of

$U(1)$ gauge field

$p = n\hbar$

Useful fact

On any M, if $F \in \Omega^2(M)$
so that $dF = 0$
and $\int_\Sigma F = 2\pi i \times \text{integer}$

Σ is a closed surface in M

$\Rightarrow F$ is the field strength of

U(1) gauge field

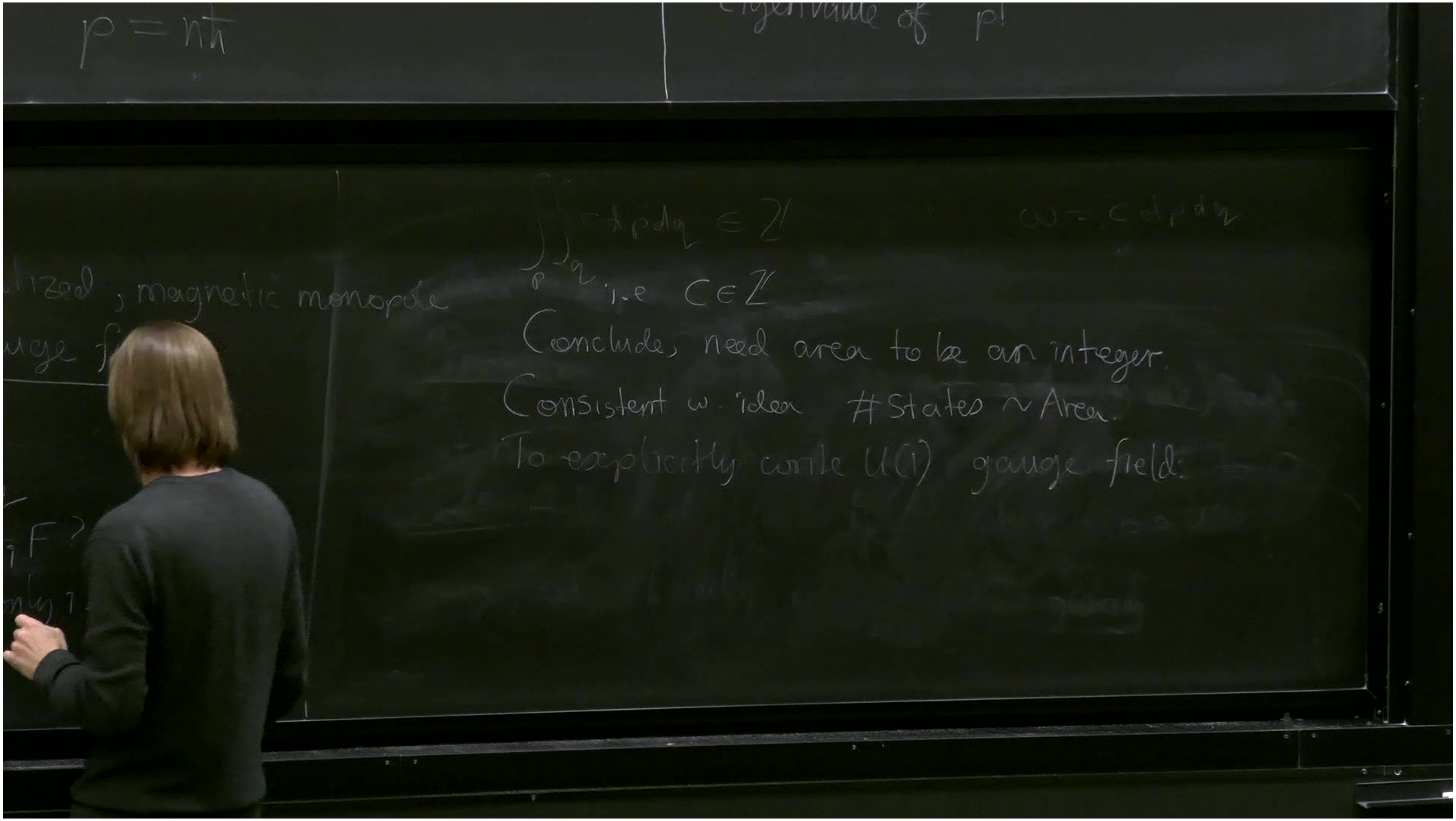
Correctly normalized, magnetic monopole is a U(1) gauge field.

$M = S^1 \times S^1$

$\omega = c\, dp\, dq$

When is this $\frac{1}{2\pi i} F$?

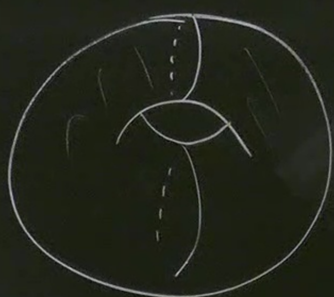
Ans If and only if


p = nħ
eigenvalue of p!
ω = c dp dq
i.e c ∈ ℤ
Conclude, need area to be an integer.
Consistent w. idea # States ~ Area.
To explicitly write U(1) gauge field:
lized , magnetic monopole

2 patches on $S^1 \times S^1$:

$U = \{0 < p < 1\}$

$V = \{p \neq \frac{1}{2}\}$



$U \cap V =$ 

$A_U = 2\pi i \, p \, dq$ on U

$A_V = 2\pi i(\tilde{p}) \, dq$ on V

$\tilde{p}$ goes from $\frac{1}{2}$ to $\frac{3}{2}$

$A_U - A_V = 2\pi i \, dq$ on region $(0, \frac{1}{2})$

p

$\tfrac{1}{2}$ $\tilde{p}$ $\tfrac{3}{2}$

$$2\pi i \, dq = \chi^{-1} d\chi$$

where $\chi = e^{2\pi i q}$

χ valued in $U(1)$.

Bohr-Sommerfeld

M symplectic

$M \xrightarrow{\pi} N$ is a Lagrangian fibration

If $n \in N$, $\pi^{-1}(n)$ is the fibre

$\omega = \frac{F}{2\pi i}$, Field strength of $U(1)$ gauge field.

BS condition is the following.

- If we assume $F = dA$ on $\pi^{-1}(n)$

then, $\int_{S^1} A \in 2\pi i \mathbb{Z}$

$\forall S^1$ in $\pi^{-1}(n)$

If $A \to x^{-1} dx = d\log x$

then $\int_{S^1} A \to \int d\log x \in 2\pi i \mathbb{Z}$

because $\log \chi$ has branch cuts
where jump by $2\pi i \mathbb{Z}$

Fancy:

$\exp\left(\int_{S^1} A\right)$ is called the Wilson loop.

We've just seen this is gauge invariant.

$M = S^1 \times S^1$

$\omega = n\, dp\, dq$

$n \in \mathbb{Z}$

Polarize by $S^1 \times S^1 \to S^1$

$(p, q) \longrightarrow q$

Hilbert space is spanned by those q's which are BS.

orbits

$A = -q\, dp \cdot n \cdot 2\pi i$

$\int_p A \in 2\pi i \cdot \mathbb{Z}$

$\int_p A = -2\pi i\, q \cdot n$

So, $q \in \frac{1}{n}\mathbb{Z}$

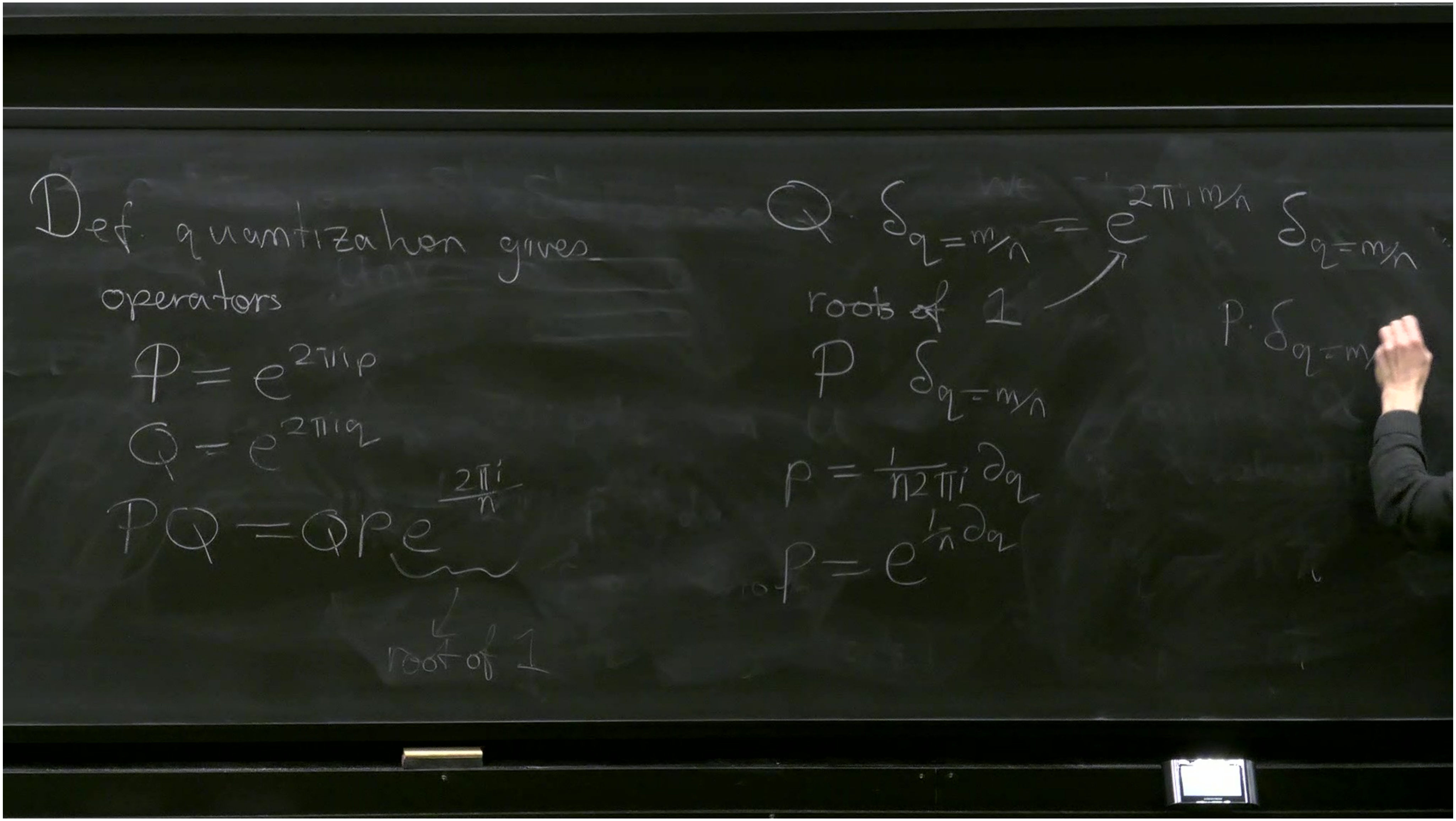
$n \cdot 2\pi i$

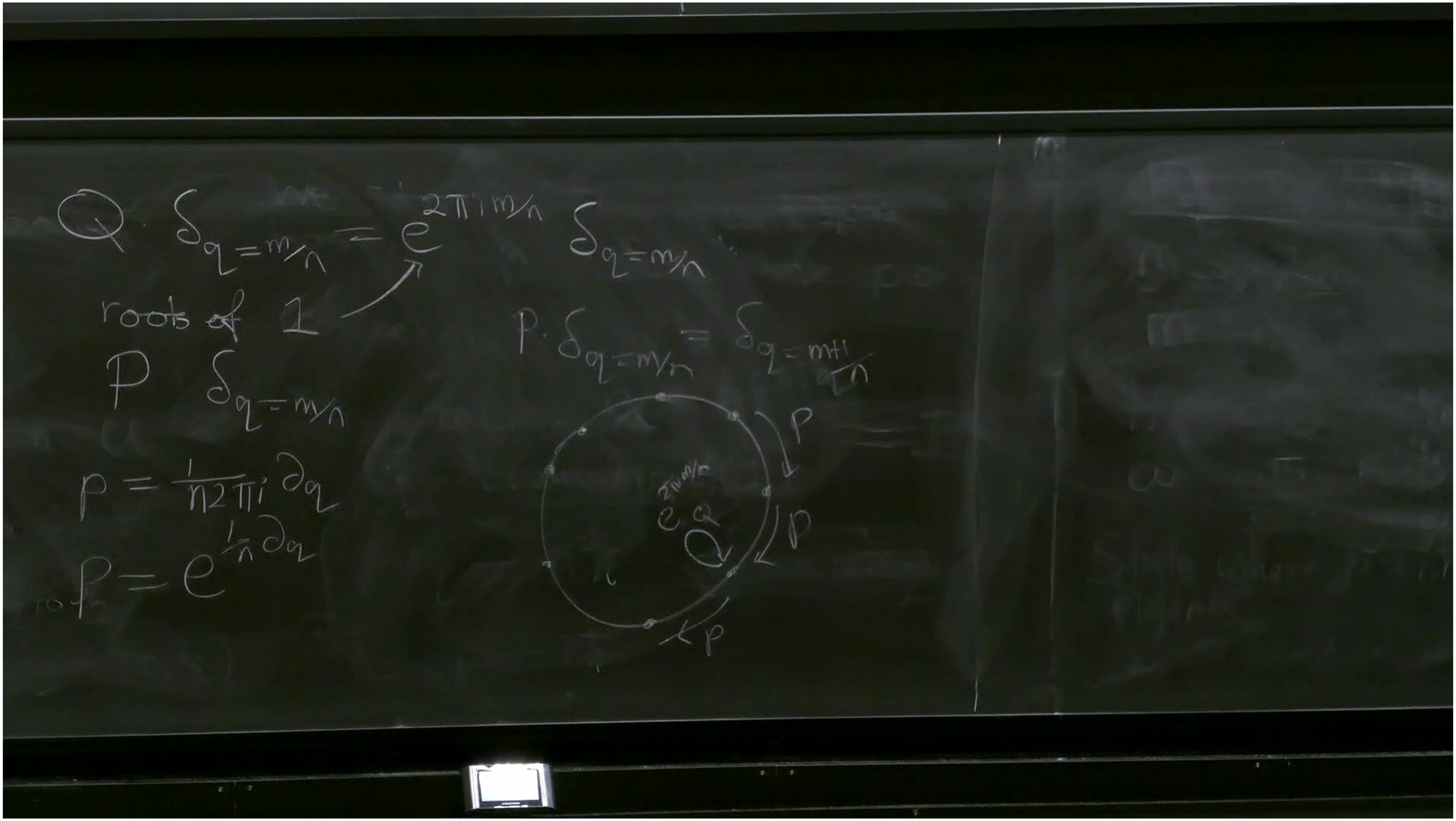
n

i.e. $q \in \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\}$

Conclude Hilbert space is of dim^{n} n, spanned by

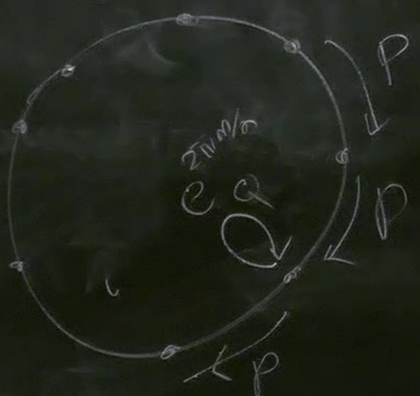
$\delta_{q=m/n}$ $\quad m=0,\dots,n-1$


Def. quantization gives operators
$P = e^{2\pi i p}$
$Q = e^{2\pi i q}$
$PQ = QP\underbrace{e^{\frac{2\pi i}{n}}}_{\text{root of } 1}$
$Q \cdot \delta_{q=m/n} = e^{2\pi i m/n}\,\delta_{q=m/n}$
roots of 1
$P \quad \delta_{q=m/n}$
$p = \frac{1}{n 2\pi i}\partial_q$
$P = e^{\frac{1}{n}\partial_q}$
$P \cdot \delta_{q=m/}$


$Q \cdot \delta_{q=m/n} = e^{2\pi i m/n} \delta_{q=m/n}$
roots of 1
$P \quad \delta_{q=m/n}$
$p = \frac{1}{n 2\pi i} \partial_q$
$P = e^{\frac{1}{n}\partial_q}$
$P \cdot \delta_{q=m/n} = \delta_{q=\frac{m+1}{n}}$
P
$e^{2\pi i m/n}$
Q
P
P

$\ldots\pi i\, m/n \quad \delta_{q=m/n}$

$$P\cdot\delta_{q=m/n} = \delta_{q=\frac{m+1}{n}}$$



$$PQ\,\delta_{q=m/n} = \delta_{q=\frac{m+1}{n}}\, e^{2\pi i m/n}$$

$$QP\,\delta_{q=m/n} = \delta_{q=\frac{m+1}{n}}\, e^{2\pi i (m+1)/n}$$

$$PQ = QP\,e^{-2\pi i/n}$$