Title: Quantum Gravity Lecture

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RECAP

Lagrangian $\underline{\mathcal{L}} = \mathcal{L}\,\underline{\epsilon} \in \Omega^{top,0}(M \times \mathcal{F})$

$$d\underline{\mathcal{L}} = \underline{E} + d\underline{\Theta}$$

$\underline{E}$: Eq. of Motion

$\underline{\Theta}$: presympl pot. c.

1) The shell: $\overline{\mathcal{F}} := \{\underline{E} = 0\} \subset \mathcal{F}$

2) $\underline{\Omega} := d\underline{\Theta}$ presymp current. $\in \Omega^{top-1,2}(M \times \mathcal{F})$

Thm $d\underline{\Omega} \approx 0$ on-shell conservation

$\Rightarrow \Omega_\Sigma := \int_\Sigma \underline{\Omega}$, $(\overline{\mathcal{F}}, \iota_{EL}^* \Omega_\Sigma)$ "covariant ph. space"

TODAY: symmetries!

$(\mathfrak{g}, [\cdot,\cdot])$ Lie alg.

$\rho : \mathfrak{g} \longrightarrow \mathfrak{X}^1(\mathcal{F})$ Lie alg. homomorph

ACTION of $\mathfrak{g}$ on $\mathcal{F}$

$$\rho(\xi) = \text{"} \int_M (\delta_\xi \varphi) \frac{\delta}{\delta \varphi} \text{"}$$

$$\rho(\xi) F = \int_M \frac{\delta F}{\delta \varphi} (\delta_\xi \varphi)$$

Ex: particle $\vec{q}(t) \in \mathcal{F}$

translations $\rho(\xi)\vec{q} = \vec{\xi} \equiv \delta_\xi \vec{q}$

$$F = \int \tfrac{1}{2} \dot{q}^2$$

$$\rho(\xi) F = \int \frac{\delta F}{\delta \varphi} (\xi)$$

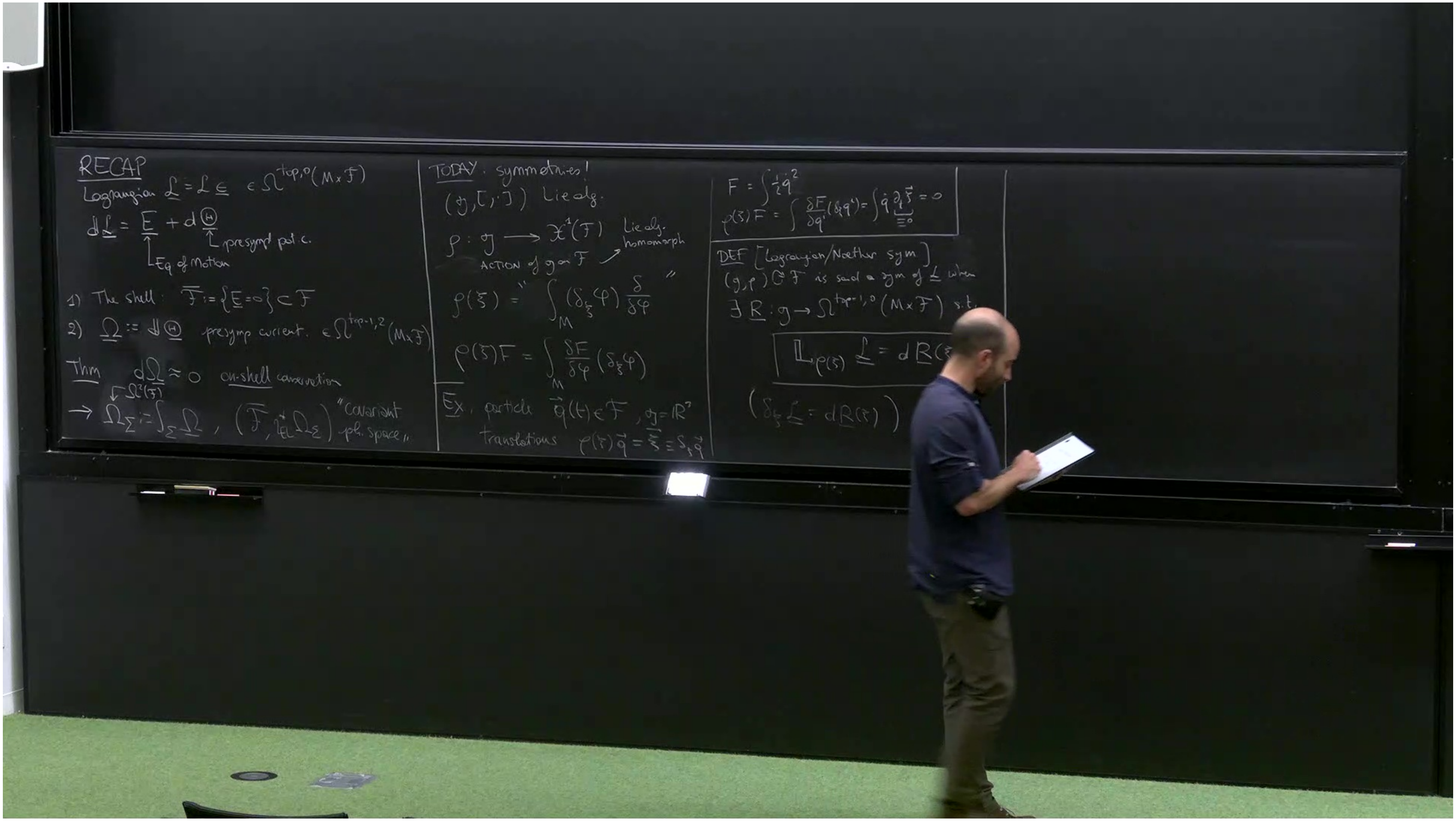
alg.
momorph

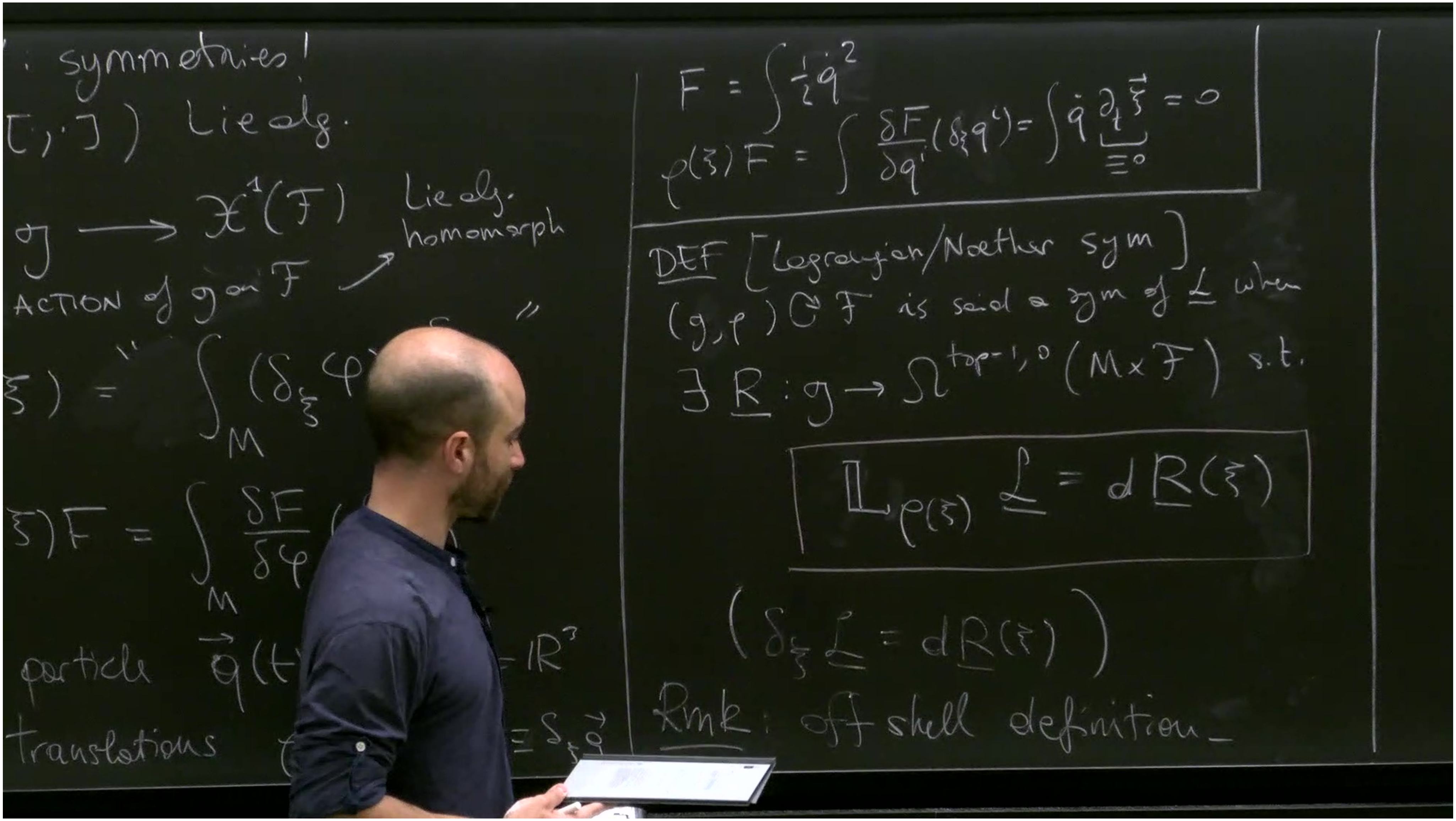
$\mathfrak{g} = \mathbb{R}^3$

$\vec{\xi} = \delta_t \vec{q}$

$$F = \int \tfrac{1}{2}\dot{q}^2$$

$$\rho(\xi) F = \int \frac{\delta F}{\delta q^i}(\delta_\xi q^i) = \int \dot{q}\, \underbrace{\partial_t \vec{\xi}}_{\equiv 0} = 0$$


RECAP
Lagrangian
presympl pot. c.
Eq of Motion
1) The shell
2) presymp current
Thm
on-shell conservation
"Covariant ph. space"
TODAY: symmetries!
Lie alg.
Lie alg. homomorph
ACTION of g on F
Ex. particle
translations
DEF [Lagrangian/Noether sym]
is said a sym of L when


symmetries!
Lie alg.
Lie alg. homomorph
ACTION of g on F
particle
translations
F = ∫ ½ q̇²
ρ(ξ) F = ∫ δF/δq^i (δ_ξ q^i) = ∫ q̇ ∂_t ξ = 0
DEF [Lagrangian/Noether sym]
(g, ρ) ↻ F is said a sym of L when
∃ R : g → Ω^{top-1,0}(M×F) s.t.
𝕃_{ρ(ξ)} L = d R(ξ)
(δ_ξ L = d R(ξ))
Rmk: off shell definition

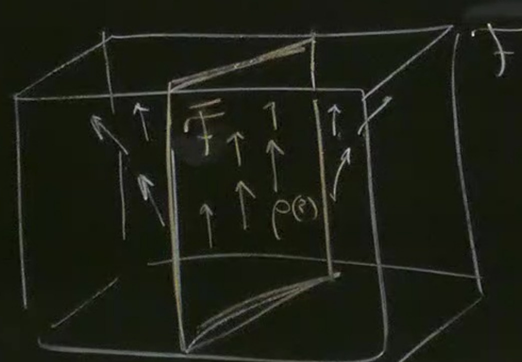
Q: is $\rho(\xi)$ mapping sol.s into sol.s ?

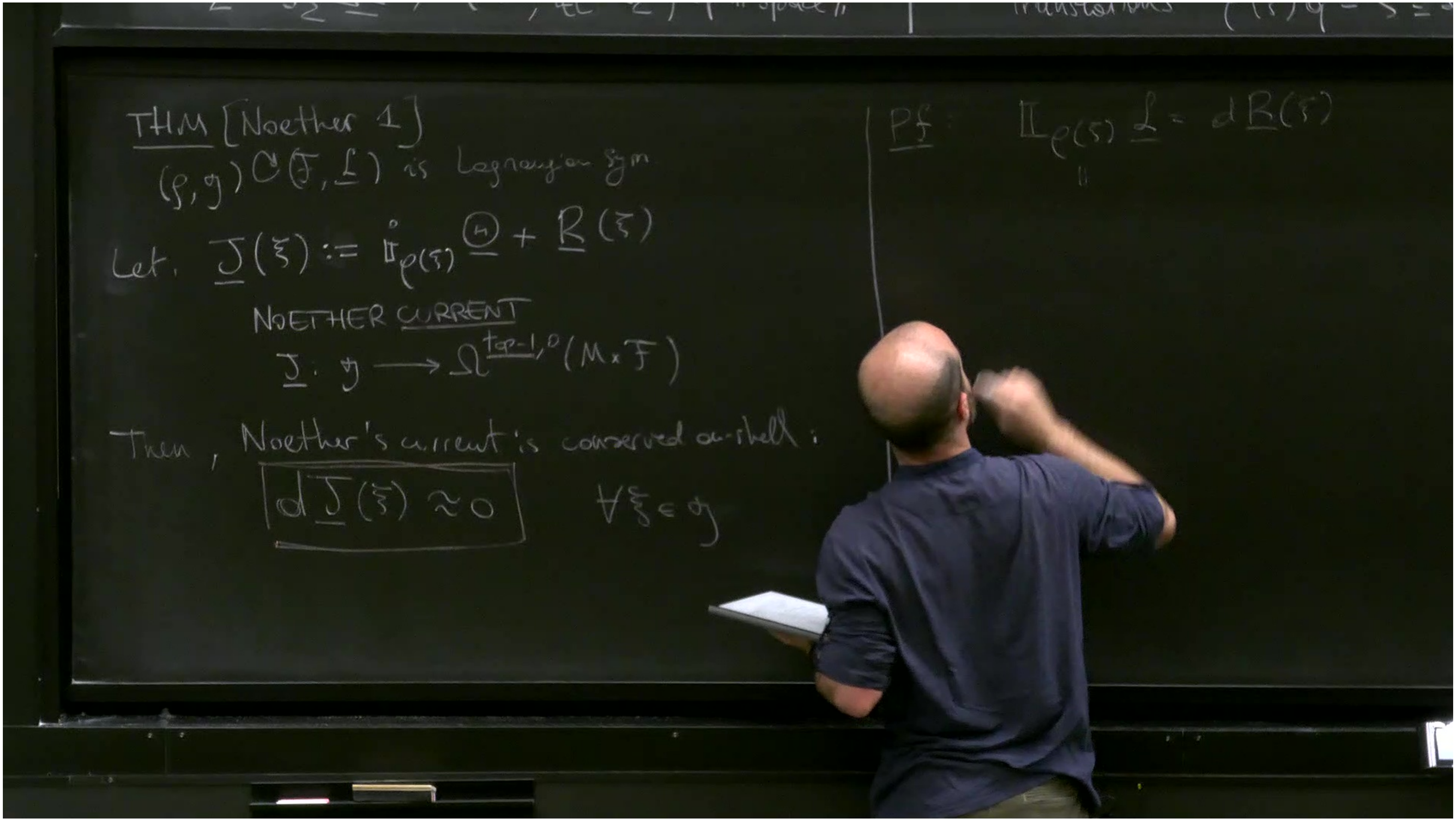
I.e. is $\rho(\xi)$ tangent to $\bar{\mathcal{F}} \subset \mathcal{F}$?

I.e. is $\rho(\xi)$ a Jacobi field, $\mathbb{L}_{\rho(\xi)} \underline{E} \approx 0$?

PROP: $\mathbb{L}_{\rho(\xi)} \underline{E} \approx 0$ $\boxed{}$

Rmk: $\imath^*_{EL}\left(\mathbb{I}_{\rho(\xi)} \Omega_\Sigma\right) = \mathbb{I}_{\rho(\xi)} \imath^*_{EL} \Omega_\Sigma$




THM [Noether 1]
$(\rho, \mathfrak{g}) \circlearrowleft (\mathcal{F}, \underline{L})$ is Lagrangian sym
Let: $\underline{J}(\xi) := \mathbb{I}_{\rho(\xi)} \underline{\Theta} + \underline{B}(\xi)$
NOETHER CURRENT
$\underline{J} : \mathfrak{g} \longrightarrow \Omega^{top-1,0}(M \times \mathcal{F})$
Then, Noether's current is conserved on-shell:
$\boxed{d\underline{J}(\xi) \approx 0}$ $\forall \xi \in \mathfrak{g}$
Pf: $\mathbb{I}_{\rho(\xi)} \underline{L} = d\underline{B}(\xi)$

THM [Noether 1]

$(\rho, \mathfrak{g}) \circlearrowright (\mathcal{F}, \underline{L})$ is Lagrangian sym.

Let. $\underline{J}(\xi) := \mathbb{I}_{\rho(\xi)}\Theta + \underline{R}(\xi)$

NOETHER CURRENT

$\underline{J} : \mathfrak{g} \to \ldots \Omega^{\mathrm{top}-1,0}(M \times \mathcal{F})$

Then, Noether's … is conserved on-shell:

$\boxed{d\underline{J}(\xi) \ldots 0} \quad \forall \xi \in \mathfrak{g}$

Pf: $\mathbb{I}_{\rho \ldots}$

$\mathbb{I}_{\rho(\xi)} \ldots$

$\boxed{\underline{E} = d\varphi^I E_I(\varphi, \partial\varphi, \partial^2\varphi)}$

<u>Pf</u>: $\mathbb{I}_{\rho(\xi)} \underline{L} = d\underline{B}(\xi)$

$$\mathring{\mathbb{I}}_{\rho(\xi)} \, d \underline{L} = \mathring{\mathbb{I}}_{\rho(\xi)} \left(\underline{E} + d \underline{\Theta} \right)$$

$$= \delta_\xi \varphi^I E_I - d \left(\mathring{\mathbb{I}}_{\rho(\xi)} \underline{\Theta} \right)$$

$$\Rightarrow d\underline{J}(\xi) \equiv d\left(\mathring{\mathbb{I}}_{\rho(\xi)} \underline{\Theta} + \underline{B}(\xi) \right) = \delta_\xi \varphi^I E_I \approx 0$$

$\boxtimes$

$$E_I(\varphi, \partial\varphi, \partial^2\varphi)$$

With bou... given sy...

on-shell:

$d\varphi^I \, E_I(\varphi, \partial\varphi, \partial^2\varphi)$

Pf: $\mathbb{I}_{\rho(\xi)} \underline{L} = d\underline{B}(\xi)$

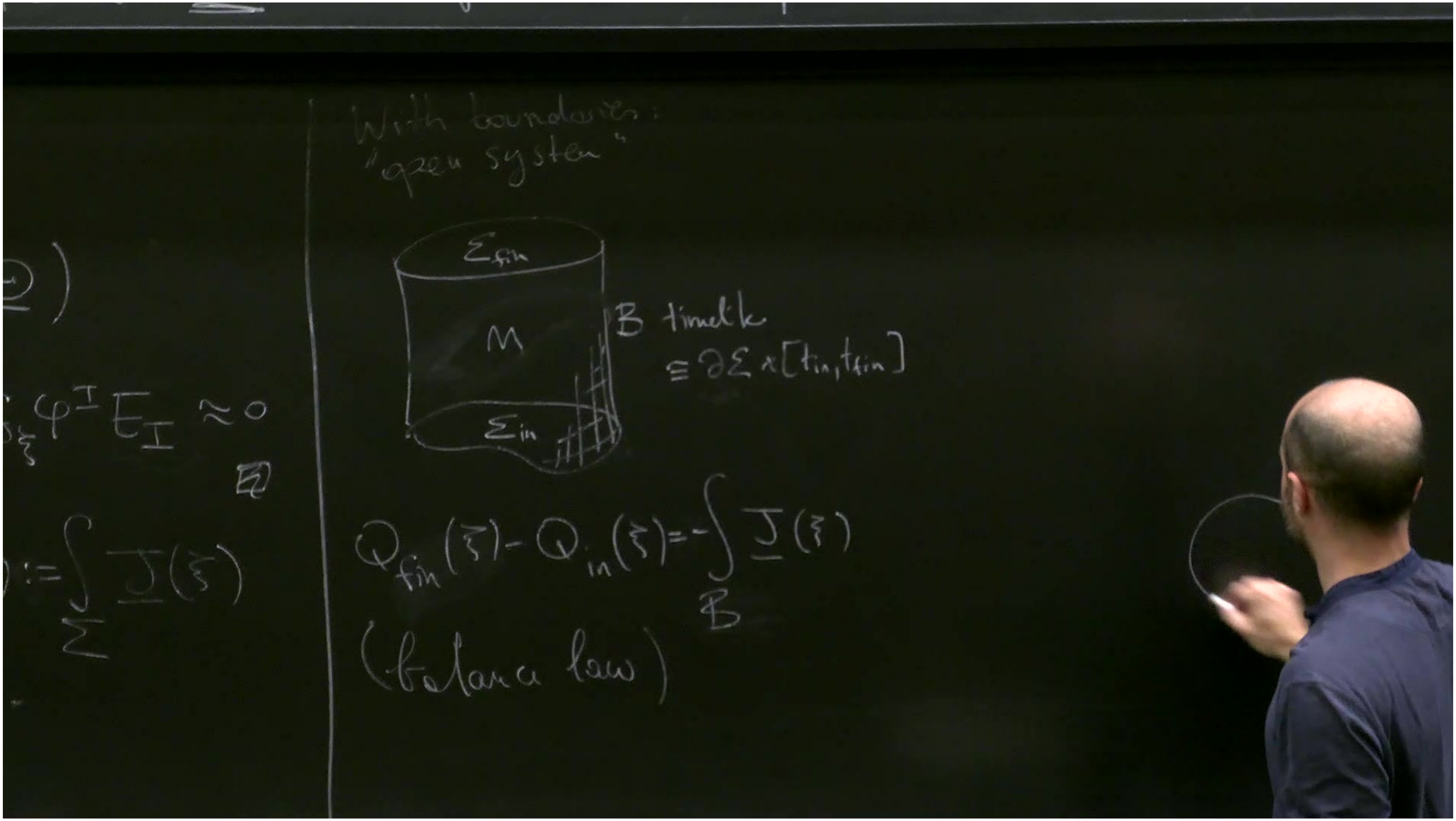
$i_{\rho(\xi)} \, d\underline{L} = i_{\rho(\xi)}(\underline{E} + d\underline{\Theta})$

$= \delta_\xi \varphi^I E_I - d(i_{\rho(\xi)} \underline{\Theta})$

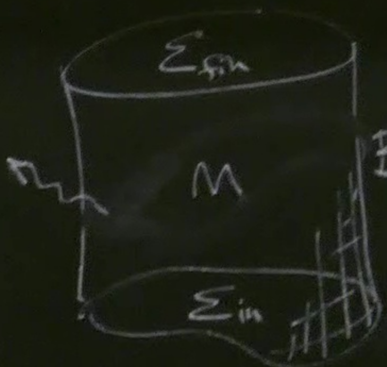
$\Rightarrow d\underline{J}(\xi) \equiv d\left(i_{\rho(\xi)}\underline{\Theta} + \underline{R}(\xi)\right) = \delta_\xi \varphi^I E_I \approx 0$

Corollary: Noether charge: $Q_\Sigma(\xi) := \int_\Sigma \underline{J}(\xi)$ is independent of $\Sigma \hookrightarrow M$.

$(\partial\Sigma = \phi)$


With boundaries:
"open system"
Σ_{fin}
M
B timelike
$\cong \partial\Sigma \times [t_{in}, t_{fin}]$
Σ_{in}
$Q_{fin}(\xi) - Q_{in}(\xi) = -\int_B \underline{J}(\xi)$
(balance law)
$\varphi^I E_I \approx 0$
$:= \int_\Sigma \underline{J}(\xi)$

With boundaries:
"open system"



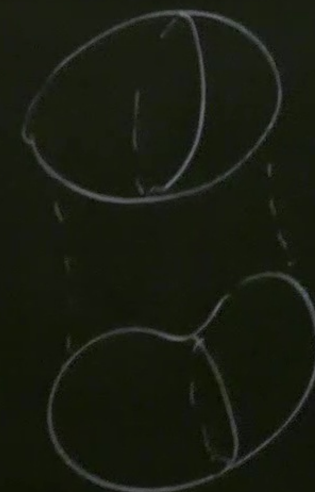
B timelike
$\cong \partial\Sigma \times [t_{in}, t_{fin}]$

≈ 0

(ξ)

$$Q_{fin}(\xi) - Q_{in}(\xi) = -\int_B \underline{J}(\xi)$$

(balance law)



$\partial\Sigma = \phi$

Question. is $Q_\Sigma(\xi)$ the Hamiltonian generator of (g,ρ) in the covariant ph. space ?

Lemma: Let $\underline{S} \in \Omega^{top-1,1}(M \times \mathcal{F})$, $d\underline{S} \approx 0$

then $\exists\, \underline{\Gamma} \in \Omega^{top-2,1}(M \times \mathcal{F})$: $\underline{S} \approx d\underline{\Gamma}$.

Rmk: I'm saying nothing about the cohom of M !

Catch: this is a 1-form on $\mathcal{F}$

$\underline{S} = \underline{S}(\varphi, \dots, \delta\varphi)$ "free"

PROP $(\rho, \mathfrak{g}) \circlearrowright (\mathcal{F}, \underline{L})$ Lagr. sym

then $\exists\, \underline{\Gamma} : \mathfrak{g} \to \Omega^{top-2,1}(M \times \mathcal{F})$ s.t.

$$\boxed{\mathring{\mathbb{I}}_{\rho(\xi)} \underline{\Omega} \approx -\mathbb{d}\underline{J}(\xi) + d\underline{\Gamma}(\xi)}$$

Pf:
$$\underline{S}(\xi) := \mathbb{L}_{\rho(\xi)} \underline{\Theta} + \mathbb{d}\underline{R}(\xi)$$
$$= \mathring{\mathbb{I}}_{\rho(\xi)} \underbrace{\mathbb{d}\underline{\Theta}}_{=:\underline{\Omega}} + \mathbb{d}\Big(\underbrace{\mathring{\mathbb{I}}_{\rho(\xi)} \underline{\Theta} + \underline{R}(\xi)}_{=:\underline{J}(\xi)}\Big)$$
$$= \mathring{\mathbb{I}}_{\rho(\xi)} \underline{\Omega} + \mathbb{d}\underline{J}(\xi)$$

Recall $\begin{cases} d\underline{J} \approx 0 \\ \mathbb{d}\underline{\Omega} \approx 0 \end{cases} \Longrightarrow d\underline{S} \approx 0 \overset{\text{lemma}}{\Longrightarrow} \underline{S} \approx d\underline{\Gamma}$ ∎

THM $(g,\rho) \circlearrowright (\mathcal{F}, \underline{L})$ Lagn sym.

$\boxed{\partial\Sigma = \phi}$

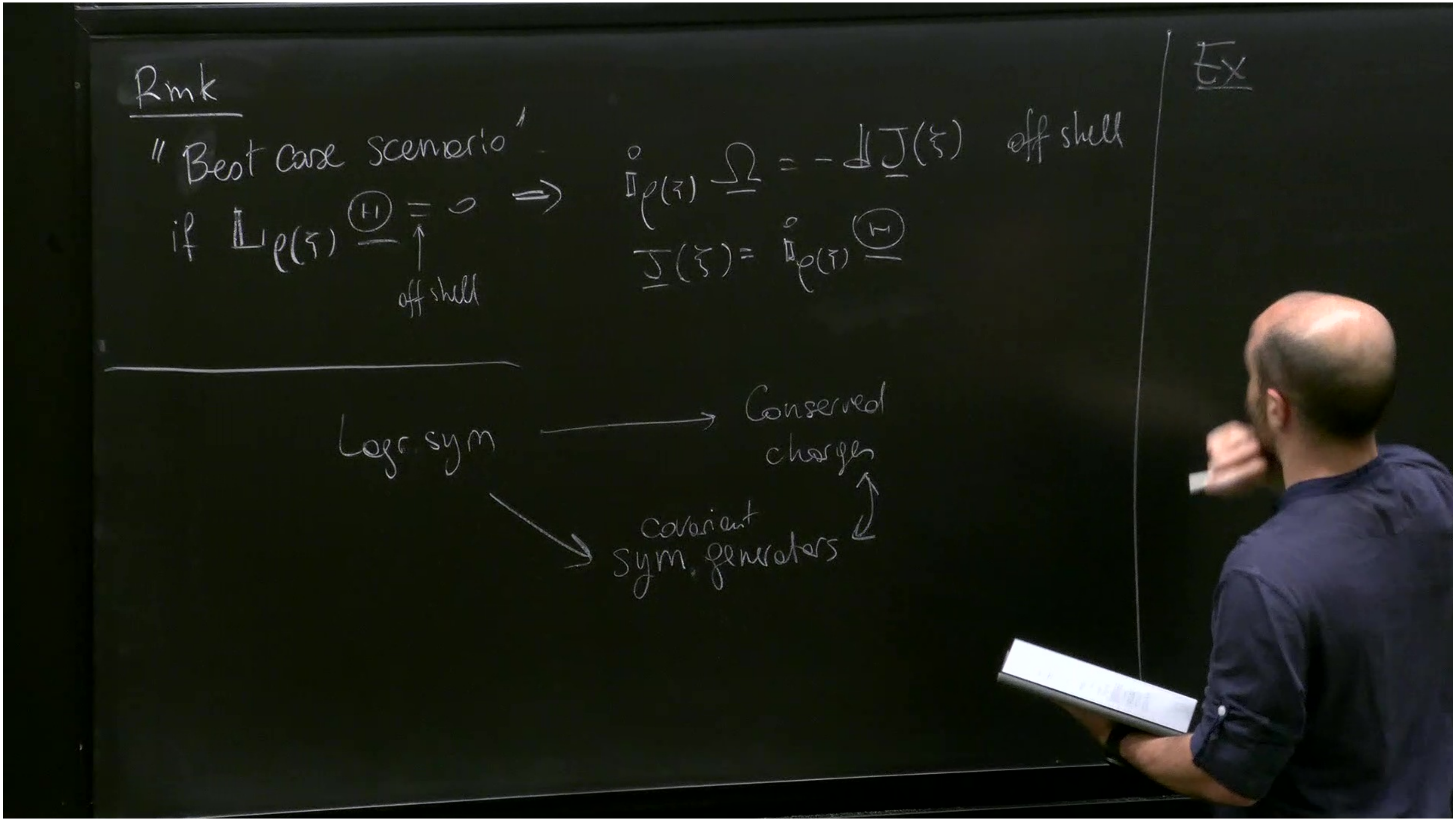
Then : $\boxed{\mathbb{I}_{\rho(\xi)} \Omega_\Sigma \approx - \mathrm{d}\!\!\mathrm{I}\, Q_\Sigma(\xi)}$

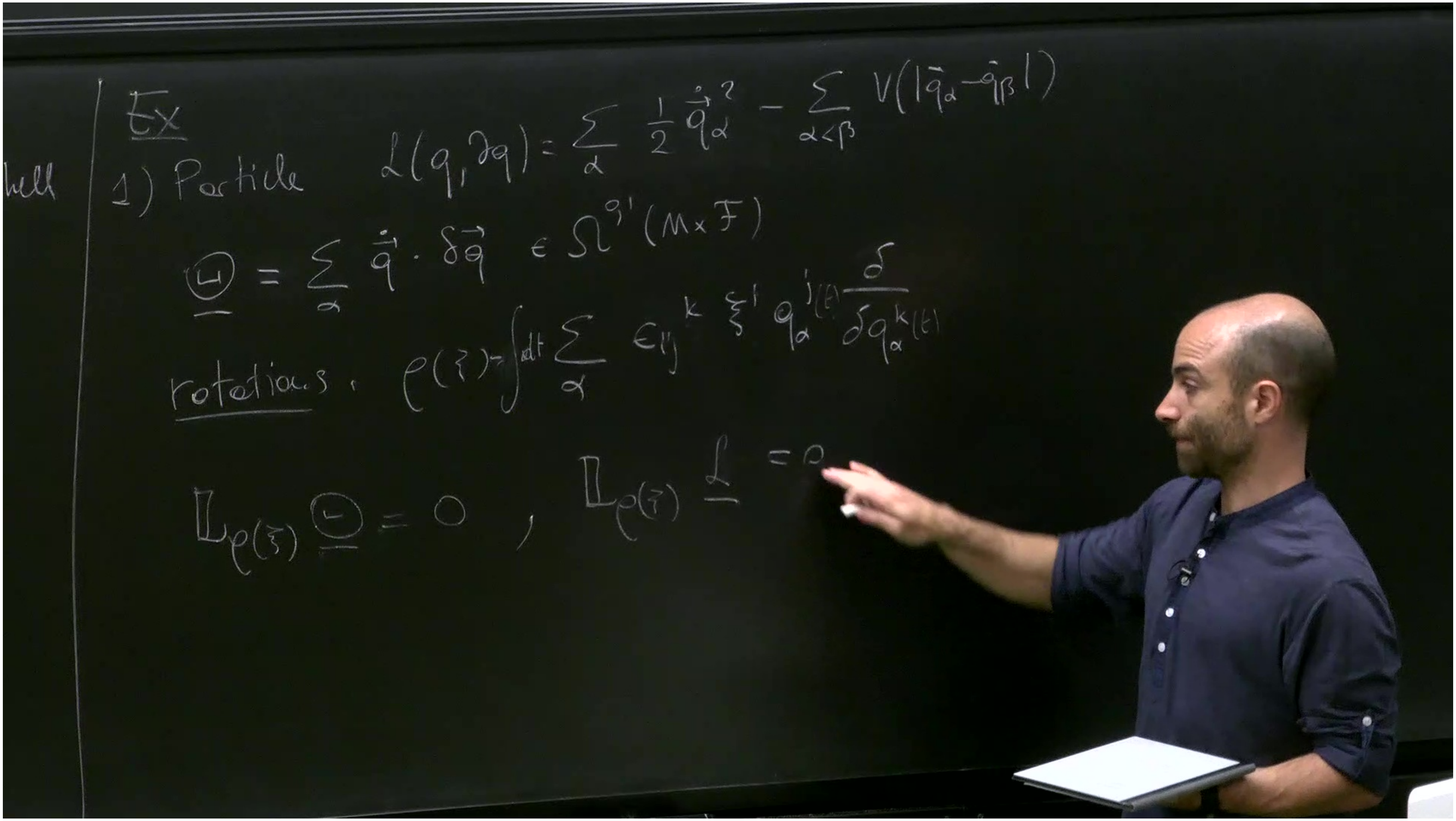
Pf : Integrate previous prop over Σ, $\partial\Sigma = \phi$

$\rightarrow$ get rid of $\mathrm{d}\underline{r}$! ☒

Rmk : Q_Σ, $\partial\Sigma = \phi$ is unambiguously defined

from $[\underline{L}] = [\underline{L} + \mathrm{d}\underline{\ell}]$ ☒


Rmk
"Best case scenario"
if $\mathbb{L}_{\rho(\xi)} \underline{\Theta} = 0$
off shell
$\Rightarrow \mathring{\mathbb{I}}_{\rho(\xi)} \underline{\Omega} = - \mathbb{d} \underline{J}(\xi)$ off shell
$\underline{J}(\xi) = \mathring{\mathbb{I}}_{\rho(\xi)} \underline{\Theta}$
Lagr. sym
Conserved charges
covariant sym. generators
Ex


Ex
1) Particle $L(q,\partial q) = \sum_\alpha \frac{1}{2} \dot{\vec{q}}_\alpha^{\,2} - \sum_{\alpha<\beta} V(|\vec{q}_\alpha - \vec{q}_\beta|)$
$\Theta = \sum_\alpha \dot{\vec{q}} \cdot \delta\vec{q} \in \Omega^{0,1}(M \times F)$
rotations: $\rho(\xi) = \int dt \sum_\alpha \epsilon_{ij}{}^{k} \xi^i q_\alpha^j(t) \frac{\delta}{\delta q_\alpha^k(t)}$
$\mathbb{L}_{\rho(\xi)} \Theta = 0 \,, \quad \mathbb{L}_{\rho(\xi)} L = 0$

(ξ) off shell

<u>Ex</u>

1) Particle $\mathcal{L}(q, \partial q) = \sum_\alpha \frac{1}{2} \dot{\vec{q}}_\alpha^{\,2} - \sum_{\alpha<\beta} V(|\vec{q}_\alpha - \vec{q}_\beta|)$

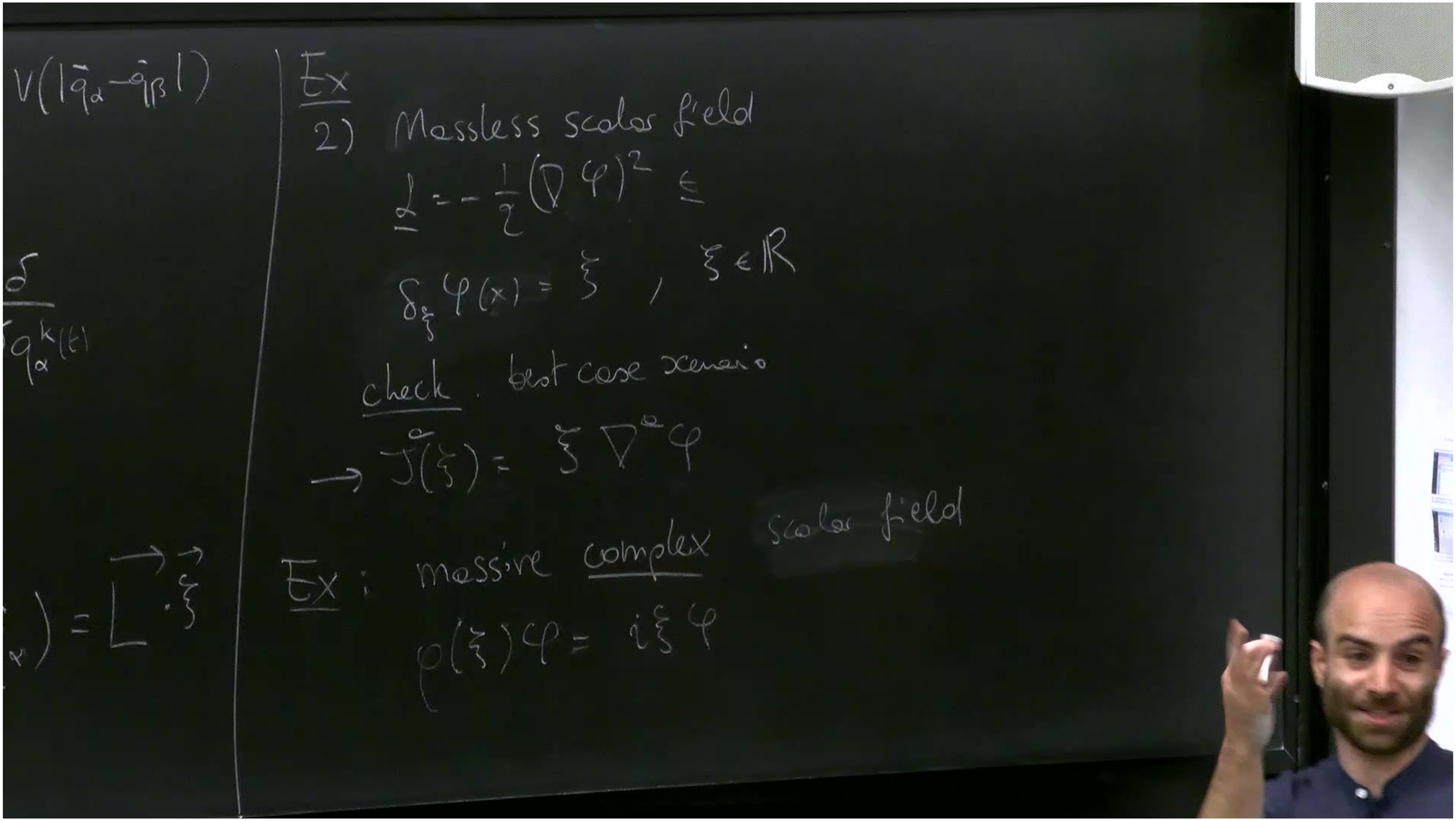
$$\underline{\Theta} = \sum_\alpha \dot{\vec{q}} \cdot \delta\vec{q} \in \Omega^{0,1}(M \times \mathcal{F})$$

<u>rotations</u>: $\rho(\xi) = \int dt \sum_\alpha \epsilon_{ij}{}^{k}\, \xi^i q_\alpha^j(t) \frac{\delta}{\delta q_\alpha^k(t)}$

$$\mathbb{L}_{\rho(\xi)} \underline{\Theta} = 0 \;, \quad \mathbb{L}_{\rho(\xi)} \underline{\mathcal{L}} = 0$$

best case scenario

$$\underline{J}(\xi) = \mathbb{I}_{\rho(\xi)} \underline{\Theta} = \sum_\alpha \vec{\xi} \cdot (\vec{q}_\alpha \times \dot{\vec{q}}_\alpha) = \vec{L} \cdot \vec{\xi}$$


$V(|\vec{q}_\alpha - \vec{q}_\beta|)$
Ex
2) Massless scalar field
$\mathcal{L} = -\frac{1}{2}(\nabla\varphi)^2 \in$
$\delta_\xi \varphi(x) = \xi \ , \quad \xi \in \mathbb{R}$
$\frac{\delta}{\delta q_\alpha^k(t)}$
check. best case scenario
$\rightarrow J^a(\xi) = \xi \nabla^a \varphi$
$) = \vec{L} \cdot \vec{\xi}$
Ex: massive complex scalar field
$\rho(\xi)\varphi = i\xi\varphi$