Title: Quantum Gravity Lecture

Speakers: Aldo Riello

Collection: Quantum Gravity 2023/24

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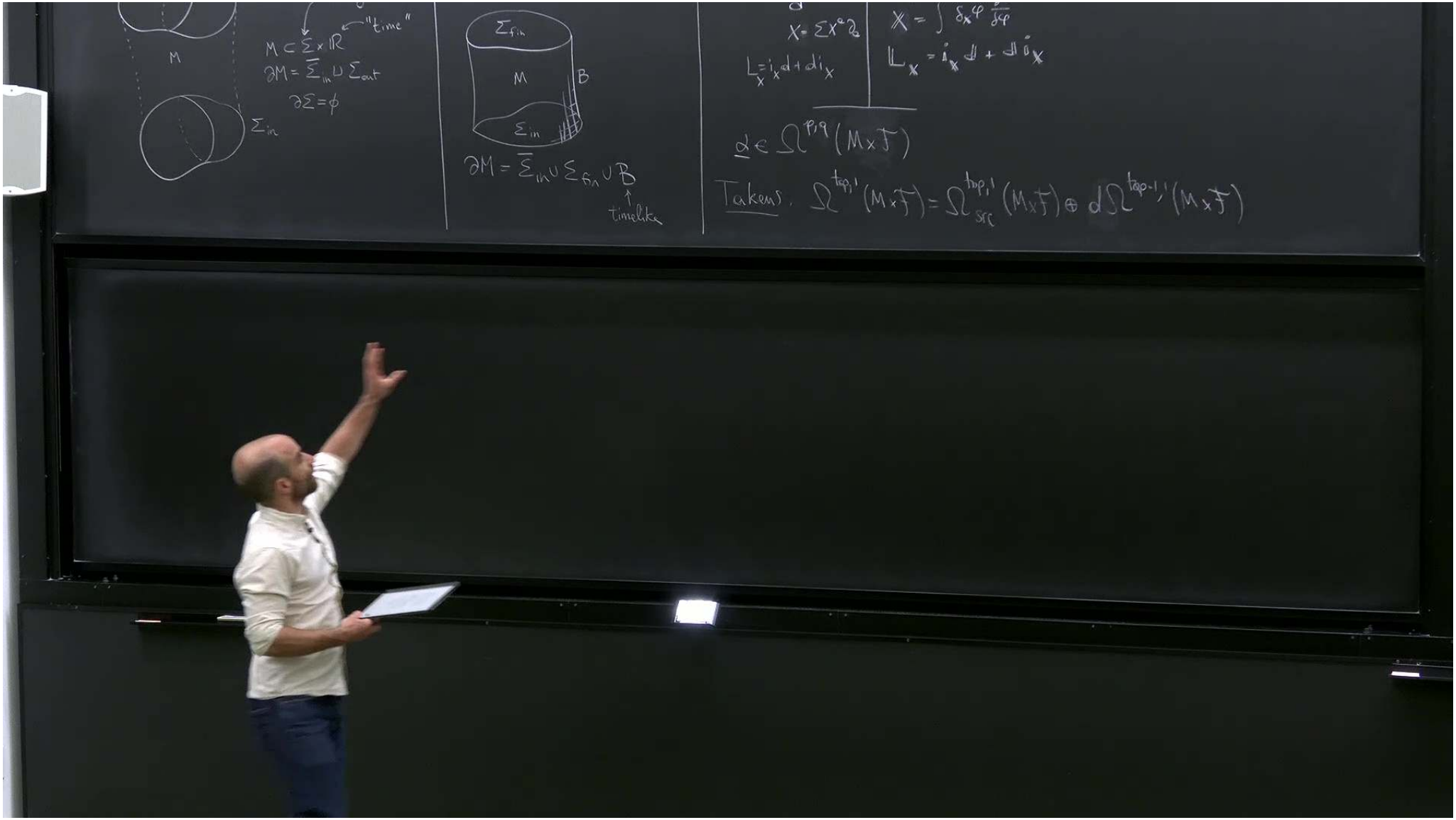
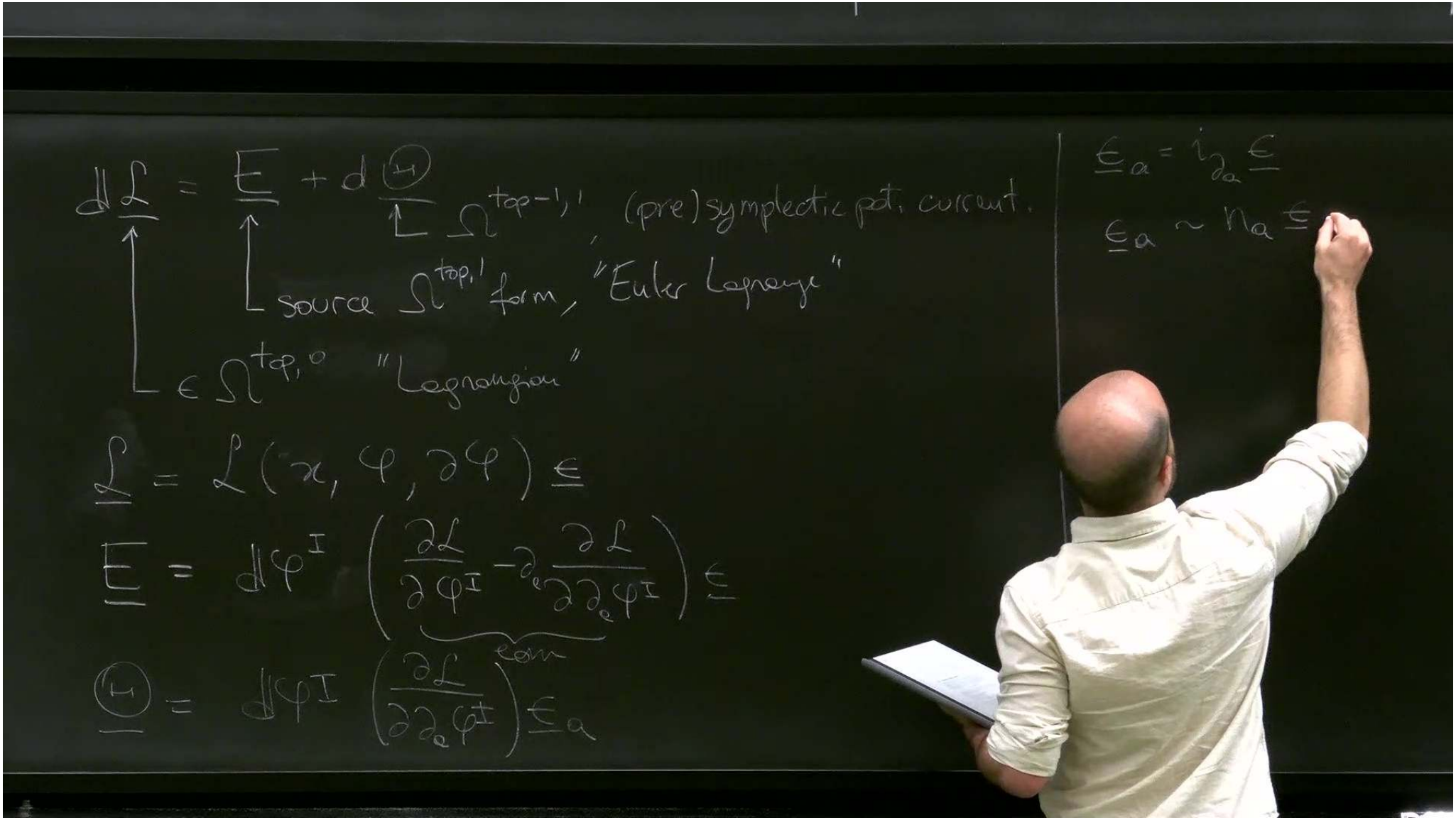

M
$\Sigma_{in}$
$M \subset \Sigma \times \mathbb{R}$
"time"
$\partial M = \overline{\Sigma}_{in} \cup \Sigma_{out}$
$\partial \Sigma = \phi$
$\Sigma_{fin}$
M
B
$\Sigma_{in}$
$\partial M = \overline{\Sigma}_{in} \cup \Sigma_{fin} \cup B$
timelike
$X = \Sigma X^a \partial_a$
$L_X = i_X d + d i_X$
$X = \int \delta_X \varphi \frac{\delta}{\delta \varphi}$
$L_X = i_X d + d i_X$
$\alpha \in \Omega^{p,q}(M \times \mathbb{T})$
Takens: $\Omega^{top,1}(M \times \mathbb{T}) = \Omega^{top,1}_{src}(M \times \mathbb{T}) \oplus d\Omega^{top-1,1}(M \times \mathbb{T})$

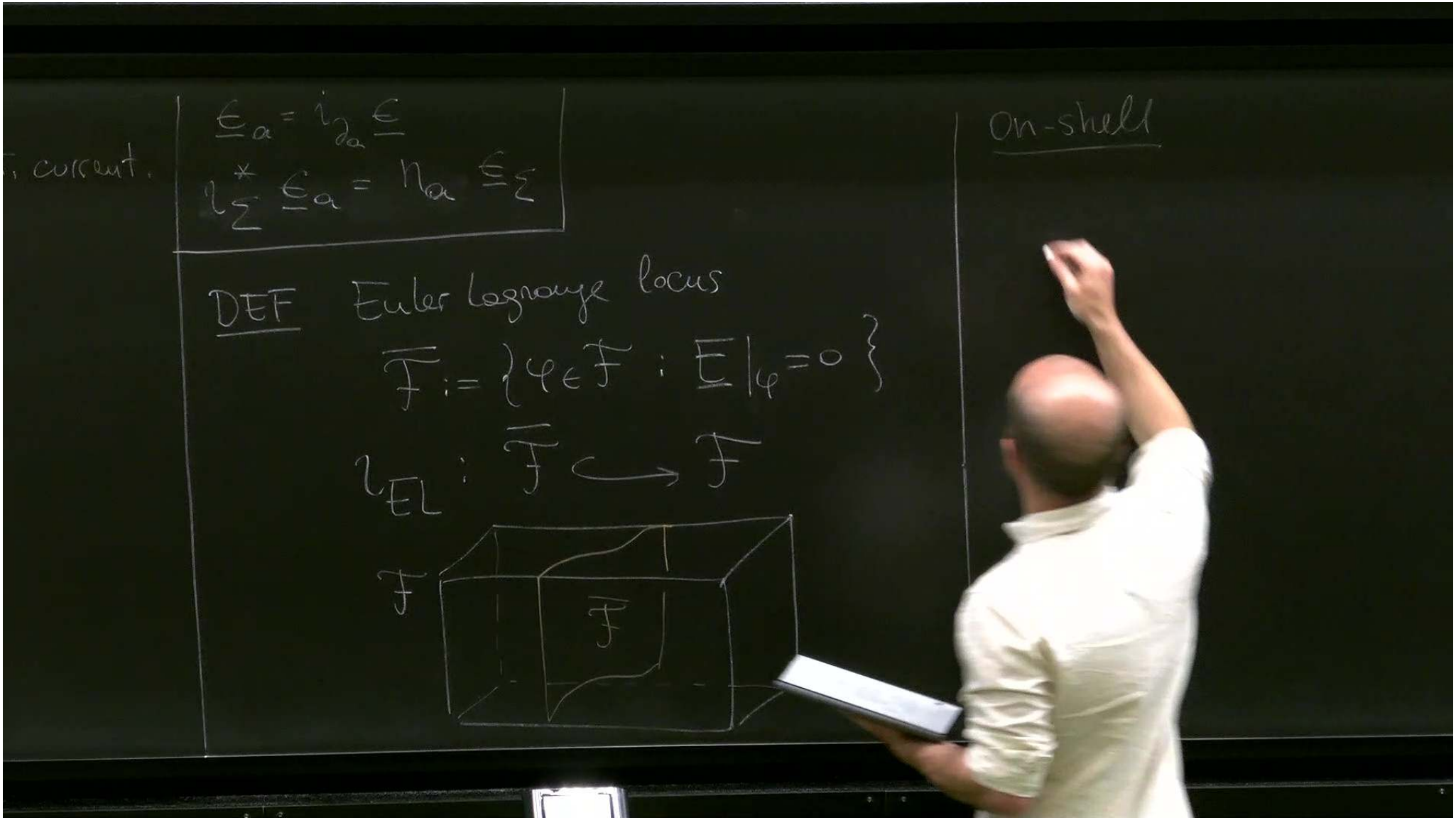

$d\underline{L} = \underline{E} + d\underline{\theta}$
$\underline{\theta} \in \Omega^{top-1,1}$ (pre) symplectic pot. current.
source $\Omega^{top,1}$ form, "Euler Lagrange"
$\in \Omega^{top,0}$ "Lagrangian"
$\underline{L} = L(x, \varphi, \partial\varphi)\,\underline{\epsilon}$
$\underline{E} = d\varphi^I \left( \frac{\partial L}{\partial \varphi^I} - \partial_a \frac{\partial L}{\partial \partial_a \varphi^I} \right) \underline{\epsilon}$
$\underline{\theta} = d\varphi^I \left( \frac{\partial L}{\partial \partial_a \varphi^I} \right)^{eom} \underline{\epsilon}_a$
$\underline{\epsilon}_a = i_{\partial_a} \underline{\epsilon}$
$\underline{\epsilon}_a \sim n_a \underline{\epsilon}$

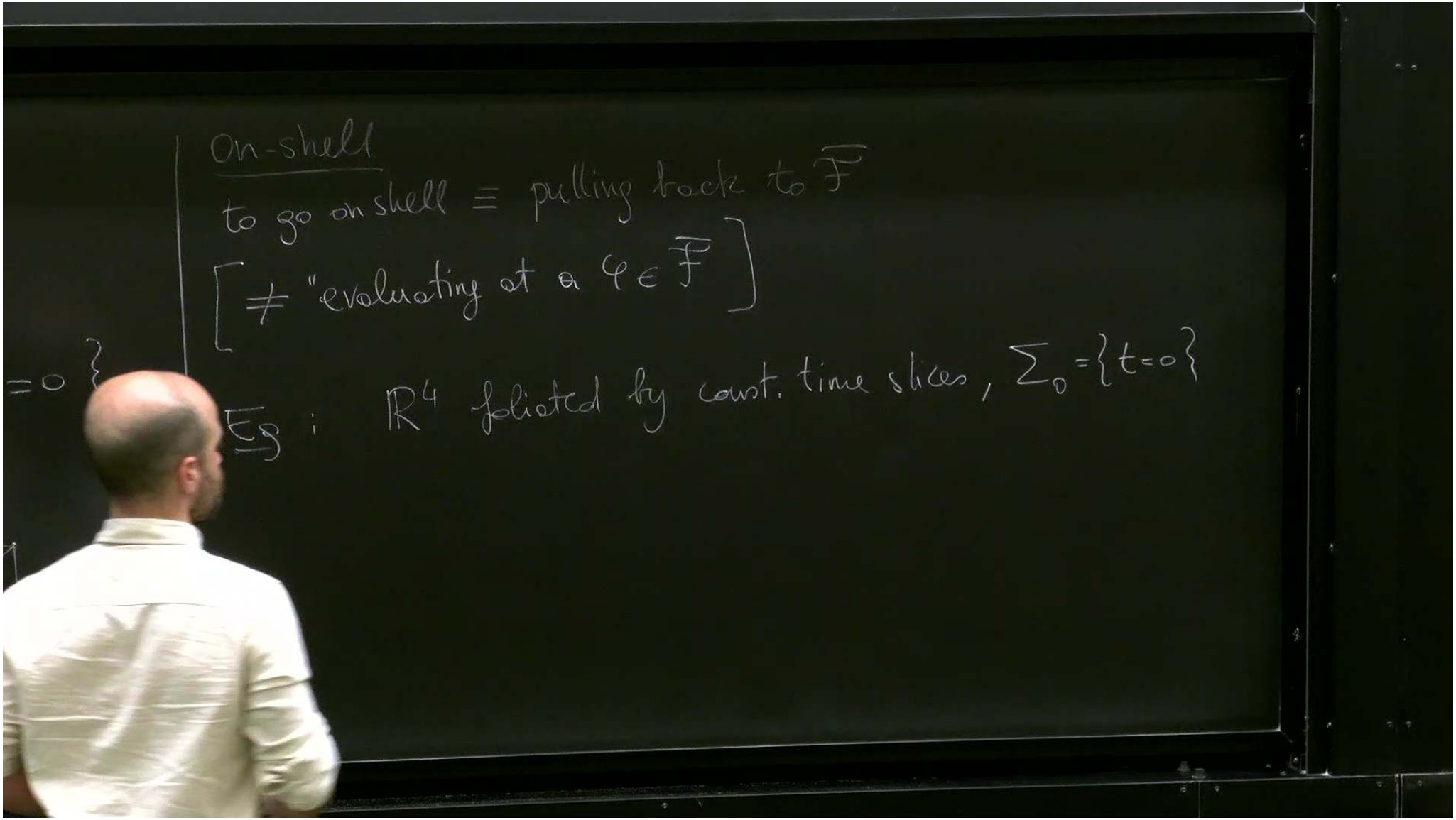
$\underline{\epsilon}_a = i_{\eta_a} \underline{\epsilon}$

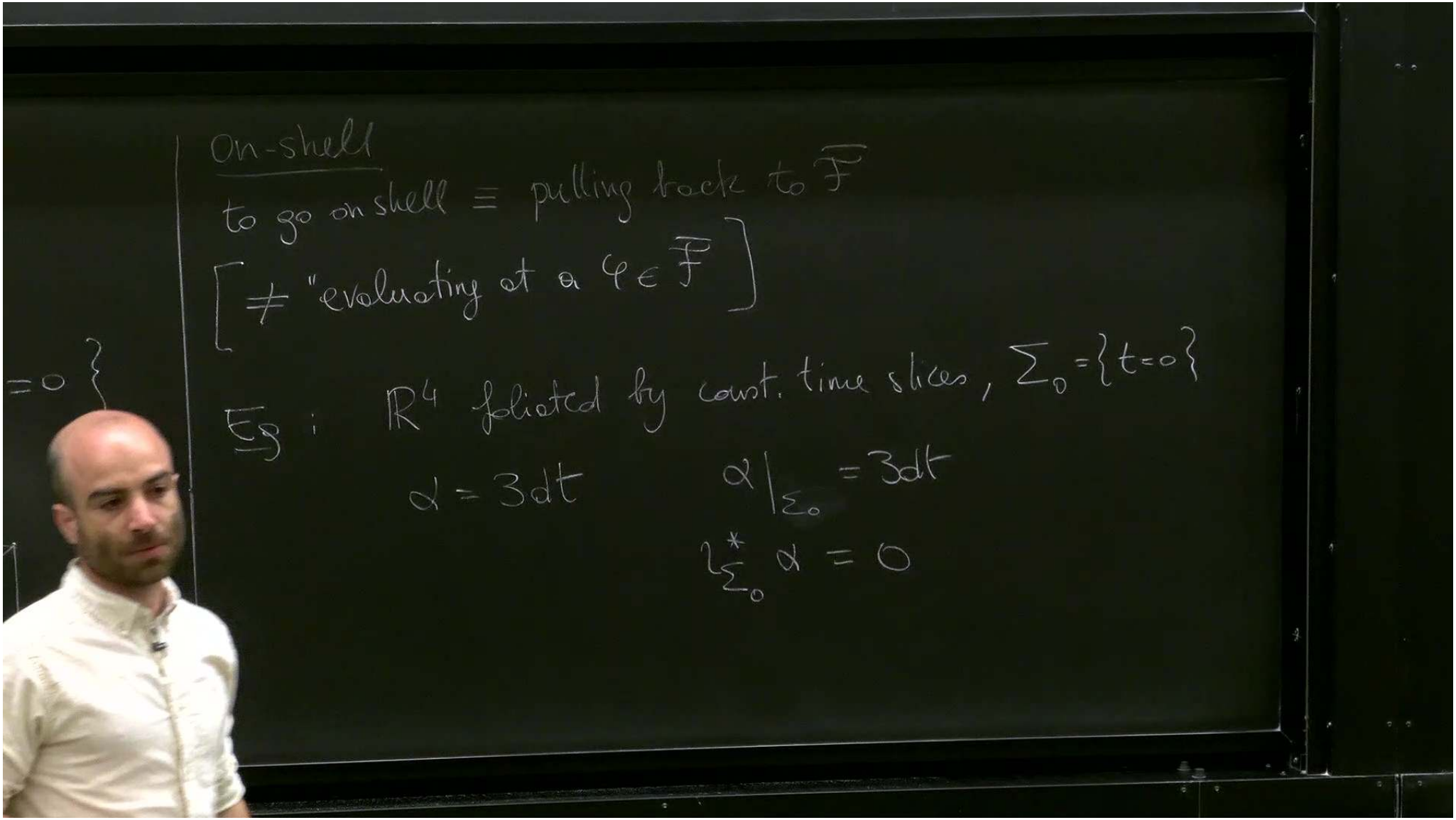
$i_\Sigma^* \underline{\epsilon}_a = \eta_a \underline{\epsilon}_\Sigma$

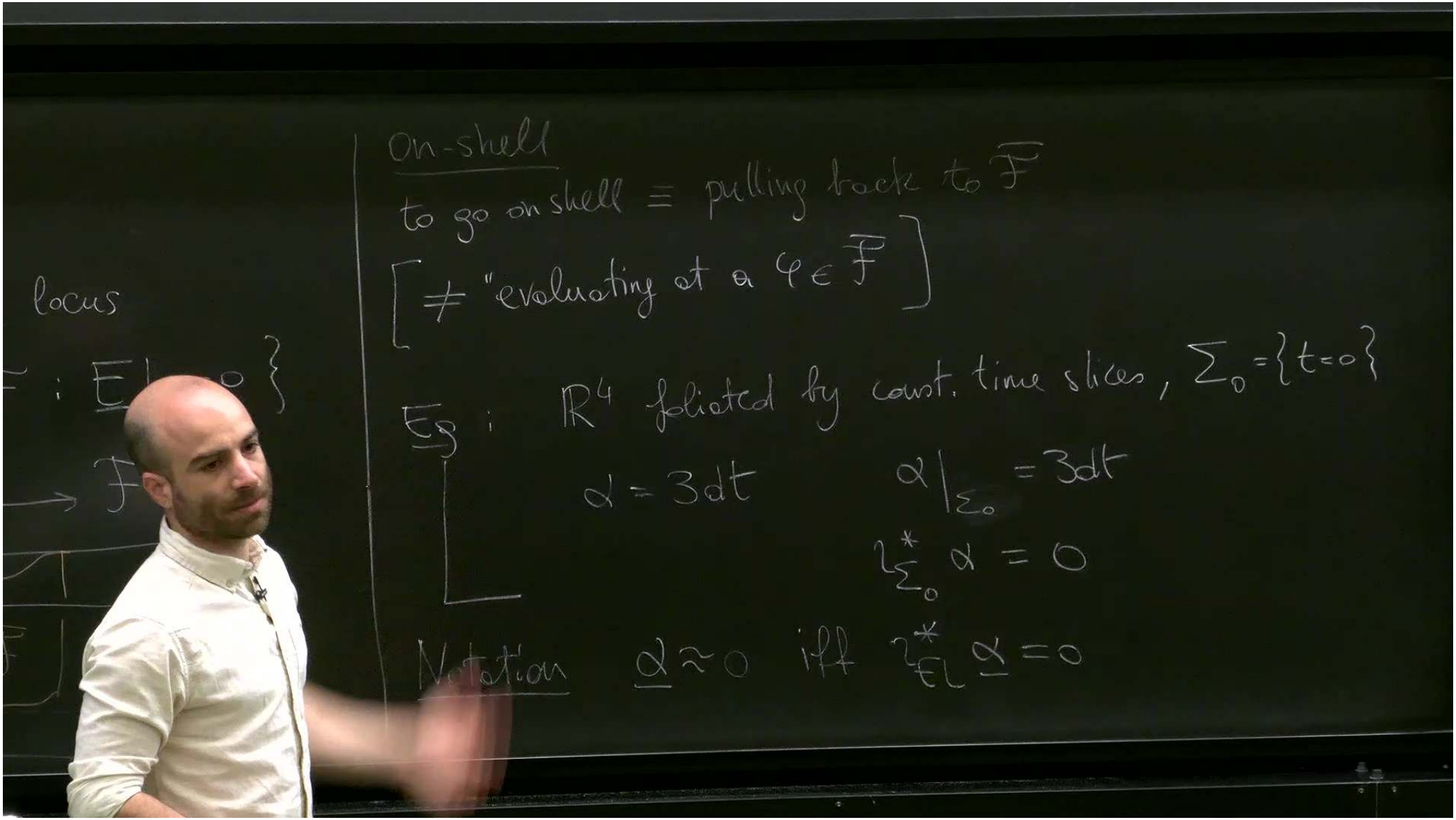
**DEF** Euler lagrange locus

$$\overline{\mathcal{F}} := \{\varphi \in \mathcal{F} : \underline{E}|_\varphi = 0\}$$


current.
$\underline{\epsilon}_a = i_{\gamma_a} \underline{\epsilon}$
$i_\Sigma^* \underline{\epsilon}_a = n_a \underline{\epsilon}_\Sigma$
DEF Euler lagrange locus
$\bar{\mathcal{F}} := \{\varphi \in \mathcal{F} : E|_\varphi = 0\}$
$i_{EL} : \bar{\mathcal{F}} \hookrightarrow \mathcal{F}$
$\mathcal{F}$
$\bar{\mathcal{F}}$
On-shell


On-shell
to go on shell ≡ pulling back to $\bar{F}$
[ ≠ "evaluating at a $\varphi \in \bar{F}$" ]
Eg : $\mathbb{R}^4$ foliated by const. time slices, $\Sigma_0 = \{t=0\}$


On-shell
to go on shell ≡ pulling back to $\bar{F}$
[ ≠ "evaluating at a $\varphi \in \bar{F}$" ]
Eg : $\mathbb{R}^4$ foliated by const. time slices, $\Sigma_0 = \{t=0\}$
$\alpha = 3dt$
$\alpha|_{\Sigma_0} = 3dt$
$i^*_{\Sigma_0} \alpha = 0$


locus
On-shell
to go on shell ≡ pulling back to $\overline{\mathcal{F}}$
[ ≠ "evaluating at a $\varphi \in \overline{\mathcal{F}}$" ]
Eg : $\mathbb{R}^4$ foliated by const. time slices, $\Sigma_0 = \{t=0\}$
$\alpha = 3dt$
$\alpha|_{\Sigma_0} = 3dt$
$\imath^*_{\Sigma_0} \alpha = 0$
Notation $\underline{\alpha} \approx 0$ iff $\imath^*_{EL} \underline{\alpha} = 0$

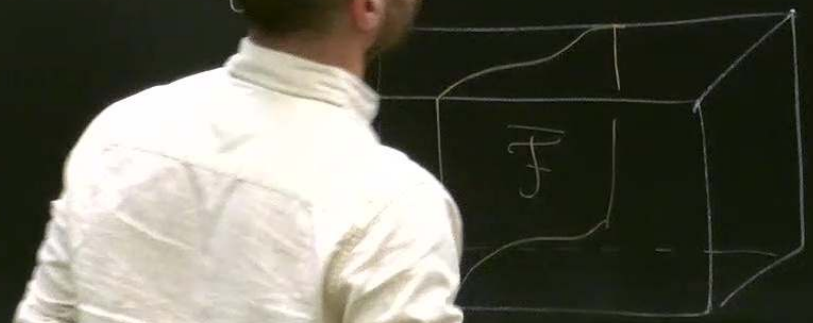
current.

$$\epsilon_a = i_{\partial_a} \epsilon$$
$$\iota_\Sigma^* \epsilon_a = n_a \epsilon_\Sigma$$

DEF Euler-Lagrange locus (the "shell")

$$\bar{\mathcal{F}} := \{\varphi \in \mathcal{F} : E|_\varphi = 0\}$$

$\bar{\mathcal{F}} \hookrightarrow \mathcal{F}$




On-shell

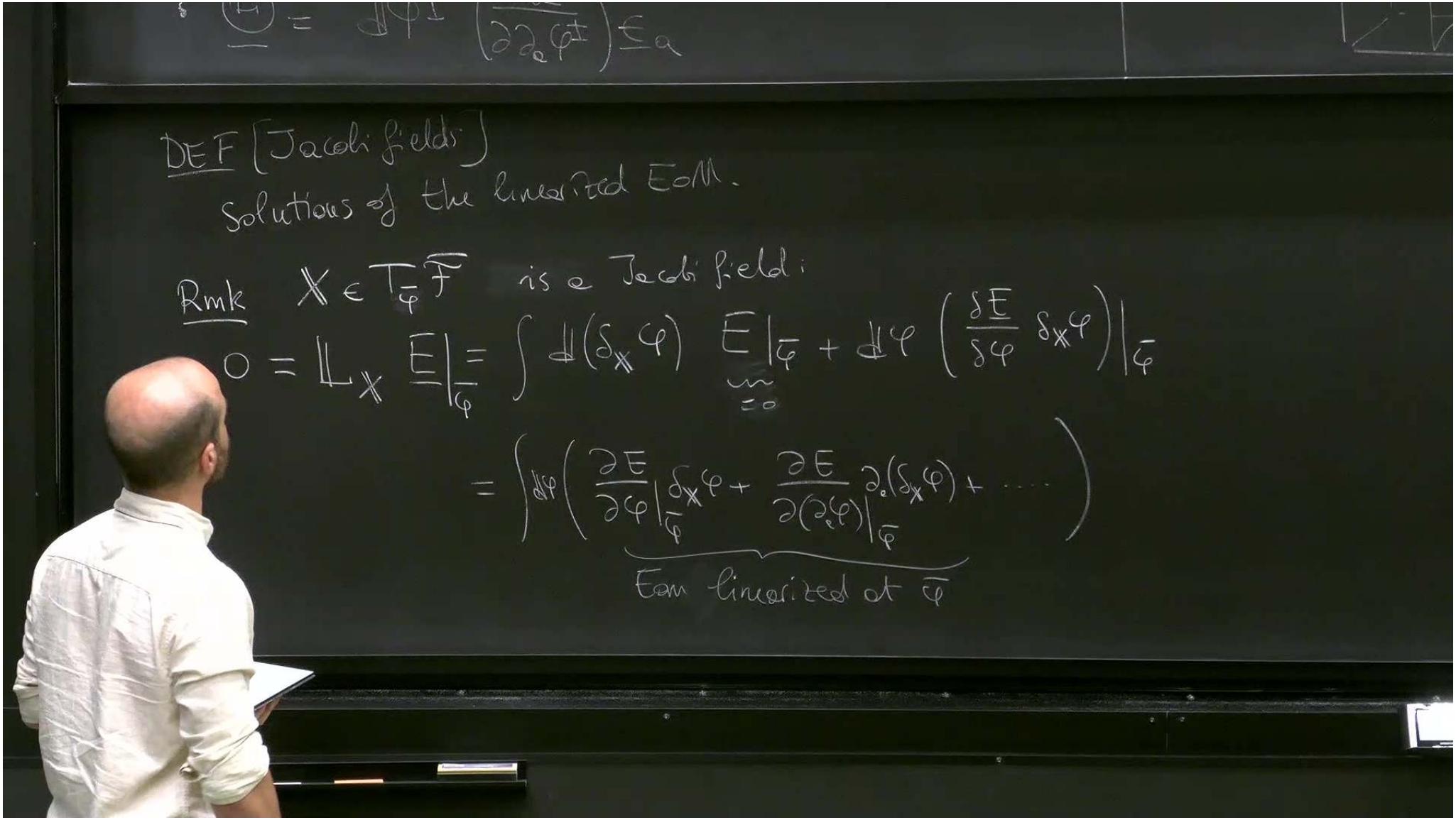
to go on shell ≡ pulling back

[≠ "evaluating at a $\varphi \in \bar{\mathcal{F}}$

$\mathcal{E}_\Sigma$ : $\mathbb{R}^4$ foliated by

$\alpha = 3dt$

Notation $\alpha \approx 0$ iff


DEF [Jacobi fields]
Solutions of the linearized EoM.
Rmk $X \in T_{\bar\varphi}\bar{\mathcal{F}}$ is a Jacobi field:
$0 = \mathbb{L}_X \underline{E}|_{\bar\varphi} = \int d(\delta_X\varphi)\ \underline{\underline{E}}|_{\bar\varphi} + d\varphi \left(\frac{\delta E}{\delta\varphi}\delta_X\varphi\right)\Big|_{\bar\varphi}$
$= \int d\varphi \left( \frac{\partial E}{\partial\varphi}\Big|_{\bar\varphi}\delta_X\varphi + \frac{\partial E}{\partial(\partial_\mu\varphi)}\Big|_{\bar\varphi}\partial_\mu(\delta_X\varphi) + \cdots \right)$
Eom linearized at $\bar\varphi$

$\underline{\Theta} = \int d\varphi^I \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^I}\right) \underline{\epsilon}_\mu$

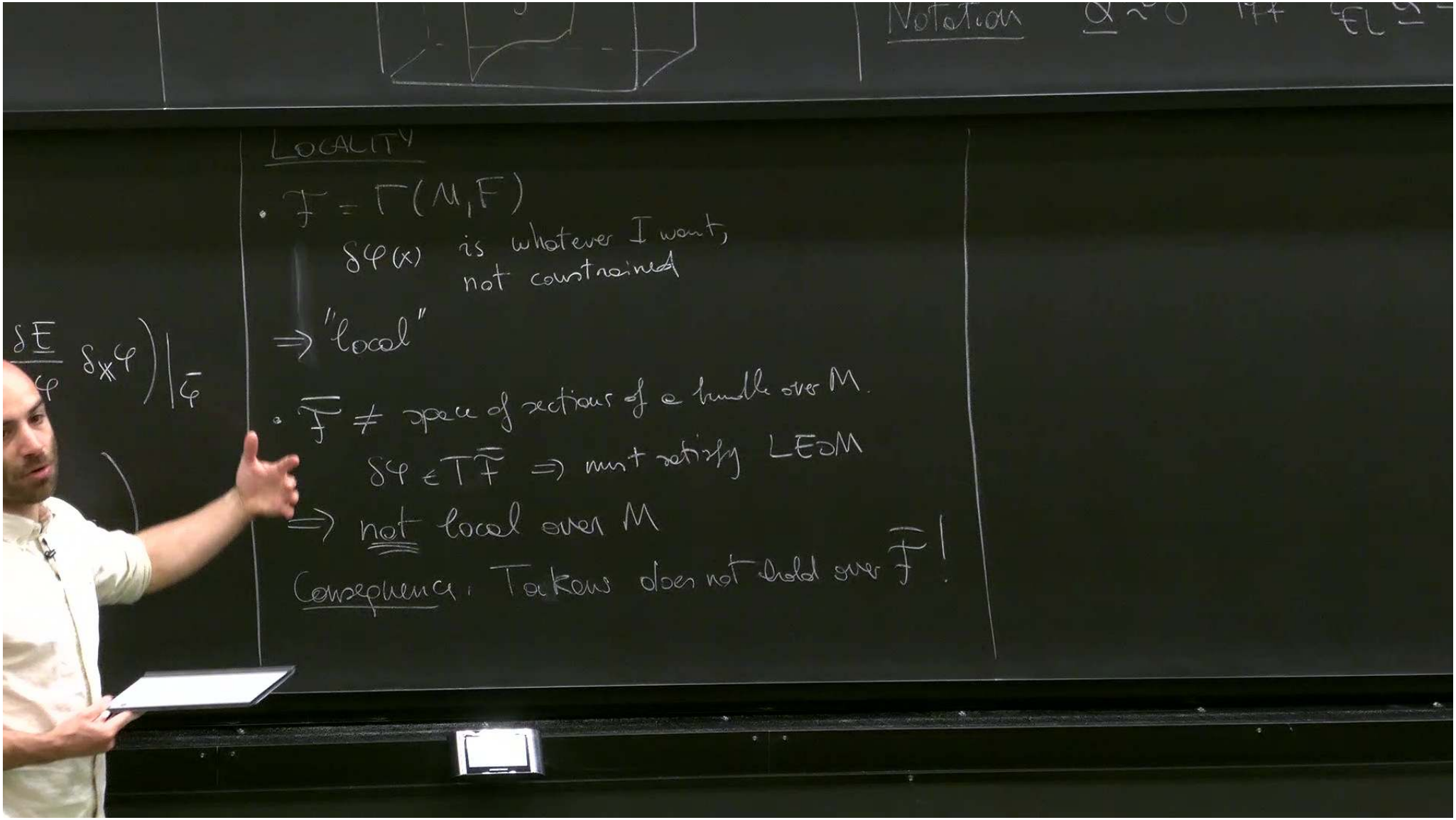
**DEF** [Jacobi fields]
Solutions of the linearized EoM.

**Rmk** $X \in T_{\bar{\varphi}} \bar{\mathcal{F}}$ is a Jacobi field:

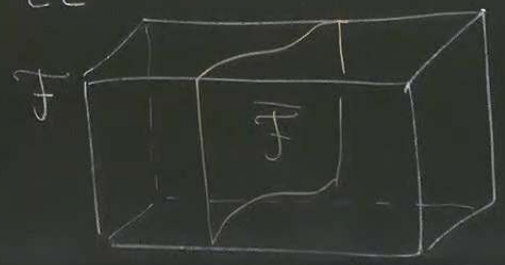
$$0 = \mathbb{L}_X \underline{E}\Big|_{\bar{\varphi}} = \int d(\delta_X \varphi) \underbrace{\underline{E}\Big|_{\bar{\varphi}}}_{=0} + d\varphi \left(\frac{\delta E}{\delta \varphi} \delta_X \varphi\right)\Big|_{\bar{\varphi}}$$

$$= \int d\varphi \Big( \underbrace{\frac{\partial E}{\partial \varphi}\Big|_{\bar{\varphi}} \delta_X \varphi + \frac{\partial E}{\partial(\partial_\mu \varphi)}\Big|_{\bar{\varphi}} \partial_\mu(\delta_X \varphi) + \cdots}_{\text{Eom linearized at } \bar{\varphi}} \Big)$$

**Assumption:**
Converse is also true


Notation
LOCALITY
$\mathcal{F} = \Gamma(M, F)$
$\delta\varphi(x)$ is whatever I want, not constrained
⇒ "local"
$\bar{\mathcal{F}}$ ≠ space of sections of a bundle over M.
$\delta\varphi \in T\bar{\mathcal{F}}$ ⇒ must satisfy LEOM
⇒ not local over M
Consequence: Takens does not hold over $\bar{\mathcal{F}}$!

$\iota_{EL} : \bar{\mathcal{F}} \hookrightarrow \mathcal{F}$




$\alpha = 3dt$ $\quad \alpha|_{\Sigma_0} = 3dt$

$\iota^*_{\Sigma_0} \alpha = 0$

Notation $\underline{\alpha} \approx 0$ iff $\iota^*_{EL} \underline{\alpha} = 0$

$)|_{\bar{\varphi}}$

LOCALITY

- $\mathcal{F} = \Gamma(M, F)$

  $\delta\varphi(x)$ is whatever I want, not constrained

  $\Rightarrow$ "local"

- $\bar{\mathcal{F}} \neq$ space of sections of a bundle over $M$.

  $\delta\varphi \in T\bar{\mathcal{F}} \Rightarrow$ must satisfy LEoM

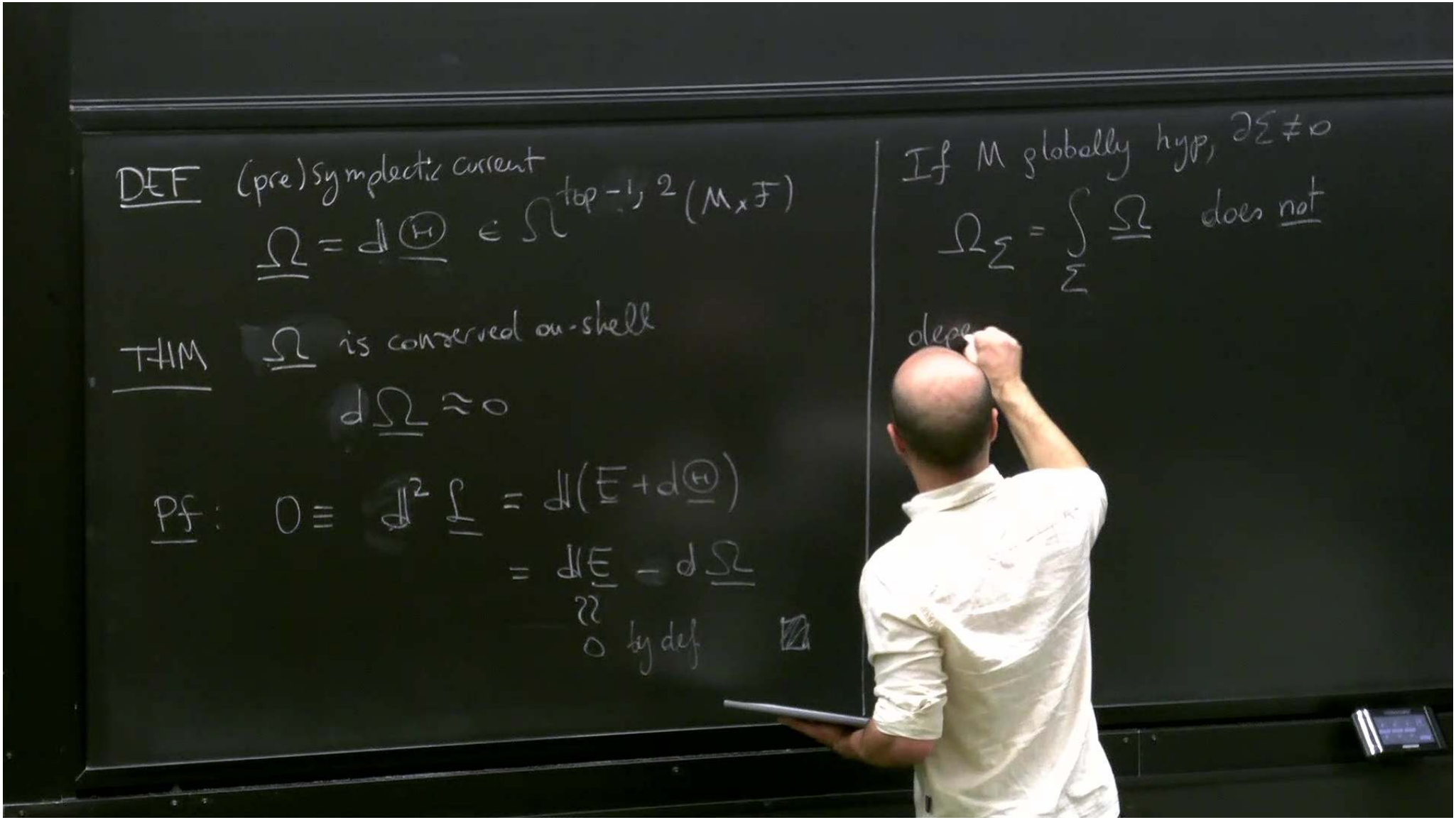
  $\Rightarrow$ not local over $M$

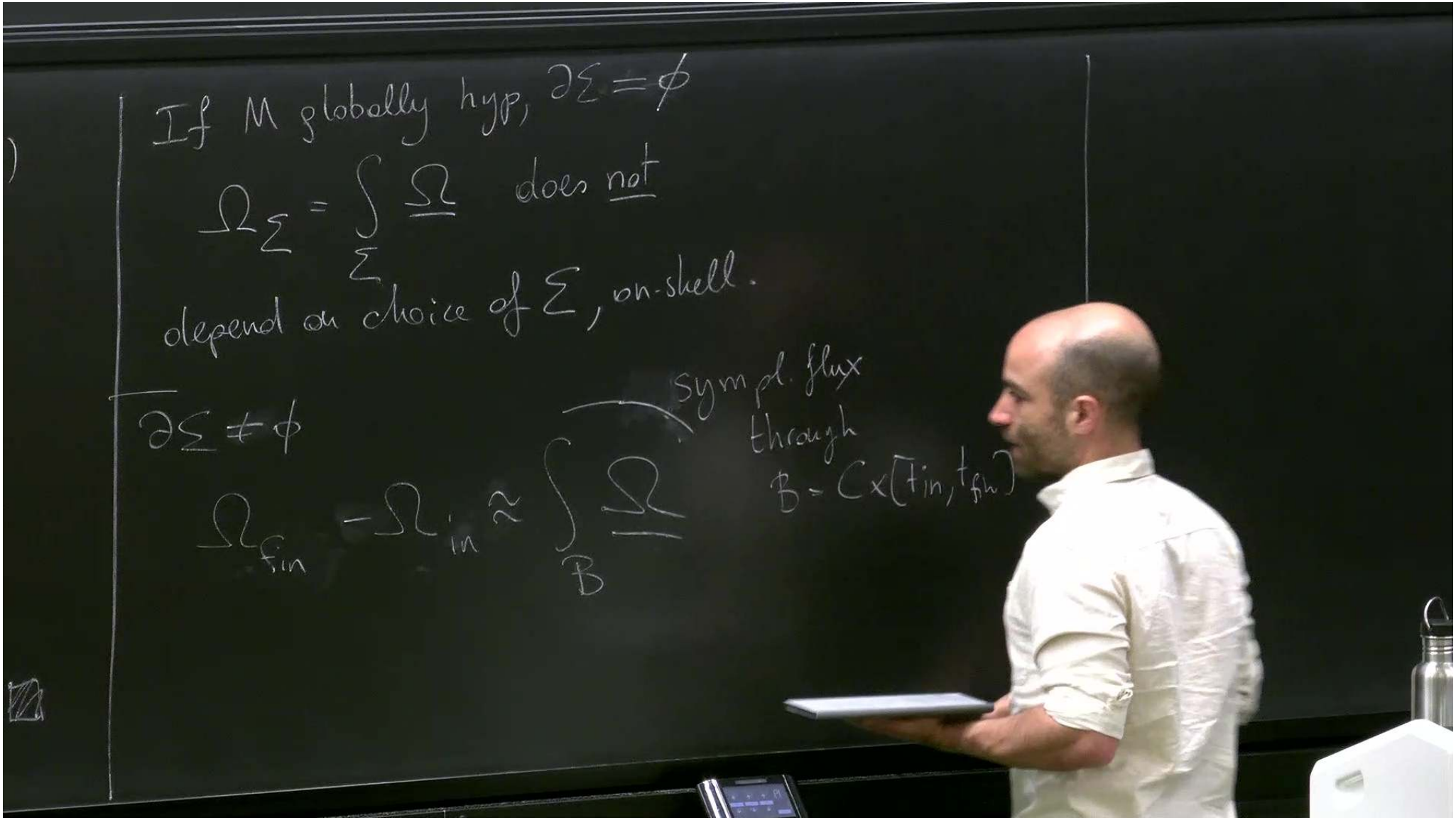
Consequence: Takens does not hold over $\bar{\mathcal{F}}$!

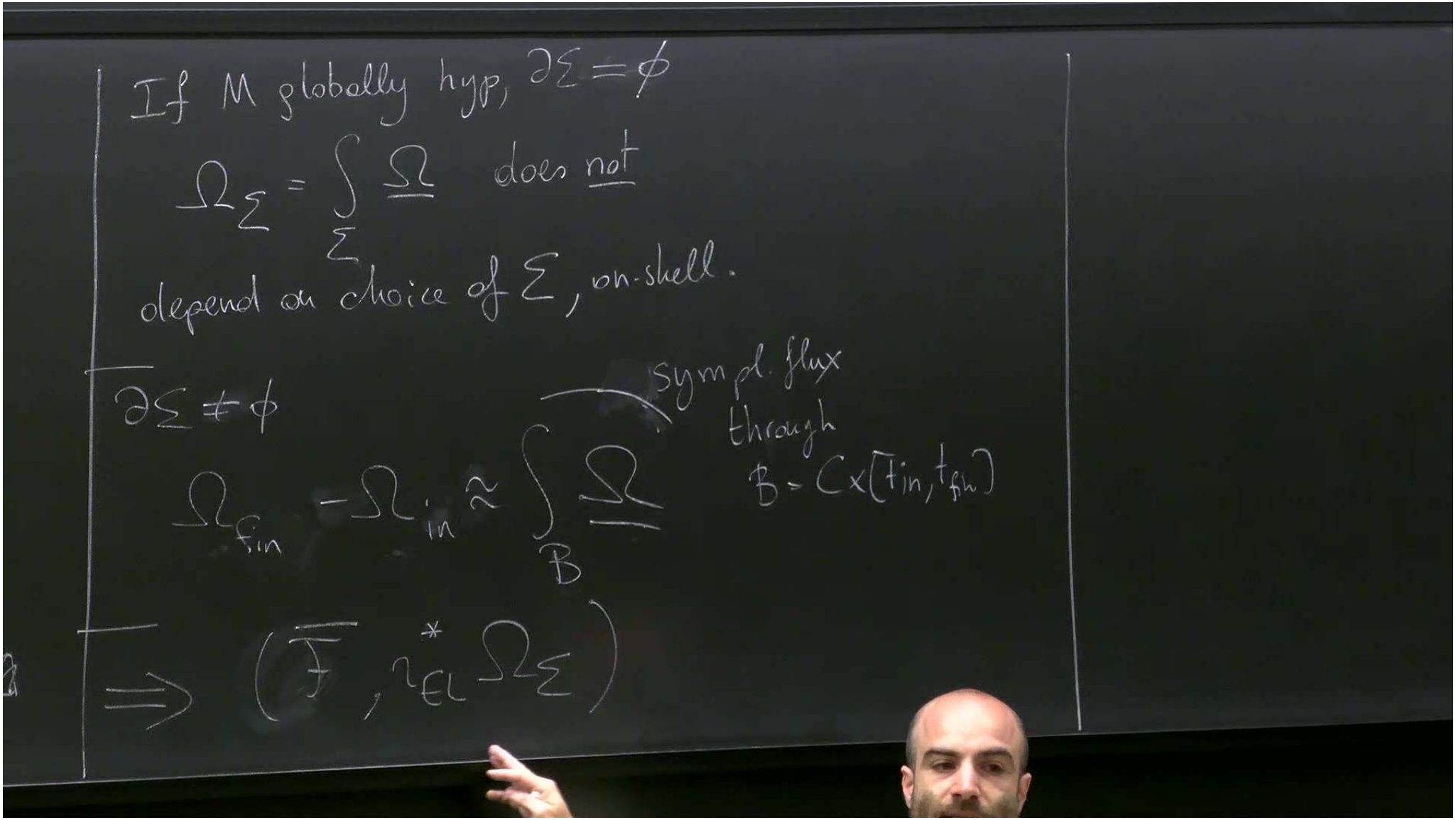
$\bar{\mathcal{F}}$ can still be isomorphic to a local space, just not one over $M$.

Usually: one over $\Sigma$

If 1-to-1 correspondence btw (unconstrained) initial conds at $\Sigma$ and $\bar{\varphi} \in \bar{\mathcal{F}}$ $\quad \uparrow$ "canonical ph. sp."


DEF (pre) symplectic current
$\underline{\Omega} = d\underline{\Theta} \in \Omega^{top-1;2}(M \times \mathcal{F})$
THM $\underline{\Omega}$ is conserved on-shell
$d\underline{\Omega} \approx 0$
Pf: $0 \equiv d^2\underline{L} = d(\underline{E} + d\underline{\Theta})$
$= d\underline{E} - d\underline{\Omega}$
$\approx 0$ by def
If M globally hyp, $\partial\Sigma \neq 0$
$\Omega_\Sigma = \int_\Sigma \underline{\Omega}$ does not


If M globally hyp, $\partial\Sigma = \emptyset$
$\Omega_\Sigma = \int_\Sigma \underline{\Omega}$ does not
depend on choice of $\Sigma$, on-shell.
$\partial\Sigma \neq \emptyset$
$\Omega_{fin} - \Omega_{in} \approx \int_B \underline{\Omega}$
sympl. flux
through
$B = C \times (t_{in}, t_{fin})$


If M globally hyp, $\partial\Sigma = \phi$
$\Omega_\Sigma = \int_\Sigma \underline{\Omega}$ does not
depend on choice of $\Sigma$, on-shell.
$\partial\Sigma \neq \phi$
$\Omega_{fin} - \Omega_{in} \approx \int_B \underline{\Omega}$
sympl. flux through $B = C \times (t_{in}, t_{fin})$
$\Rightarrow (\overline{\mathcal{F}}, \overset{*}{\gamma}_{EL} \Omega_\Sigma)$

If $M$ globally hyp, $\partial\Sigma = \emptyset$

$\Omega_\Sigma = \int_\Sigma \underline{\Omega}$ does not depend on choice of $\Sigma$, on-shell.

$\partial\Sigma \neq \emptyset$

$\Omega_{fin} - \Omega_{in} \approx \int_B \underline{\Omega}$ sympl. flux through $B = C \times (t_{in}, t_{fin})$

$\Rightarrow (\overline{\mathcal{F}}, \overset{*}{\epsilon_L} \Omega_\Sigma) \equiv$ "covariant ph. sp."

## EXAMPLES

1) Particle $\quad \mathcal{L} = \left(\frac{1}{2}\dot{q}^2 - V(q)\right) dt$

$$d\mathcal{L} = dq(t)\,(\text{eom}) + \underbrace{dt}_{=d}\,\partial_t\Big(\underbrace{\mathbb{d}q\;\dot{q}}_{\Theta \in \Omega^{\text{top}-1,1}(M\times\mathbb{R})}\Big)$$

$$\Omega(t) = d\dot{q}\wedge dq = \partial_t\, dq(t)\wedge dq(t)$$

$$\Sigma = \{t_0\} \hookrightarrow M = \mathbb{R}$$

$$\Omega_\Sigma = \mathbb{d}\dot{q}(t_0)\wedge \mathbb{d}q(t_0)$$

At $t=t_0$, $\begin{cases} x \equiv q(t_0) \\ p = \dot{q}(t_0) \end{cases}$

can be specified freely & independently

$$(x,p) \in \mathcal{P} = C^\infty(\underbrace{\Sigma}_{\{t_0\}}, \mathbb{R}^2) = \mathbb{R}^2$$

EXAMPLES

1) Particle $\quad \underline{\mathcal{L}} = \left(\frac{1}{2}\dot{q}^2 - V(q)\right) dt$

$$\mathbb{d}\mathcal{L} = \mathbb{d}q(t)\,(\text{eom}) + \underbrace{dt\,\partial_t}_{\equiv d}\Big(\underbrace{\mathbb{d}q\;\dot{q}}_{\underline{\Theta}\,\in\,\Omega^{\overset{0}{\text{top-1},1}}(M\times\mathcal{F})}\Big) \qquad M \to \mathbb{R}$$

$$\underline{\Omega}(t) = \mathbb{d}\dot{q}\wedge\mathbb{d}q = \partial_t\,\mathbb{d}q(t)\wedge\mathbb{d}q(t)$$

$$\Sigma = \{t_0\} \hookrightarrow M = \mathbb{R}$$

$$\Omega_\Sigma = \mathbb{d}\dot{q}(t_0)\wedge\mathbb{d}q(t_0)$$

At $t = t_0$, $\begin{cases} x \equiv q(t_0) \\ p = \dot{q}(t_0) \end{cases}$

can be specified freely & independently

$$(x,p) \in \mathcal{P} = C^\infty(\underset{\{t_0\}}{\Sigma}, \mathbb{R}^2) = \mathbb{R}^2$$

finite dim

## EXAMPLES

1) Particle $\quad \underline{L} = \left(\frac{1}{2}\dot{q}^2 - V(q)\right) dt$

$$d\underline{L} = dq(t)\,(\text{eom}) + \underbrace{dt\,\partial_t}_{=d}\Big(\underbrace{dq\,\dot{q}}_{\underline{\Theta} \in \Omega^{\text{top}-1,1}(M\times\mathbb{R})}\Big)$$

$$\underline{\Omega}(t) = d\dot{q}\wedge dq \overset{!}{=} \partial_t\, dq(t)\wedge dq(t)$$

$$\Sigma = \{t_0\} \hookrightarrow M = \mathbb{R}$$

$$\Omega_\Sigma = d\dot{q}(t_0)\wedge dq(t_0)$$

At $t=t_0$, $\begin{cases} x := q(t_0) \\ p := \dot{q}(t_0) \end{cases}$

can be specified freely & independently

$$(x,p) \in \mathcal{P} = C^\infty(\underbrace{\Sigma}_{\{t_0\}}, \mathbb{R}^2) = \mathbb{R}^2$$

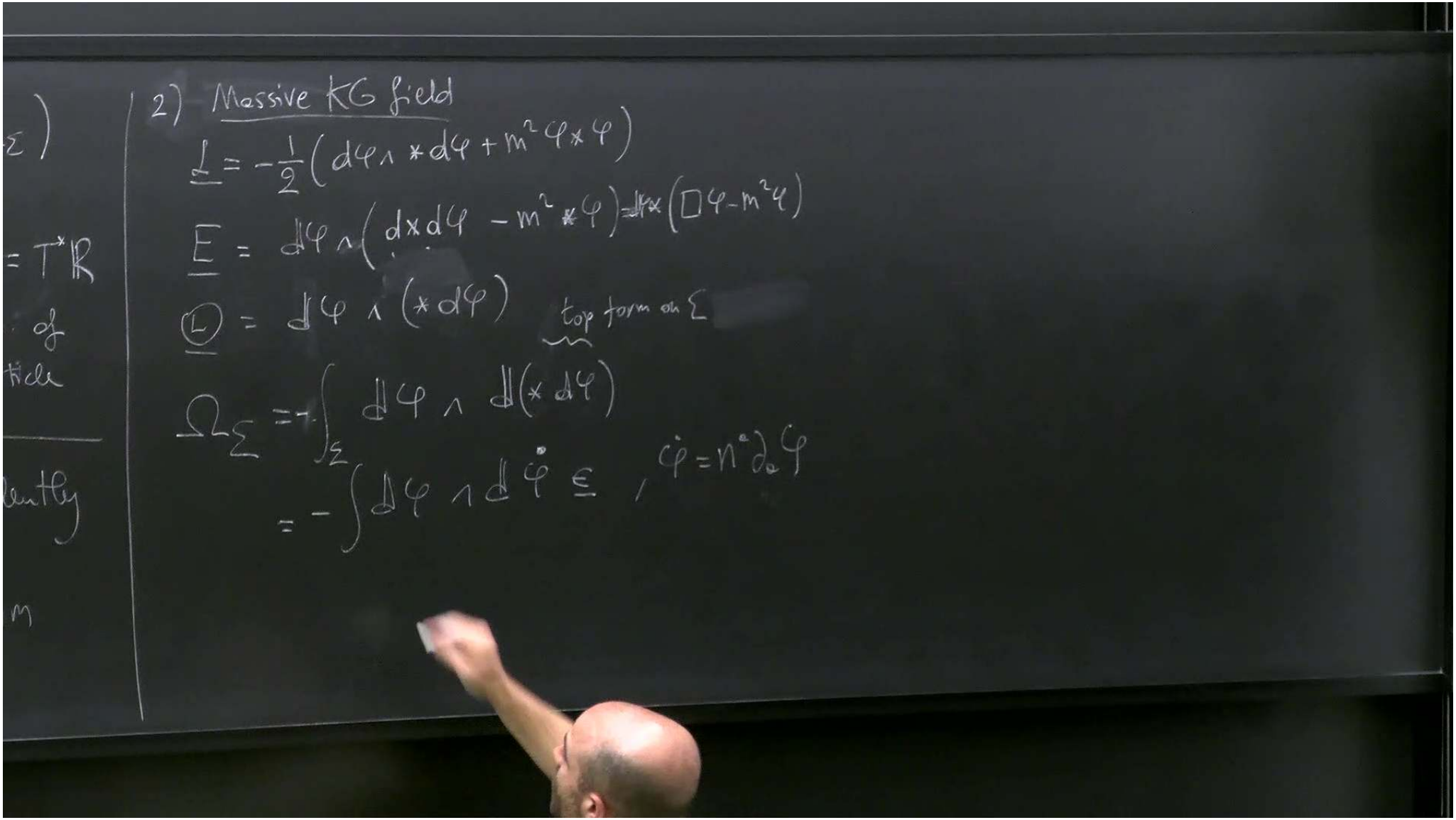
finite dim

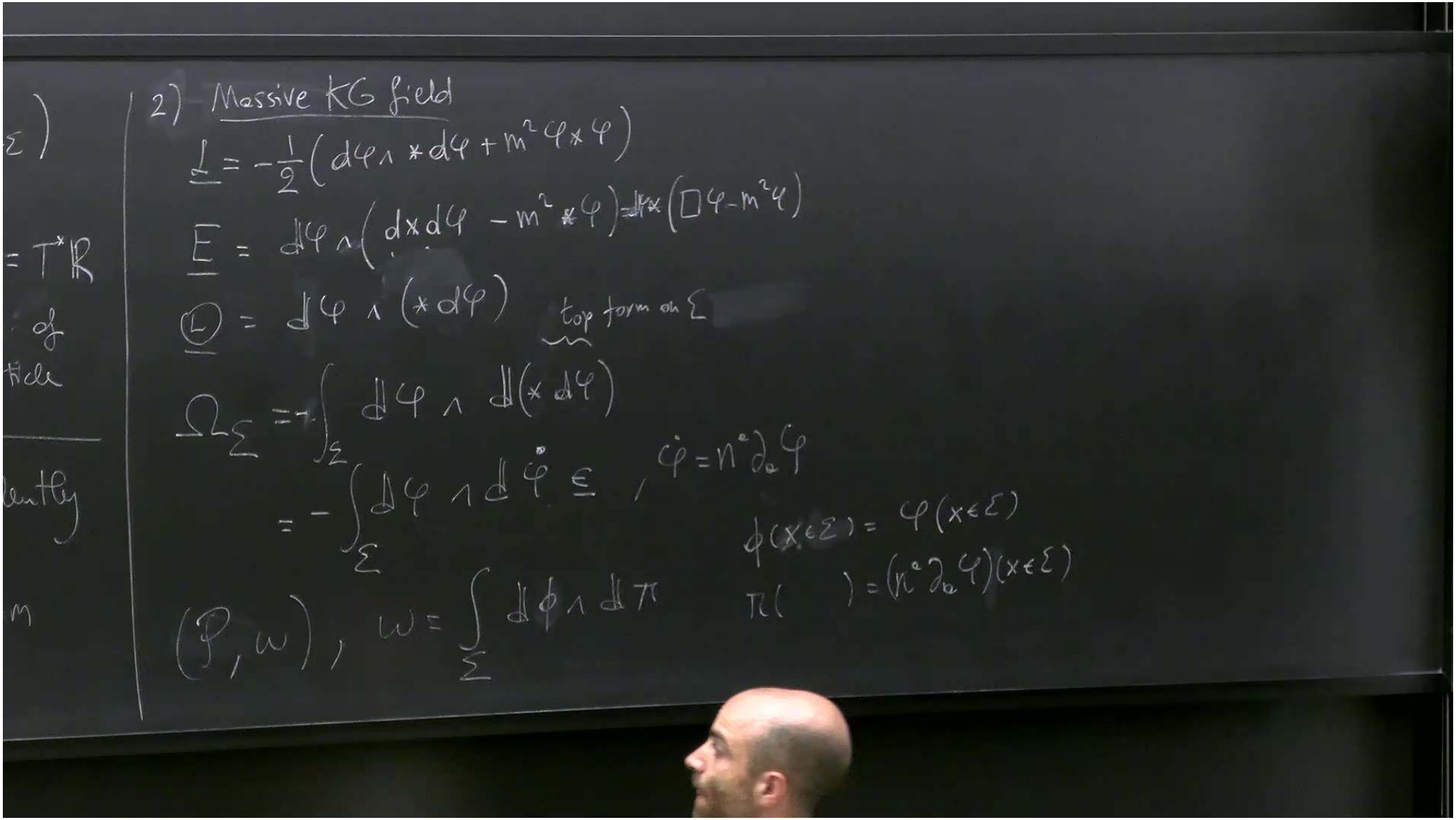
$(\bar{\mathcal{F}}, \bar{t}_{EL}^{\,*}\Omega_\Sigma)$

$\mathbb{R}$

$(\mathcal{P}, \omega) = T^*\mathbb{R}$

symplectic mfd of point particle


2) Massive KG field
$\mathcal{L} = -\frac{1}{2}(d\varphi \wedge *d\varphi + m^2 \varphi * \varphi)$
$E = d\varphi \wedge (d*d\varphi - m^2 * \varphi) = d^4x\,(\Box\varphi - m^2\varphi)$
$\mathcal{L} = d\varphi \wedge (*d\varphi)$ top form on $\Sigma$
$\Omega_\Sigma = \int_\Sigma d\varphi \wedge \delta(*d\varphi)$
$= -\int \delta\varphi \wedge d\dot{\varphi}$ $\in$ , $\dot{\varphi} = n^a \partial_a \varphi$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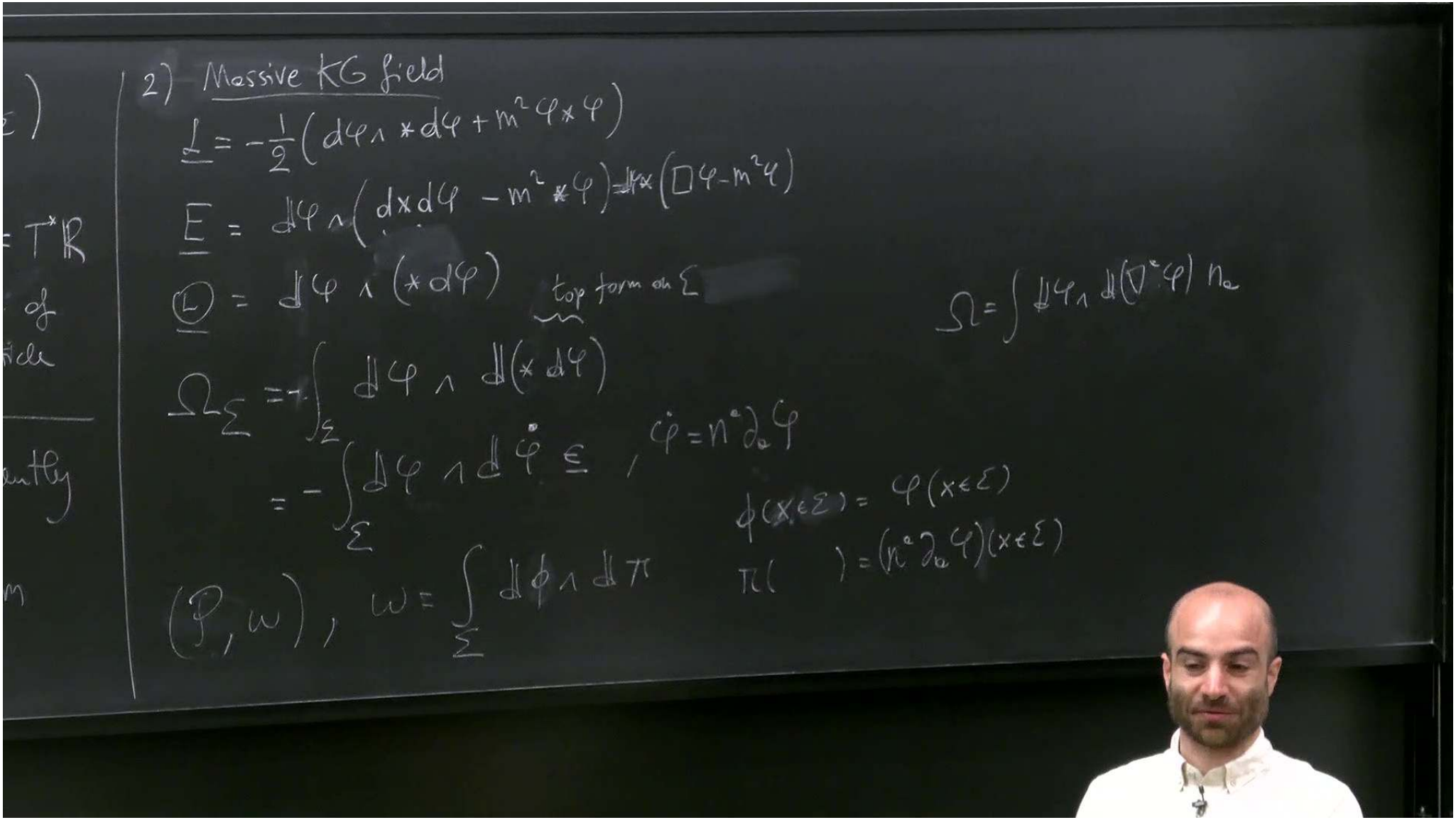
$\underline{L} = -\frac{1}{2}(d\varphi \wedge *d\varphi + m^2 \varphi * \varphi)$
$\underline{E} = d\varphi \wedge (d*d\varphi - m^2 * \varphi) = d*(\Box\varphi - m^2\varphi)$
$\underline{(L)} = d\varphi \wedge (*d\varphi)$ top form on $\Sigma$
$\Omega_\Sigma = \int_\Sigma d\varphi \wedge d(*d\varphi)$
$= -\int_\Sigma d\varphi \wedge d\dot{\varphi}$ $\in$ , $\dot{\varphi} = n^a \partial_a \varphi$
$\phi(x \in \Sigma) = \varphi(x \in \Sigma)$
$\pi(\quad) = (n^a \partial_a \varphi)(x \in \Sigma)$
$(\mathcal{P}, \omega)$, $\omega = \int_\Sigma d\phi \wedge d\pi$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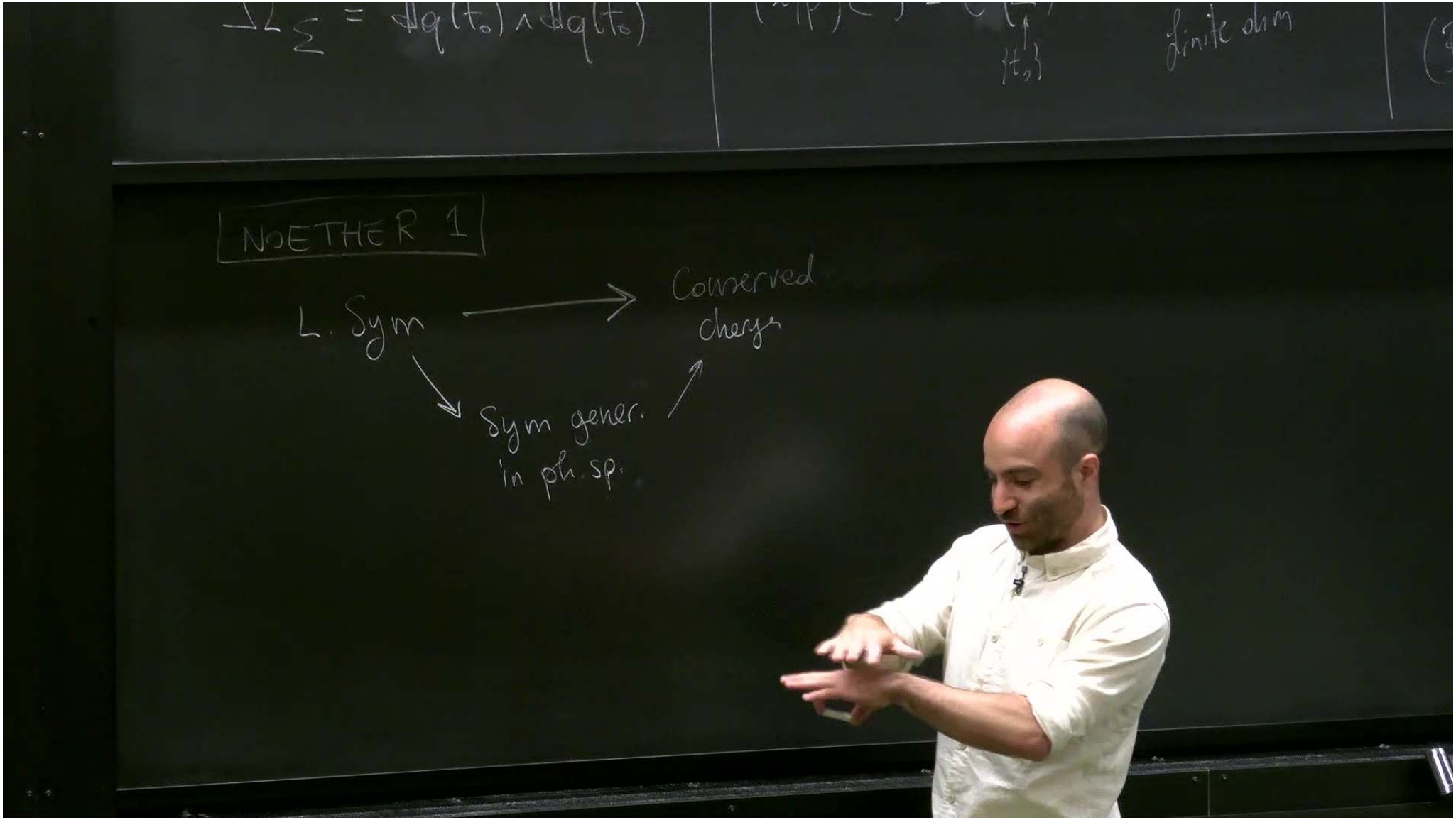
$\underline{L} = -\frac{1}{2}(d\varphi \wedge * d\varphi + m^2 \varphi * \varphi)$
$\underline{E} = d\varphi \wedge (d * d\varphi - m^2 * \varphi) = d\varphi * (\Box\varphi - m^2\varphi)$
$\underline{\Theta} = d\varphi \wedge (* d\varphi)$
top form on $\Sigma$
$\Omega = \int d\varphi \wedge d(\nabla^a \varphi)\, n_a$
$\Omega_\Sigma = -\int_\Sigma d\varphi \wedge d(* d\varphi)$
$= -\int_\Sigma d\varphi \wedge d\dot{\varphi}\, \underline{\epsilon}$ , $\dot{\varphi} = n^a \partial_a \varphi$
$\phi(x \in \Sigma) = \varphi(x \in \Sigma)$
$\pi(\quad) = (n^a \partial_a \varphi)(x \in \Sigma)$
$(\mathcal{P}, \omega)$, $\omega = \int_\Sigma d\phi \wedge d\pi$


NOETHER 1
L. Sym
Conserved charge
Sym gener. in ph. sp.