Title: Quantum Gravity Lecture

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Collection: Quantum Gravity 2023/24

Date: April 17, 2024 - 9:00 AM

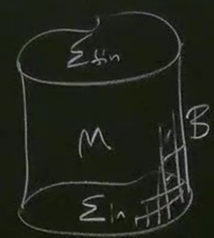
URL: https://pirsa.org/24040027

RECAP (Noether 1)

$$\mathbb{d}\underline{L} = \underline{E} - d\underline{\Theta}$$

$$\underline{L} = L\,\underline{\epsilon}, \quad \mathbb{d}L = E + \nabla_a \tilde{\Theta}^a$$

$$\underline{\Theta} = \tilde{\Theta}^a \underline{\epsilon}_a$$



$$\rho : \mathfrak{g} \longrightarrow \mathfrak{X}^1(\mathcal{F})$$ Lie alg. homomorph.

Lagrangian sym if $\mathbb{L}_{\rho(\xi)}\underline{L} = d\underline{R}(\xi)$

"invariance up to boundary"

$$\underline{\Omega} = \mathbb{d}\underline{\Theta}$$

- $d\underline{\Omega} \approx 0$

$$\Omega_{fin} - \Omega_{in} \approx \int_B \underline{\Omega} \underset{\text{if } \partial\Sigma = \phi}{=} 0$$

- $d\underline{J}(\xi) \approx 0$ $\quad \underline{J}(\xi) := i_{\rho(\xi)}\underline{\Theta} - \underline{R}(\xi)$

Noether current.

$$\boxed{d\underline{J}(\xi) = -(\delta_\xi \varphi^I)\underline{E}_I} \quad (N1)$$

$$Q_{fin}(\xi) - Q_{in}(\xi) \approx \int_B \underline{J}(\xi) \underset{\text{if } \partial\Sigma = \phi}{=} 0$$

$$\bullet \quad \mathbb{i}_{\rho(\xi)} \underline{\Omega} = -\mathbb{d}\,\underline{J}(\xi) + \underline{S}(\xi) \qquad (*)$$

$$\underline{S}(\xi) := \mathbb{i}_{\rho(\xi)}\underline{\Omega} + \mathbb{d}\underline{J}(\xi) \equiv \mathbb{L}_{\rho(\xi)}\underline{\Theta} - \mathbb{d}\underline{R}(\xi)$$

$\uparrow$ obstruction for $\underline{J}$ to "look like" a momentum map.

Thm $\underline{S}(\xi) \simeq d\underline{r}(\xi)$, $\underline{r} : \mathfrak{g} \to \Omega^{\text{top}-2,1}(M \times \mathcal{F})$

Obstruction is a boundary term (corner)

Cor: integrate $(*)$ over $\Sigma \hookrightarrow M$, $\underline{\partial\Sigma = \phi}$: $\boxed{\mathbb{i}_{\rho(\xi)}\Omega_\Sigma \simeq -\mathbb{d}Q_\Sigma(\xi)}$

In absence of corners: N-charge generates the sym in the covariant ph. space (on-shell!)

recall $(\bar{\mathcal{F}}, \iota^*_{EL}\Omega_\Sigma)$ covariant ph. space

- $\mathbb{I}^{\circ}_{\rho(\xi)} \underline{\Omega} = - \mathbb{d} \underline{J}(\xi) + \underline{S}(\xi) \qquad (*)$

$\underline{S}(\xi) := \mathbb{I}_{\rho(\xi)} \Omega + \mathbb{d} J(\xi) \equiv \mathbb{L}_{\rho(\xi)} \underline{\Theta} - \mathbb{d} \underline{R}(\xi)$

$\uparrow$ obstruction for $\underline{J}$ to "look like" a momentum map.

$\underline{\text{Thm}}$ $\quad \underline{s}(\xi) \simeq d\underline{r}(\xi)$, $\underline{r} : \mathfrak{g} \to \Omega^{top-2, 1}(M \times \mathcal{F})$

Obstruction is a boundary term (corner)

$\underline{\text{Cor}}$: integrate $(*)$ over $\Sigma \hookrightarrow M$, $\underline{\partial \Sigma = \phi}$: $\boxed{\mathbb{I}^{\circ}_{\rho(\xi)} \Omega_\Sigma \simeq - \mathbb{d} Q_\Sigma(\xi)}$

In absence of corners: N-charge generates the sym in the covariant ph. space (on-shell!)

recall $(\overline{\mathcal{F}}, i^*_{\Sigma} \Omega_\Sigma)$ covariant ph. space

On top of this, if $\partial\Sigma = \phi$

- Ω_Σ, Q_Σ indep of Σ
- — " — unambiguosly determined by $[\underline{L}] = [\underline{L} + d\underline{\ell}]$

Ex (spacetime symms of a scalar field)

$$\mathcal{L} = -\frac{1}{2}\nabla_a\varphi\nabla^a\varphi$$

$$\rho(\chi)\varphi = \chi^a\nabla_a\varphi\,,\quad \mathcal{L}_\chi g_{ab} = 0$$

killing $\leftrightarrow \nabla_a\chi_b + \nabla_b\chi_a = 0$

$$\begin{cases}\tilde{\Theta}^a = -d\varphi\,\nabla^a\varphi\\ E = \Box\varphi\end{cases}$$

killing

$$\mathbb{L}_{\rho(\chi)}\mathcal{L} = \dots = \nabla_a\tilde{R}^a(\chi)\,,\quad \tilde{R}^a(\chi) = -\frac{1}{2}\chi^a(\nabla\varphi)^2 = \chi^a\mathcal{L} \neq 0$$

Stress energy t.

$$\tilde{J}^a = \mathbb{I}_{\rho(\chi)}\tilde{\Theta}^a - \tilde{R}^a(\chi) = \dots = \chi^b t_b{}^a\,,\quad t_b{}^a = -\nabla_b\varphi\nabla^a\varphi + \frac{1}{2}\delta_b^a(\nabla\varphi)^2$$

$$\tilde{S}^a(x) = \mathcal{L}_{\xi(x)} \bar{\Theta}^a - ȷ \tilde{R}^a(x)$$

$$= - ȷ\varphi (\Box\varphi) x^a + \nabla_b \underbrace{(-2x^{[b}\nabla^{a]}\varphi ȷ\varphi)}$$

(Killing) ≈ 0 $\qquad \tilde{r}^{ba}(x) = \tilde{r}^{[ba]}(x) \sim$ (top-2) form

$$\tilde{S}^a(x) \approx \nabla_b \tilde{r}^{ba}, \quad \underline{r}(x) = \frac{1}{2} \tilde{r}^{ba}(x) \underline{\epsilon}_{ba}$$

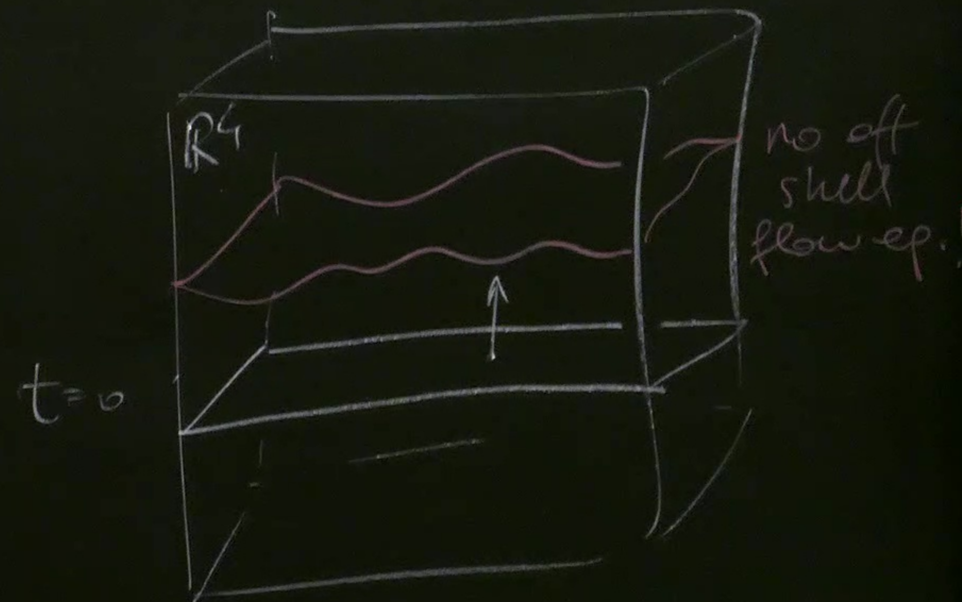
$\partial\Sigma = \phi$

$$\mathbb{i}_{\xi(X)} \Omega_\Sigma = - ȷ \underbrace{Q_\Sigma(X)}_{\int_\Sigma X^b t_b{}^a n_a} - \int_\Sigma \sqrt{h}\, \boxed{n_a X^a} (\Box\varphi)\, ȷ\varphi$$

$\epsilon)^2$

as in box

$$\boxed{n_e \chi^e} (\underline{\Box\varphi}) \, d\varphi \qquad \Box\varphi \overset{!}{=} 0$$



- $\chi^e \propto \partial_t$
- $\chi^e \propto \partial_i \hookrightarrow$ off shell flow eq.

$= \mathbb{d}\tilde{R}^e(\chi)$

$\mathbb{d}\varphi(\Box\varphi)\chi^e + \nabla_b\underbrace{(-2\chi^{[b}\nabla^{e]}\varphi\,\mathbb{d}\varphi)}_{\tilde{r}^{be}(\chi) = \tilde{r}^{[be]}(\chi)}$ → (top-2) form

≈ 0

$\tilde{r}^{be}$, $\underline{r}(\chi) = \frac{1}{2}\tilde{r}^{be}(\chi) \in \mathbb{be}$

$\Rightarrow 0$ as in box

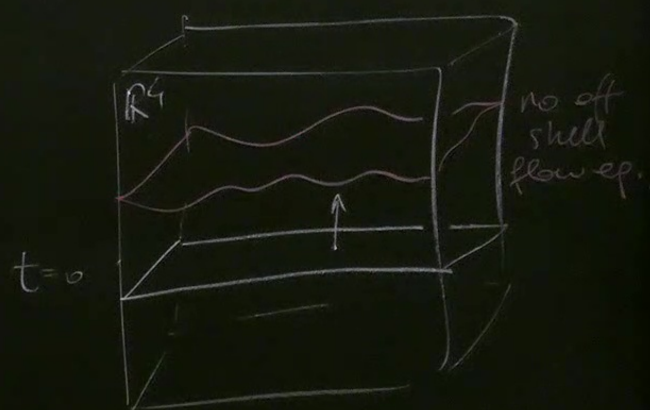
$\Sigma = -\mathbb{d}\underbrace{Q_\Sigma(\chi)}_{\int_\Sigma \chi^b t_b{}^e n_e} - \int_\Sigma \sqrt{h}\,\boxed{n_e\chi^e}\,\underline{(\Box\varphi)}\,\mathbb{d}\varphi$

↑ timelike X

has never an off shell generator

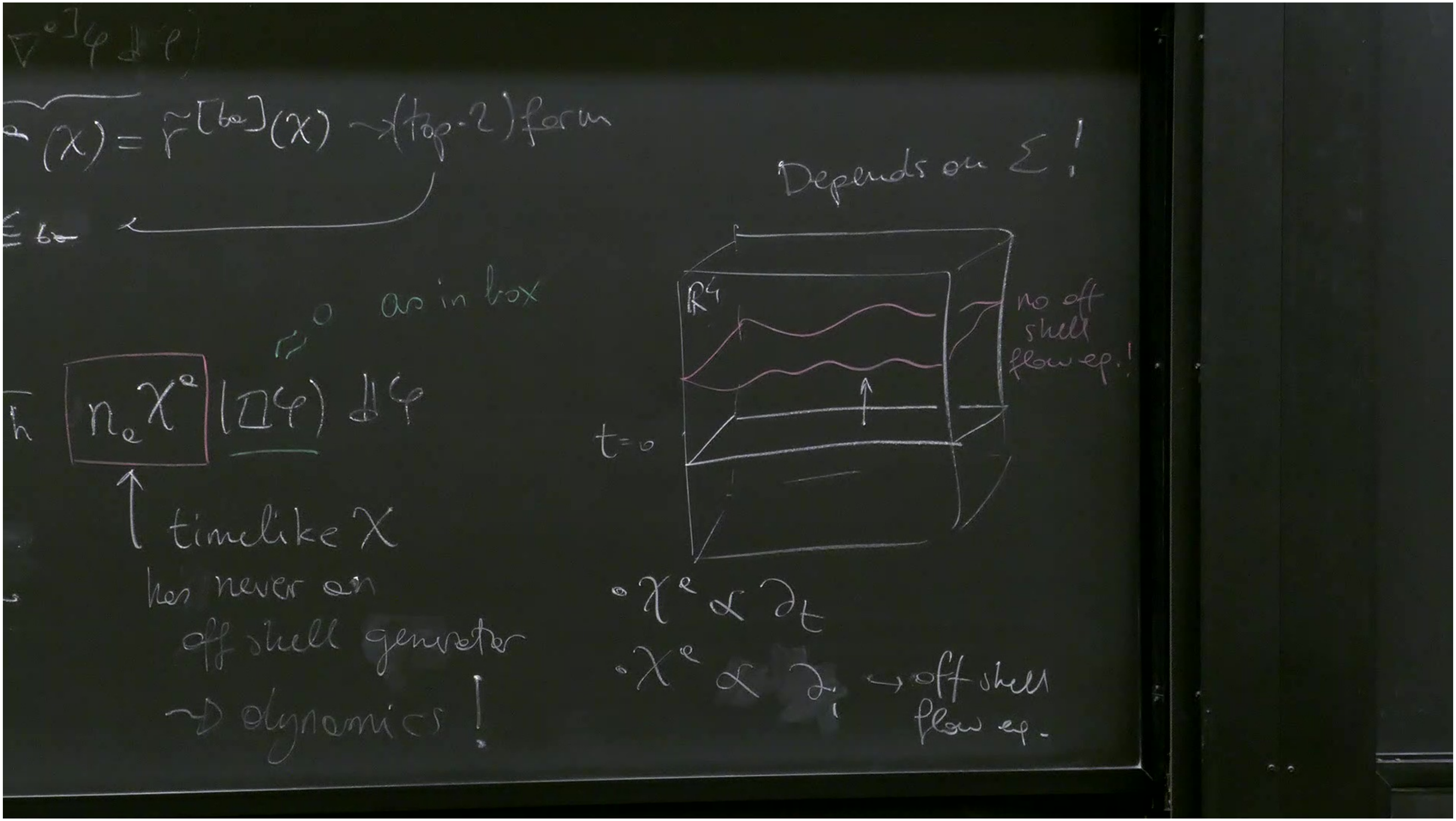
→ dynamics

Depends on Σ!



• $\chi^e \propto \partial_t$

• $\chi^e \propto \partial_i$ → off shell flow eq.


(χ) = r̃^[be](χ) → (top-2) form
Depends on Σ!
as in box
R^4
no off shell flow eq.!
$n_e \chi^e (\Box\varphi)$
t=0
timelike χ
has never an off shell generator
→ dynamics!
$\chi^e \propto \partial_t$
$\chi^e \propto \partial_i$ → off shell flow eq.

LOCAL SYMS

Notation: $\mathfrak{g} \rightsquigarrow \mathfrak{G}$

finite dim Lie alg

Typically: $\mathfrak{G} = C^\infty(M, \mathfrak{g})$ (YM)

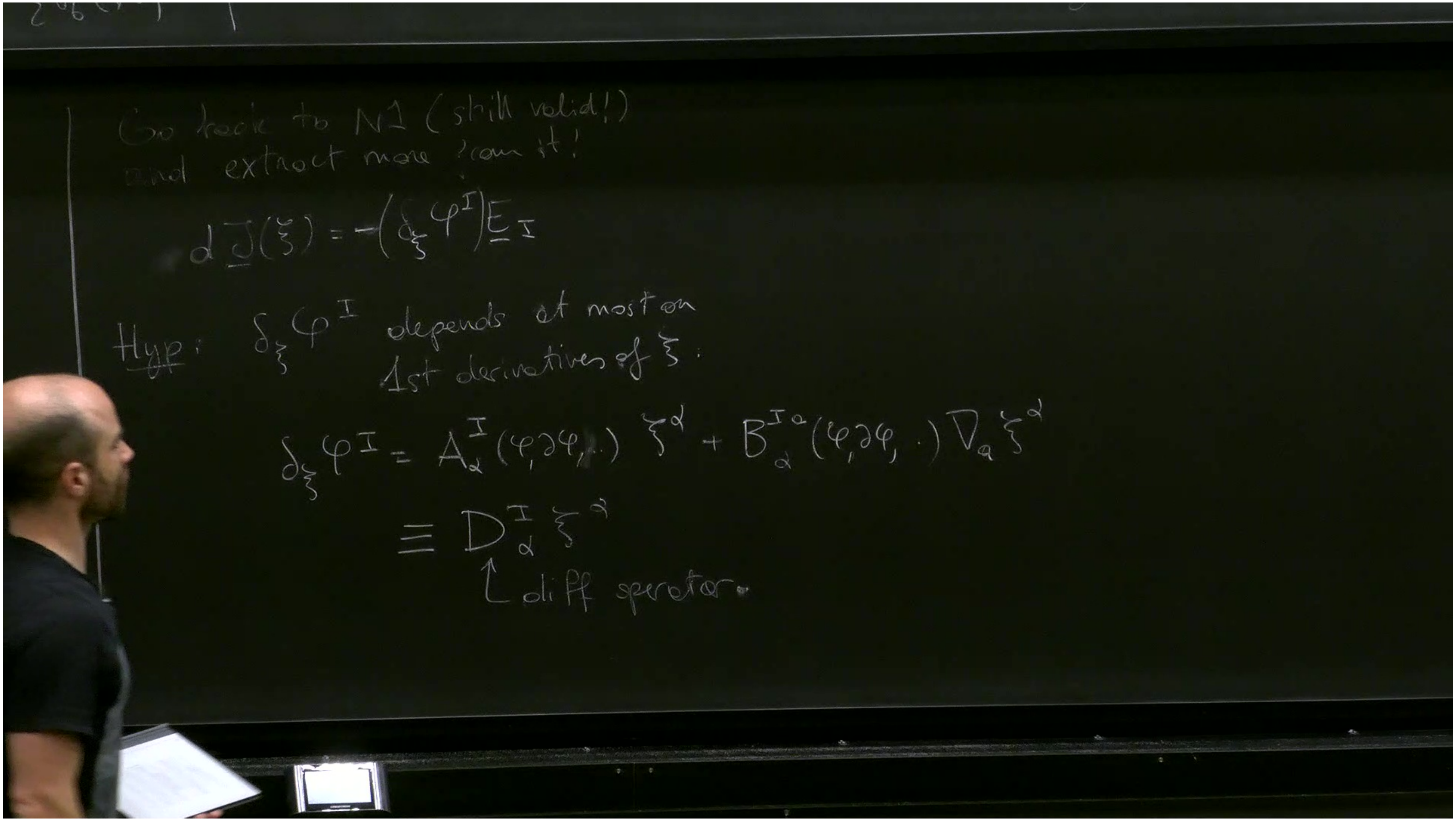
$\mathfrak{G} = \mathfrak{X}^1(M)$ (GR)

$\xi \in \mathfrak{G}$ can be freely chosen locally

(Rmk: killing v.f. are a "global" subalg of $\mathfrak{X}^1(M)$)

DEF I call a local sym "gauge" if $\underline{S} \equiv 0$

off-shell obj.


Go back to N1 (still valid!)
and extract more from it!
$d\underline{J}(\xi) = -(\delta_\xi \varphi^I)\underline{E}_I$
Hyp: $\delta_\xi \varphi^I$ depends at most on
1st derivatives of ξ:
$\delta_\xi \varphi^I = A^I_\alpha(\varphi, \partial\varphi, ..)\, \xi^\alpha + B^{I a}_\alpha(\varphi, \partial\varphi, .)\, \nabla_a \xi^\alpha$
$\equiv D^I_\alpha \xi^\alpha$
diff operator

st on

of ξ:

$\xi^\alpha + B^{I a}_\alpha(\varphi, \partial\varphi, \cdot)\nabla_a \xi^\alpha$

operator.

$$\int E_I \delta_\xi \varphi^I = \int E_I D^I_\alpha \xi^\alpha$$

for all ξ of comp. support in $\mathrm{Int}(M)$

$$= \int \xi^\alpha (D^\dagger)^I_\alpha E_I$$

$$(D^\dagger)^I_\alpha := (A^I_\alpha - \nabla_b B^{bI}_\alpha) - B^{Ib}_\alpha \nabla_b$$

(i.b.p.)

Thm (Noether 2)

(G, ρ) is a local sym, the EOM are not all linearly indep: $(D^\dagger)^I_a E_I = 0$

Rmk: we have less eom ($\# I - \# \alpha$)
then d.o.f. ($\# I$)
$\longrightarrow$ (mod out gauge!)

Pf: ξ of cpt support in Int(M)

$$0 = \int_M dJ(\xi) \overset{\boxed{N1}}{=} \int \delta_\xi \varphi^I E_I = \int \xi^\alpha (\mathbb{D}^+)^I_\alpha E_I \; \xi$$

For all $\xi(x)$ $\Rightarrow$ $(\mathbb{D}^+)^I_\alpha E_I = 0$ $\quad \square$

local sym!!!

Now, go back to J!

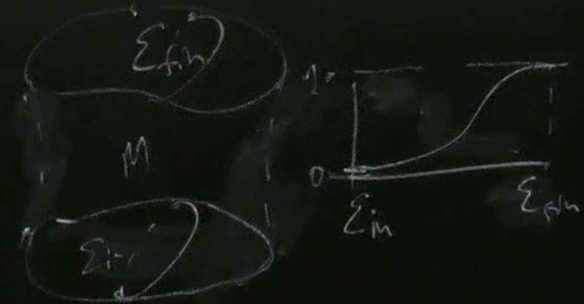
PROP (Vanishing charges)

Now, go back to J!

PROP (Vanishing charges)

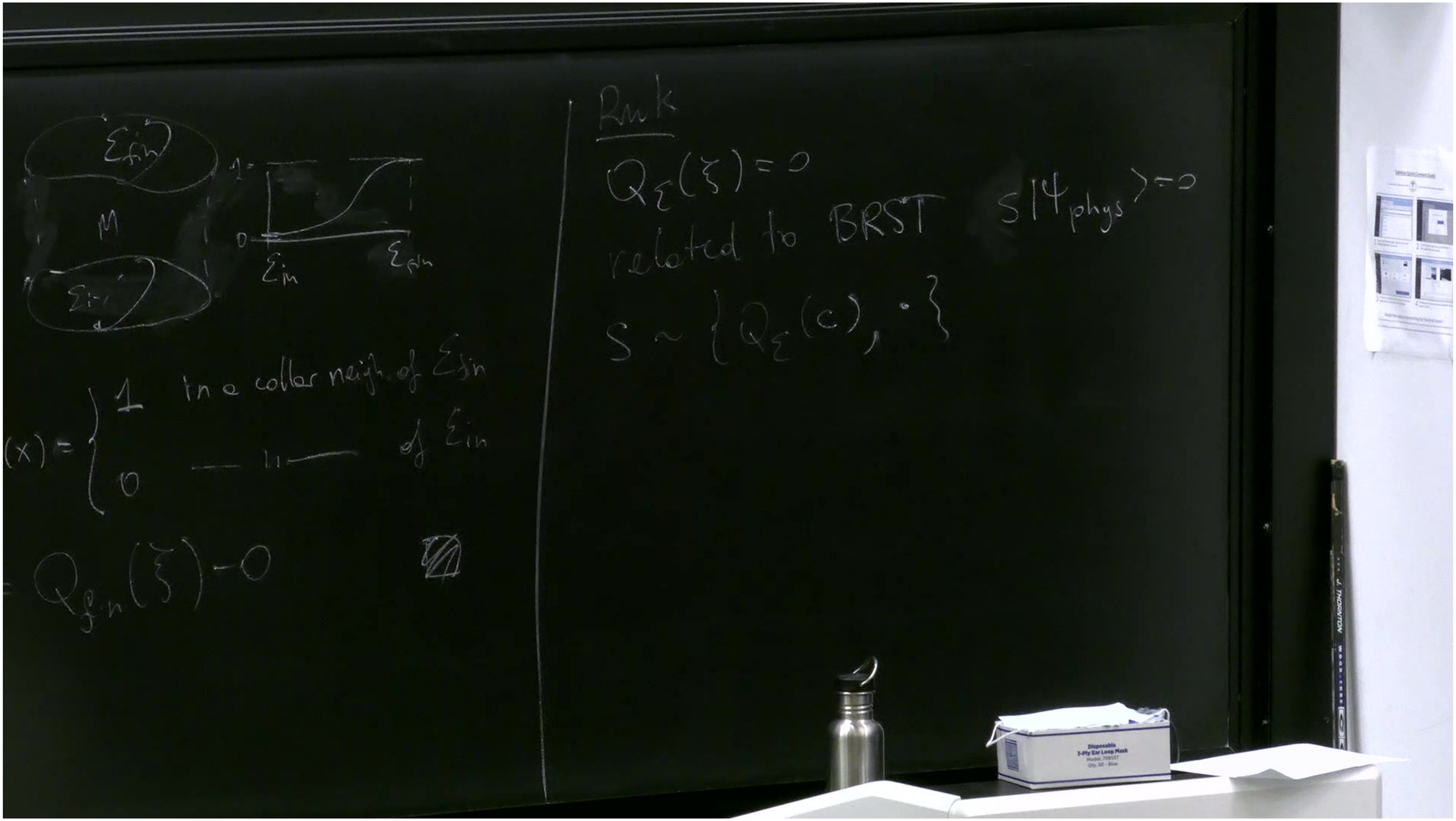
$\partial\Sigma = \emptyset$, (G, ℓ) is local sym

$\Longrightarrow \boxed{Q_\Sigma(\xi) \approx 0}$



Pf: Let $\hat{\xi}(x) = f(x)\,\xi(x)$, $f(x) = \begin{cases} 1 & \text{in a collar neigh. of } \Sigma_{fin} \\ 0 & \text{— " — of } \Sigma_{in} \end{cases}$

$$\underset{\boxed{N1}}{0 \approx} \int_M d\,J(\hat{\xi}) = Q_{fin}(\hat{\xi}) - Q_{in}(\hat{\xi}) = Q_{fin}(\xi) - 0 \qquad \blacksquare$$


Rmk
Q_Σ(ξ) = 0
related to BRST
s|Ψ_phys⟩ = 0
s ~ {Q_Σ(c), ·}
in a collar neigh. of Σ_fin
of Σ_in
Q_fin(ξ) = 0