Title: Mathematical Physics Lecture

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2d scalar

$$\int_{S^1 \times \mathbb{R}} \varphi \left(\partial_t^2 - \partial_\theta^2\right) \varphi$$

$\theta \sim \theta + 1$

Operators

$$q_n = \int_{t=0,\,\theta=0}^{1} e^{-2\pi i n \theta}\, \varphi(\theta, 0)$$

$P_n =$

Wan

$= \int e^{-2\pi i n \theta} \partial_t \varphi(\theta, 0)$

ant to show

$\{ q_n, p_m \} = \delta_{n+m}$

Technique

$\lim_{t \to 0^+} q_n(0)\, p_m(t)$

$- \lim_{t \to 0^-} q_n(0)\, p_m(t) = \{ q_n, p_m \}$

Sourced

Use propagator with one leg on $q_n(0)$

one leg on $p_m(t)$

$q_n(0)$ $p_m(t)$

The propagator with one leg on $q_n(0)$ is the field sourced by the operator $q_n(0)$

Solve the EOM for Lagrangian with source term

$$\int_{t,\theta} \frac{1}{2}\varphi(\partial_t^2 - \partial_\theta^2)\varphi - \int_{t=0,\theta} e^{-2\pi i n\theta}\, \varphi(\theta,0)$$

$q_n(0)p_m(t)$: measuring $p_m(t)$ of field sourced by $q_n(0)$

$-\delta_{t\geq 0}\, t\, e^{-2\pi i n\theta}\left(1\right.$

Source eq^n is

$$\left(\partial_t^2-\partial_\theta^2\right)\varphi(\theta,t)=\delta_{t=0}\,e^{-2\pi i n\theta}$$

Sol^n: 1^{st} Approx

$$\varphi(\theta,t)=-\delta_{t\geq 0}\, t\, e^{-2\pi i n\theta}$$

Correct by: $\delta_{t\geq 0}\dfrac{t^3}{6}(2\pi i n\theta)^2 e^{-2\pi i n\theta}$

$+ \ldots\ldots$

$$\lim_{t\to 0^+} q_n(0)\, p_m(t) = \delta_{n+m}$$

$$\lim_{t\to 0^-} q_n(0)\, p_m(t) = 0$$

$$\{q_n, p_m\} = \delta_{n+m}$$

Note

$\{q_n, \partial_t^k q_n\}$ can be complicated!!

Not surprising: $\partial_t = \{H, -\}$ and $\{H\{H,q\}\}$ can be complicated.

$$\lim_{t\to 0^+} q_n(0)\, p_m(t) = \delta_{n+m}$$

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What is H?

$$\{H, q_n(0)\} = \partial_t q_n(0) = p_n(0)$$

$$\{H, p_n(0)\} = \partial_t p_n(0) = \partial_t^2 q_n(0)$$

$$= \partial_\theta^2 q_n = (2\pi i n)^2 q_n$$

$$H = -\frac{1}{2}\sum_n p_n p_{-n} + \frac{1}{2}\sum_n q_n q_{-n} (2\pi i n)^2$$

$$= -\frac{1}{2} p_0^2 - \sum_{n>0} |p_n|^2 - \sum_{n>0} (2\pi n)^2 |q_n|^2$$

Canonical commutation

$$[\varphi(\theta), \partial_t \varphi(\theta')] = \delta_{\theta-\theta'}$$

$\varphi(\theta)$ operator measures φ at $(\theta, 0)$ $\longrightarrow$ $\int_{\theta'} \delta_{\theta=\theta'} d\theta \; \varphi(\theta, 0)$

$q_n(0)$ measures n^{th} fourier mode

$$\int e^{-2\pi i n\theta'} \varphi(\theta', 0)\, d\theta'$$

$$\delta_{\theta=\theta'} = \sum_n e^{2\pi i n(\theta-\theta')}$$

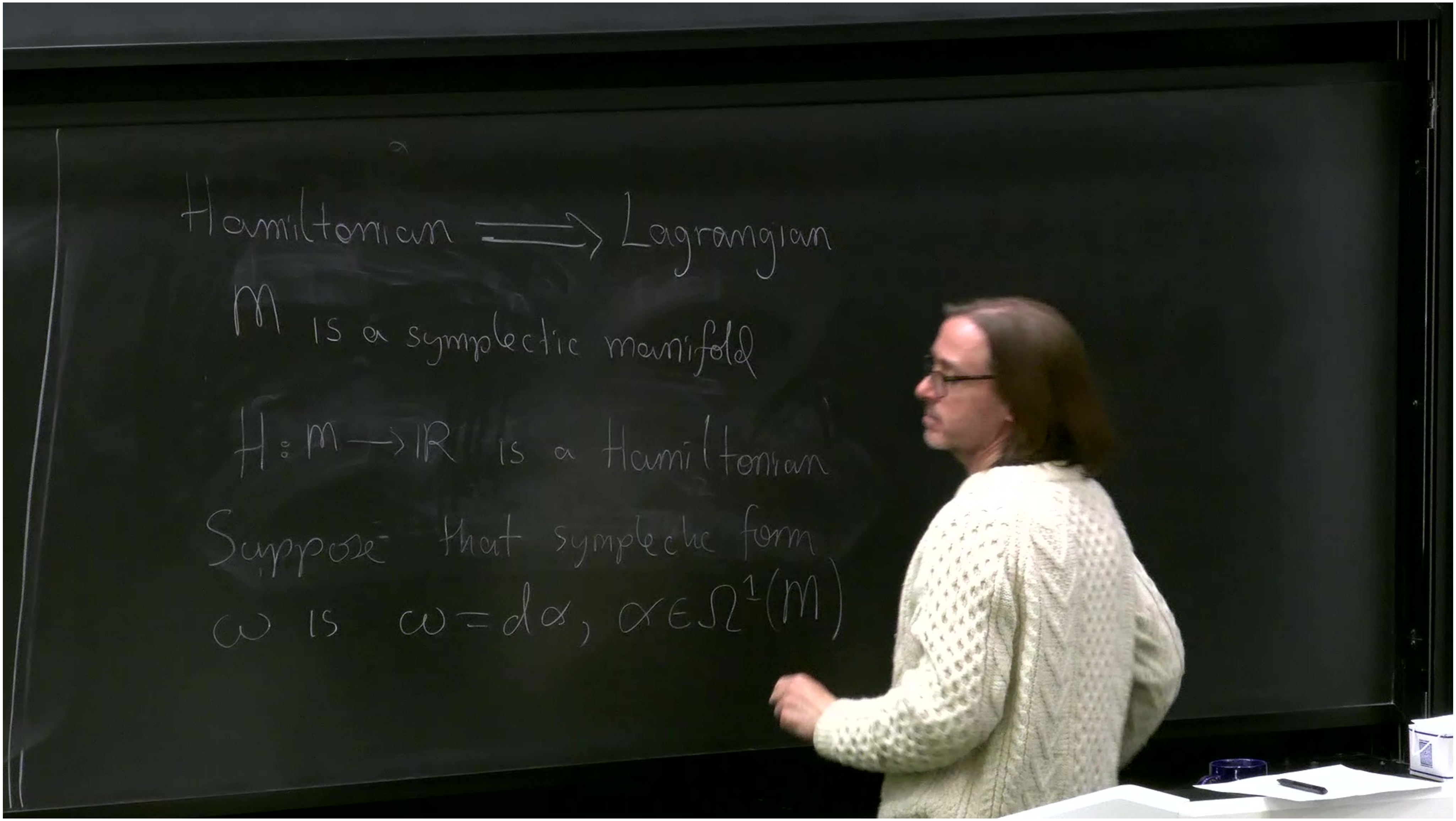
$\Rightarrow$ $\varphi(\theta)$ is the sum $\sum_n e^{2\pi i n\theta} q_n$

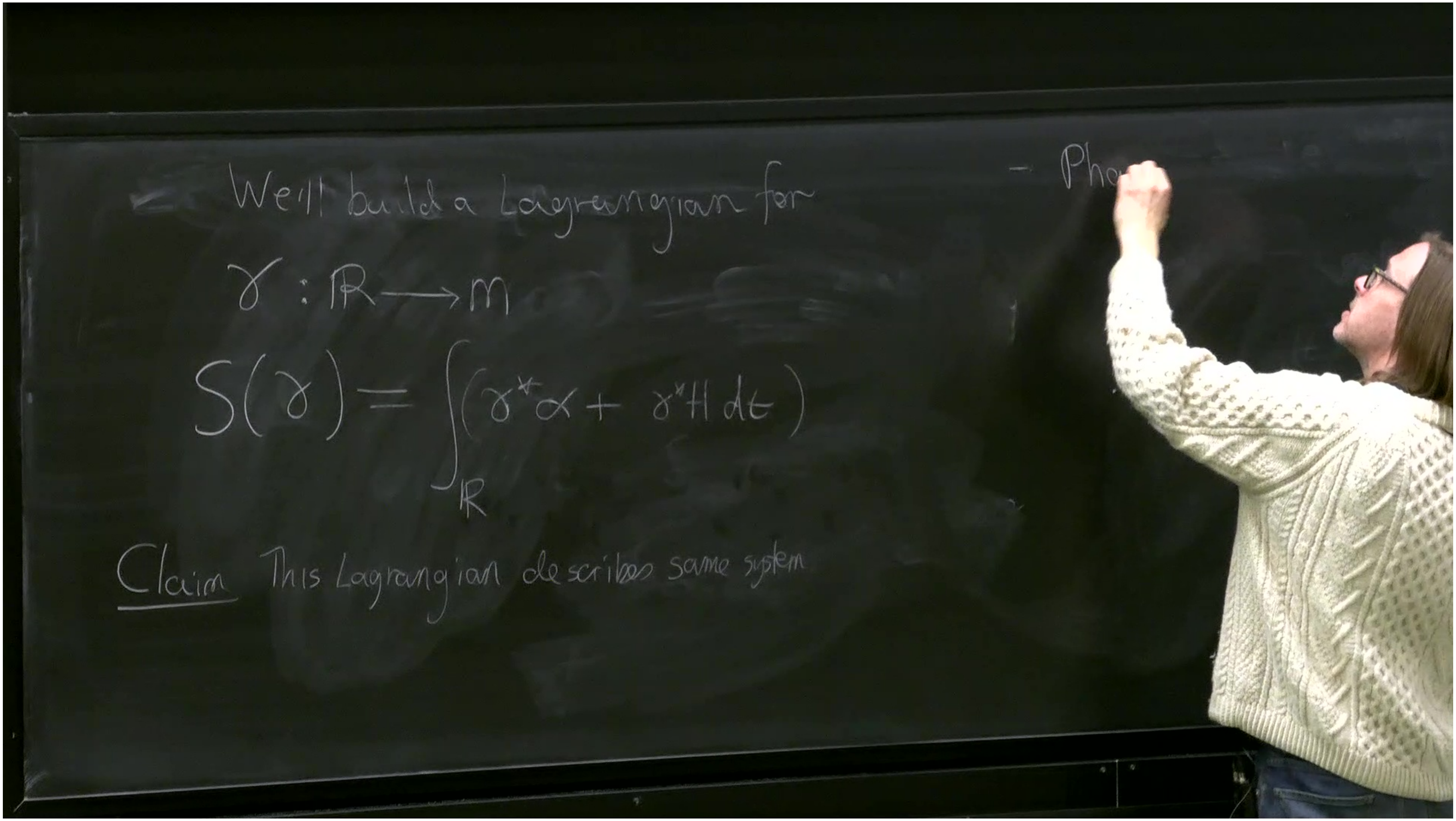
$$\partial_t \varphi(\theta) = \sum e^{2\pi i n\theta} p_n$$

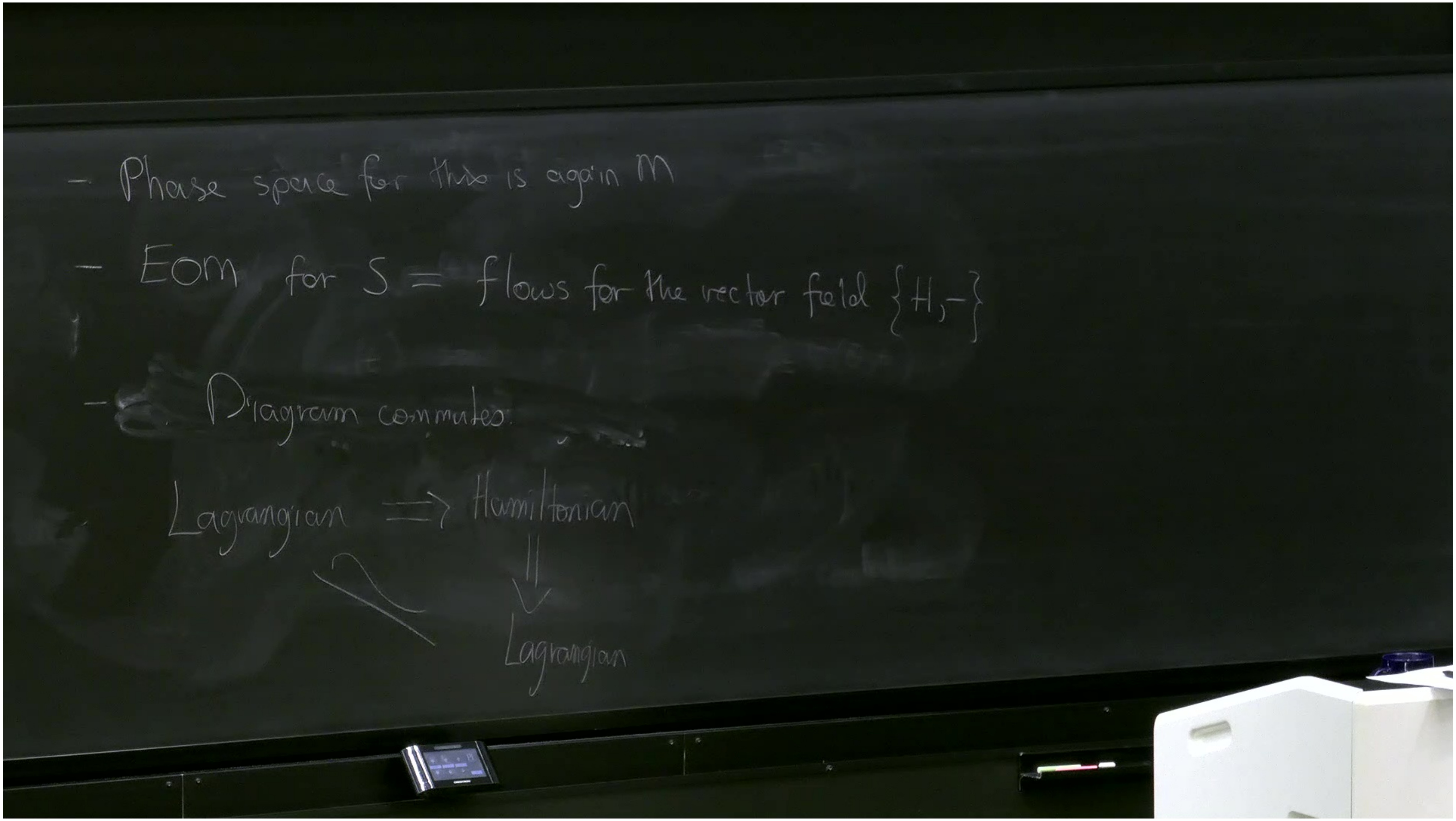
$$\{\varphi(\theta), \partial_t \varphi(\theta')\} = \sum_{n,m} e^{2\pi i(n\theta + m\theta')} \{q_n, p_m\}$$

$$= \sum_n e^{2\pi i n(\theta - \theta')}$$

$$= \delta_{\theta = \theta'}$$


Hamiltonian ⟹ Lagrangian
M is a symplectic manifold
H: M → ℝ is a Hamiltonian
Suppose that symplectic form
ω is ω = dα, α ∈ Ω¹(M)


We'll build a Lagrangian for
$\gamma : \mathbb{R} \longrightarrow M$
$S(\gamma) = \int_{\mathbb{R}} (\gamma^* \alpha + \gamma^* H \, dt)$
Claim This Lagrangian describes same system


- Phase space for this is again M
- EOM for S = flows for the vector field {H,-}
- Diagram commutes:
Lagrangian ⟹ Hamiltonian
Lagrangian

In a patch of M where $\omega = dp_i \wedge dq_i$

$$\alpha = p_i \wedge dq_i$$

$\gamma(t)$ is then a time-dependent p_i, q_i

$$\gamma(t) = (p_i(t), q_i(t))$$

Lagrangian is

$$\int_{\mathbb{R}} p_i(t) \frac{\partial}{\partial t} q_i(t)\, dt + H(p_i(t), q_i(t))\, dt$$

EOM: vary p_i, we find

$$\frac{\partial q_i}{\partial t} + \frac{\partial H}{\partial p_i}(p_i(t), q_i(t)) = 0$$

Vary q_i:

$$-\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q_i}(p, q) = 0$$

Recall the vector field

$\{H, -\}$ is $\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$

Component pointing in q_i direction is $\frac{\partial H}{\partial p_i}$, ---- p_i direction is $-\frac{\partial H}{\partial q_i}$

$\mathbb{R}$

$\frac{\partial q_i}{\partial t}$: component of tangent vector to path points in q_i direction

$= -\frac{\partial H}{\partial p_i}$ = component of $\{H, -\}$

This is a first order Lagrangian

$\Longrightarrow$ Cauchy data is just the initial value, no derivatives

Phase space $= M$

Also, $\{p_i, q_j\} = \delta_{ij}$ (in these coordinates)

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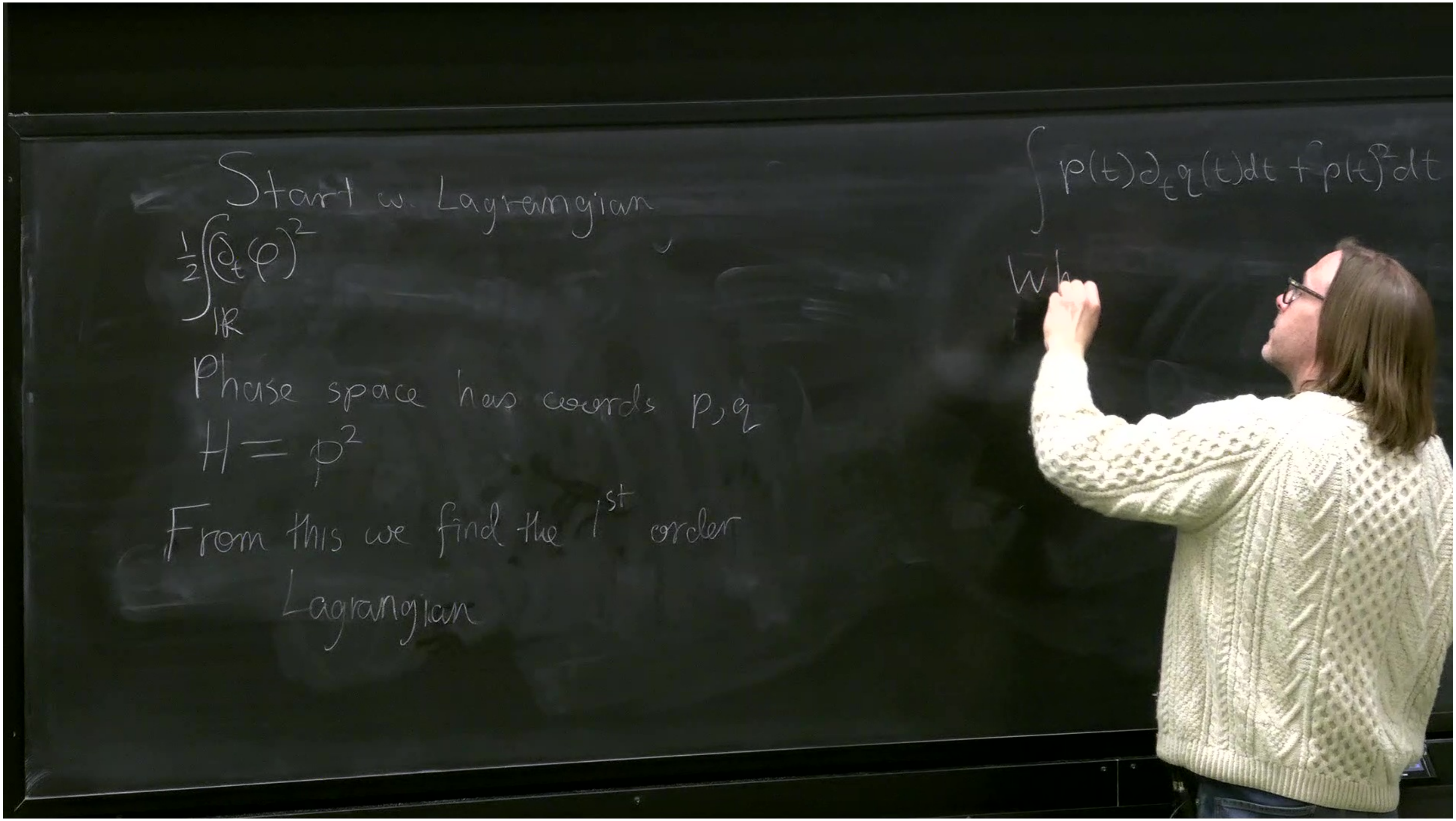
Phase space $= \mathcal{M}$

Also, $\{p_i, q_j\} = \delta_{ij}$ (in these coordinates)

Forget H for now. The field sourced by $p_i(0)$

satisfies $\frac{\partial}{\partial t} q_i(t) = \delta_{t=0}$, so it is $q_i(t) = \delta_{t \geq 0}$


Start w. Lagrangian
$\frac{1}{2}\int_{\mathbb{R}} (\partial_t \varphi)^2$
Phase space has coords p, q
$H = p^2$
From this we find the 1st order Lagrangian
$\int p(t)\partial_t q(t)dt + \int p(t)^2 dt$
Wh

$$\int p(t)\partial_t q(t)dt + \int p(t)^2 dt$$

Why is this the same as 2nd order Lagrangian?

Ans: Can integrate out p.

$$\lambda(t) = p(t) + \tfrac{1}{2}\partial_t q(t)$$

Lagrangian becomes

$$\int \left(\lambda(t) - \tfrac{1}{2}\partial_t q\right)^2 + \left(\lambda - \tfrac{1}{2}\partial_t q\right)\partial_t q$$

$$= \int \lambda^2 - \tfrac{1}{4}(\partial_t q)^2$$

Now λ does not propagate, or interact; $\int$ it out leaving

$$-\tfrac{1}{4}\int (\partial_t q)^2$$