Title: Quantum Gravity Lecture

Speakers: Aldo Riello

Collection: Quantum Gravity 2023/24

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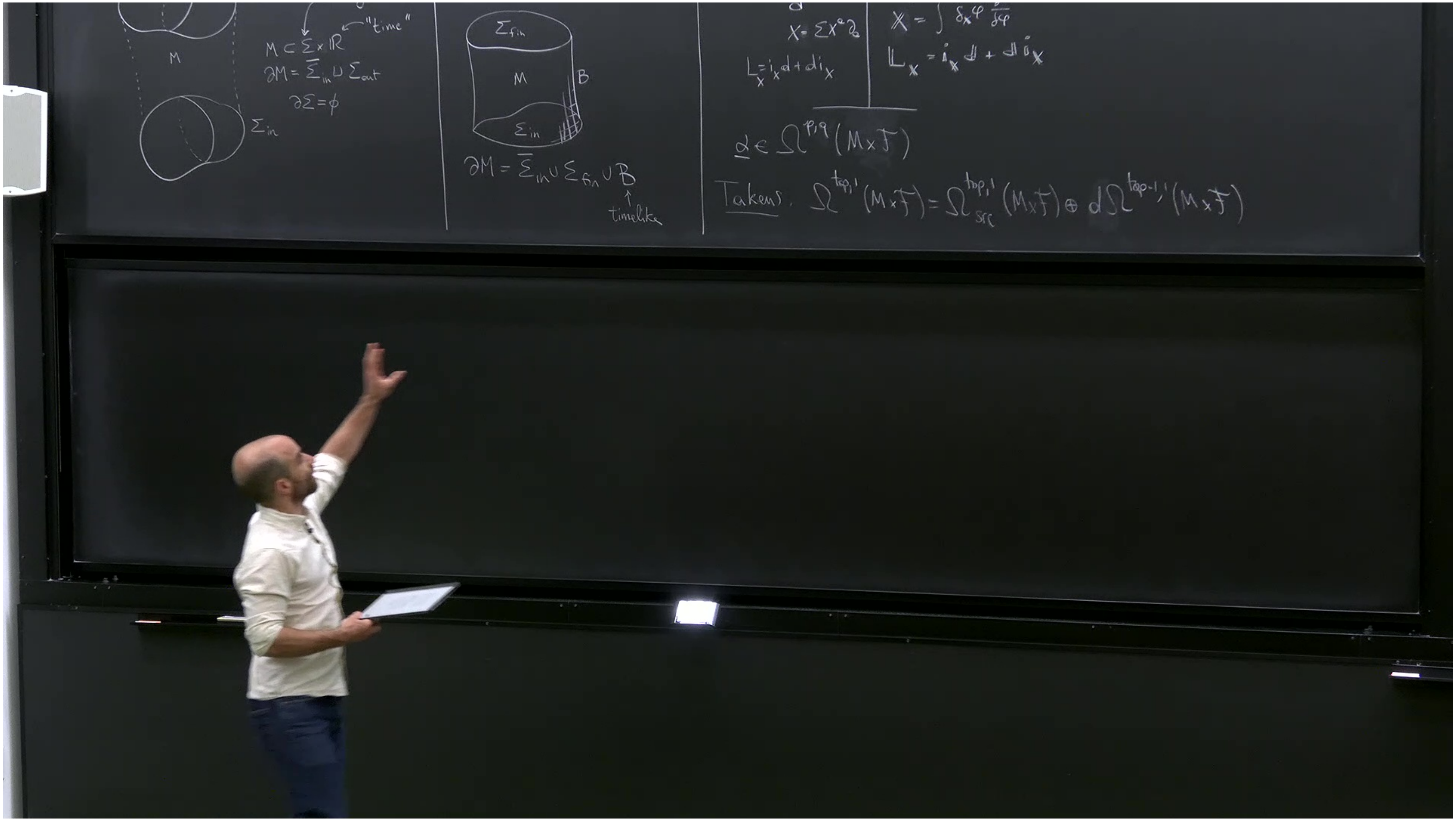
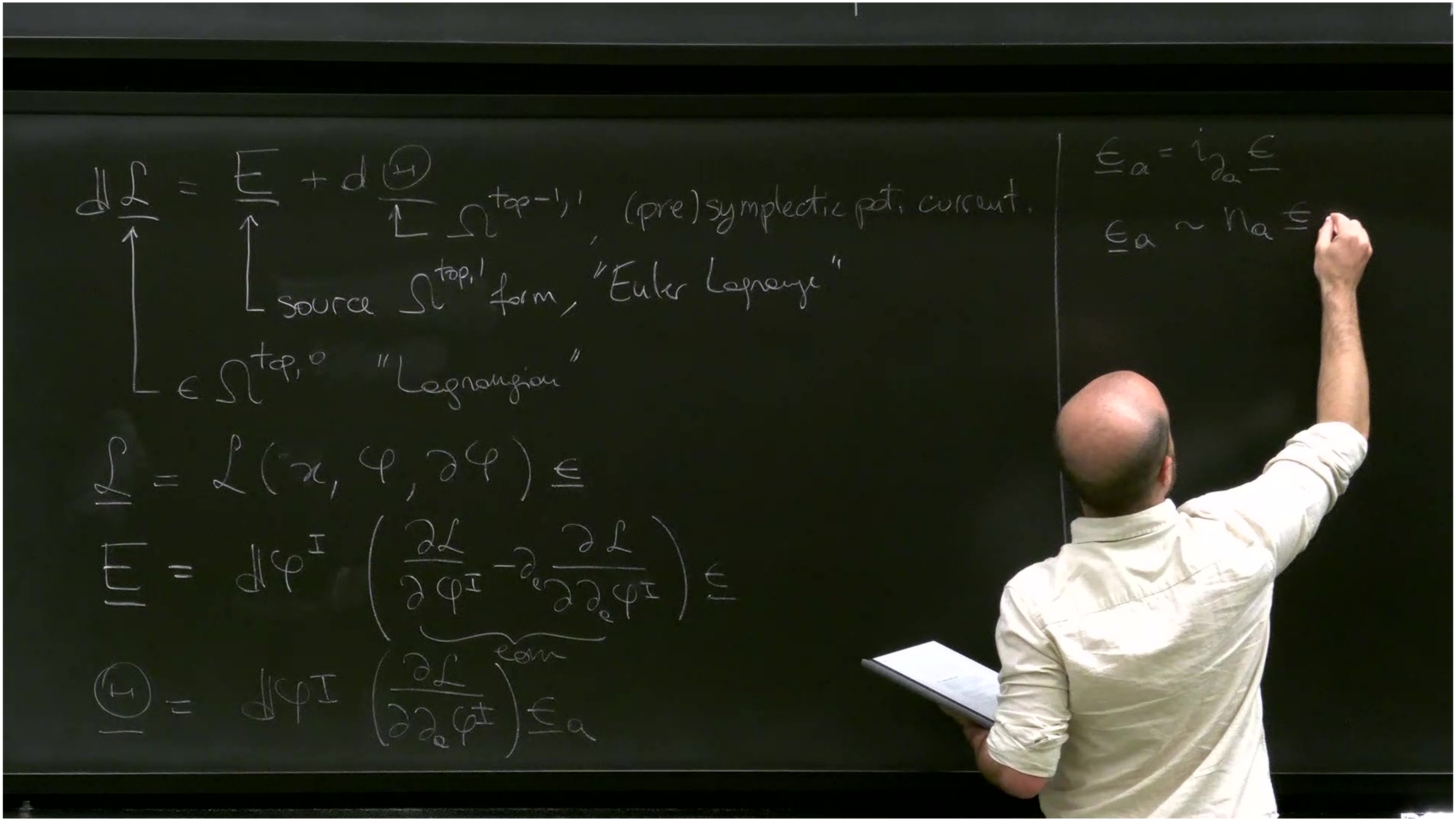

"time"
$M \subset \Sigma \times \mathbb{R}$
$\partial M = \overline{\Sigma}_{in} \cup \Sigma_{out}$
$\partial \Sigma = \phi$
M
Σ_{in}
Σ_{fin}
M
B
Σ_{in}
$\partial M = \overline{\Sigma}_{in} \cup \Sigma_{fin} \cup B$
timelike
$X = \Sigma X^a \partial_a$
$L_X = i_X d + d i_X$
$\mathbb{X} = \int \delta_X \varphi \frac{\delta}{\delta \varphi}$
$\mathbb{L}_X = i_{\mathbb{X}} \mathbb{d} + \mathbb{d} i_{\mathbb{X}}$
$\alpha \in \Omega^{p,q}(M \times F)$
Takens: $\Omega^{top,1}(M \times F) = \Omega^{top,1}_{src}(M \times F) \oplus d\Omega^{top-1,1}(M \times F)$


dL = E + dΘ
Ω^top−1,1 (pre)symplectic pot. current.
source Ω^top,1 form, "Euler Lagrange"
∈ Ω^top,0 "Lagrangian"
L = L(x, φ, ∂φ) ε
E = dφ^I (∂L/∂φ^I − ∂_e ∂L/∂∂_eφ^I) ε
eom
Θ = dφ^I (∂L/∂∂_eφ^I) ε_a
ε_a = i_∂a ε
ε_a ~ n_a ε

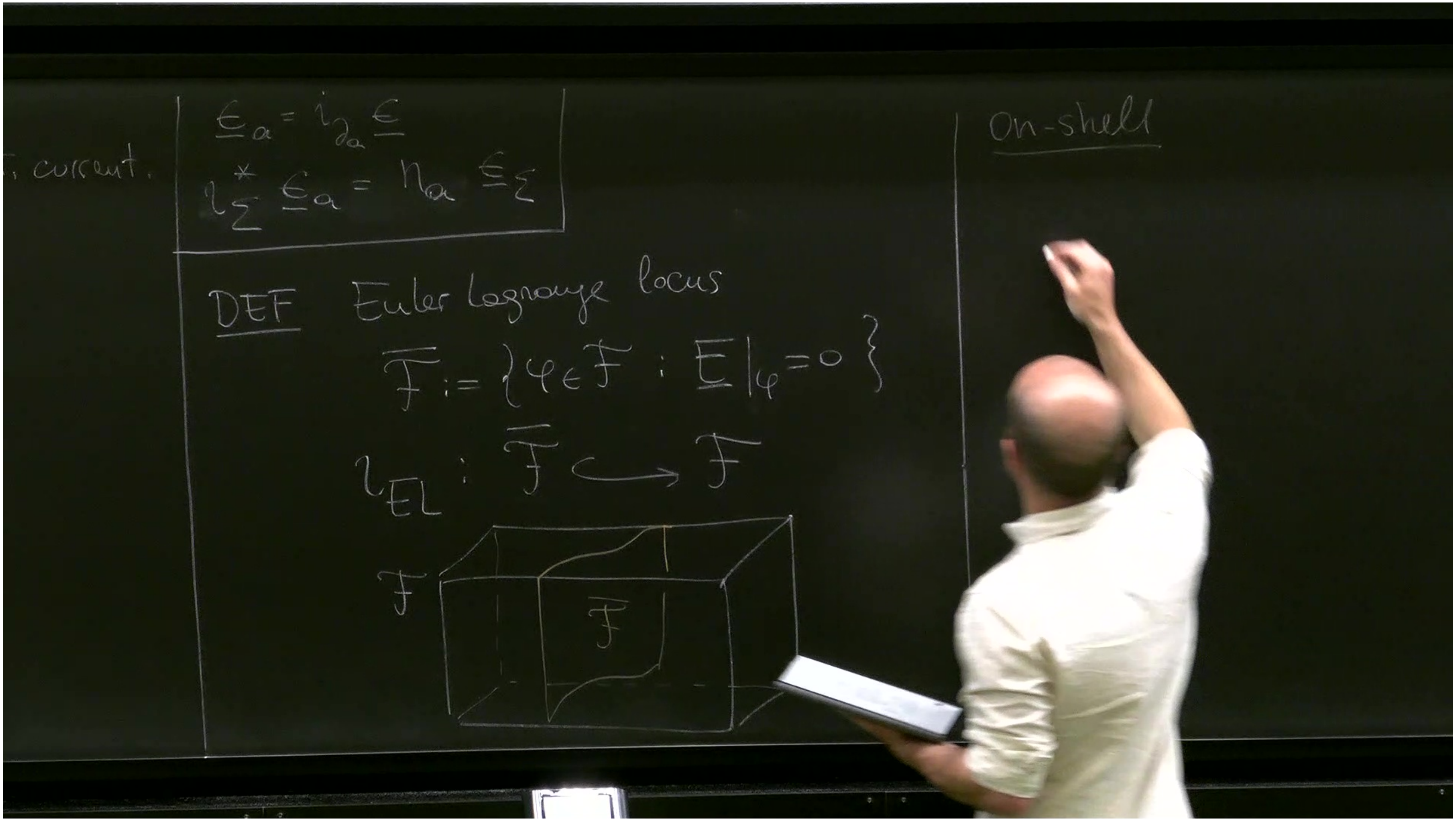
...t. current.

$$\underline{\epsilon}_a = i_{\partial_a} \underline{\epsilon}$$

$$i_\Sigma^* \underline{\epsilon}_a = n_a \underline{\epsilon}_\Sigma$$

DEF Euler Lagrange locus

$$\overline{F} := \{ \varphi \in F : \underline{E}|_\varphi = 0 \}$$


current.
$\underline{\epsilon}_a = i_{\partial_a} \underline{\epsilon}$
$\iota_\Sigma^* \underline{\epsilon}_a = n_a \underline{\epsilon}_\Sigma$
DEF Euler Lagrange locus
$\overline{\mathcal{F}} := \{ \varphi \in \mathcal{F} : \underline{E}|_\varphi = 0 \}$
$\iota_{EL} : \overline{\mathcal{F}} \hookrightarrow \mathcal{F}$
$\mathcal{F}$
$\overline{\mathcal{F}}$
On-shell

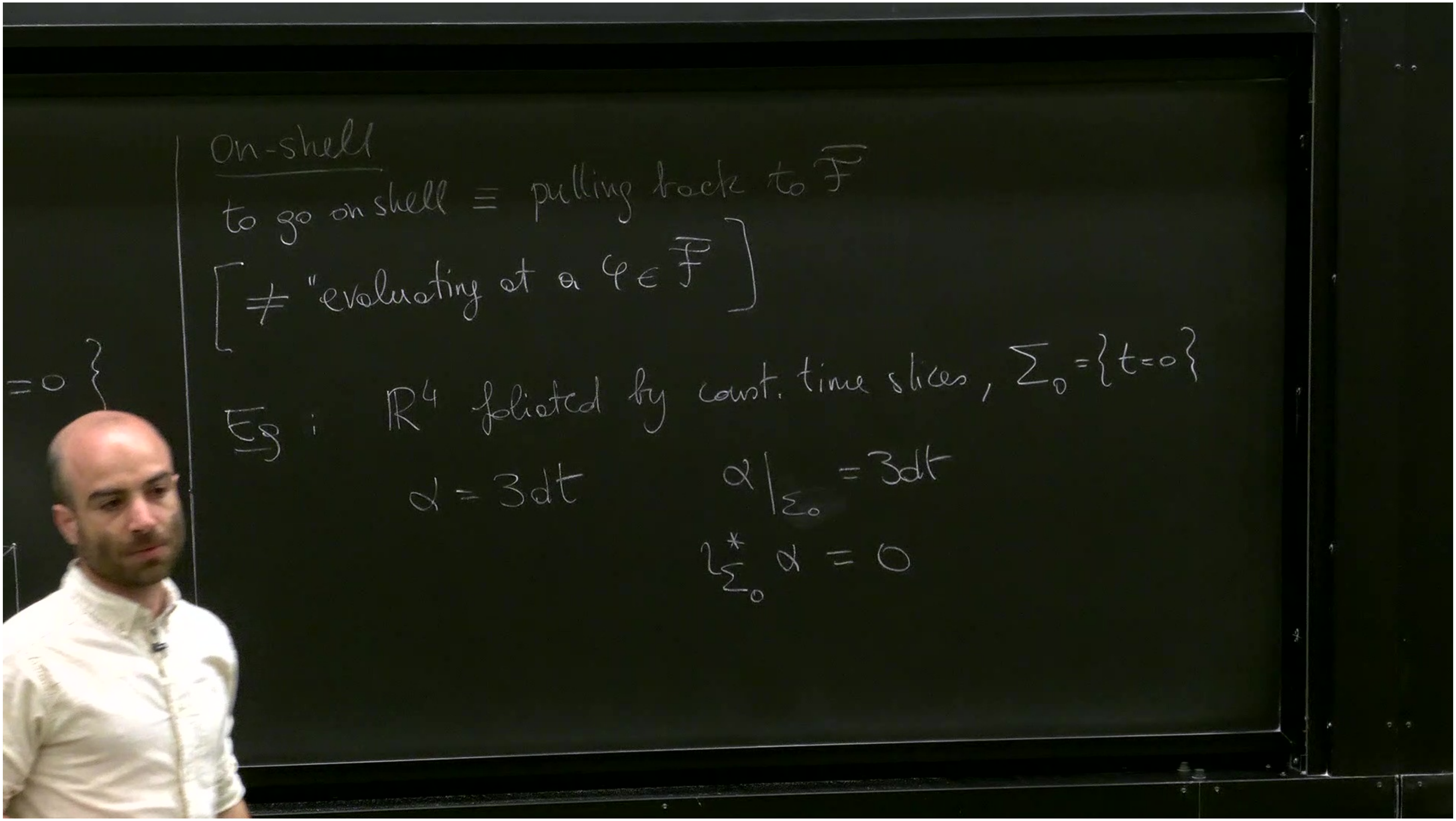
On-shell

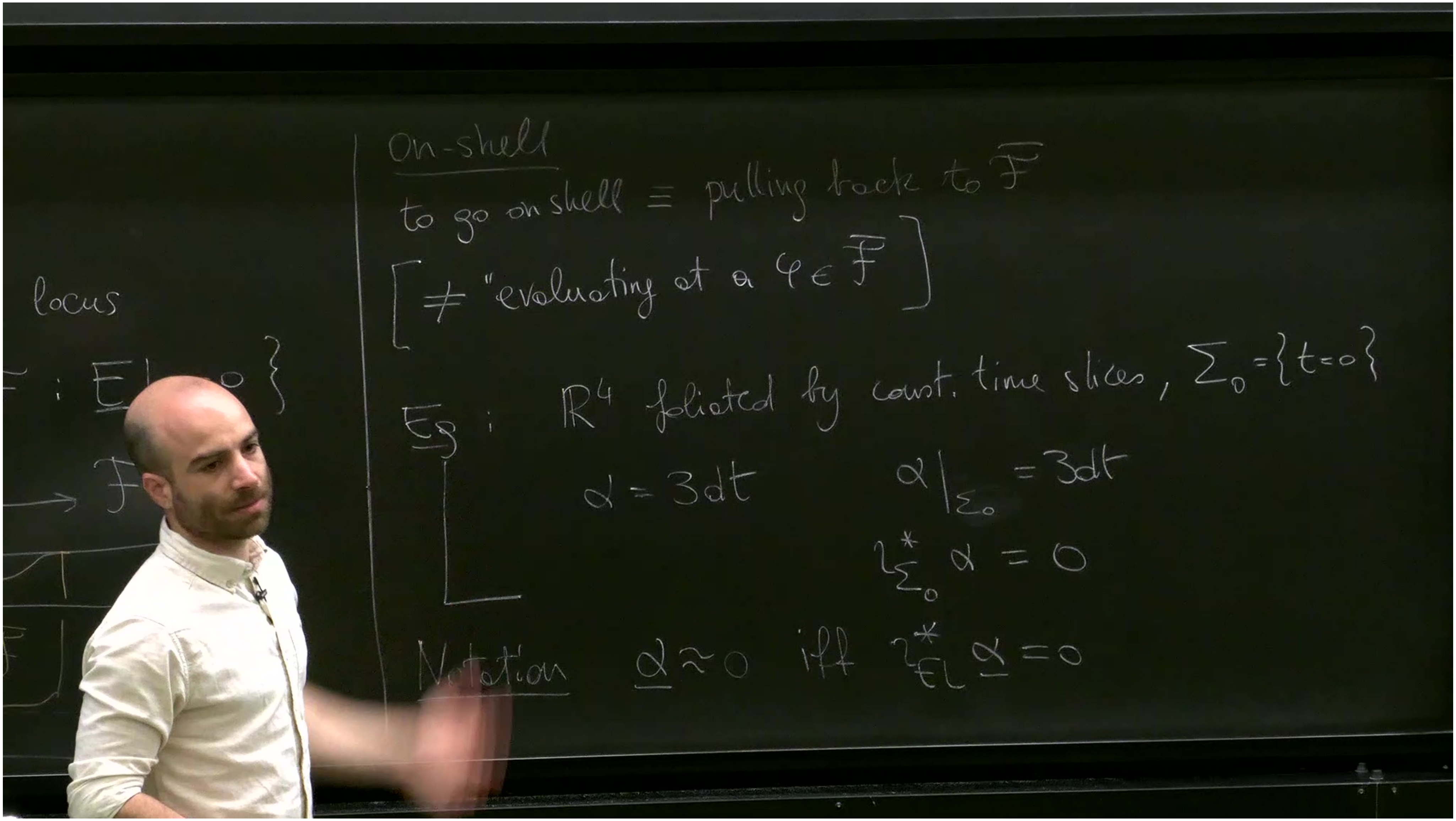
to go on shell $\equiv$ pulling back to $\mathcal{F}$

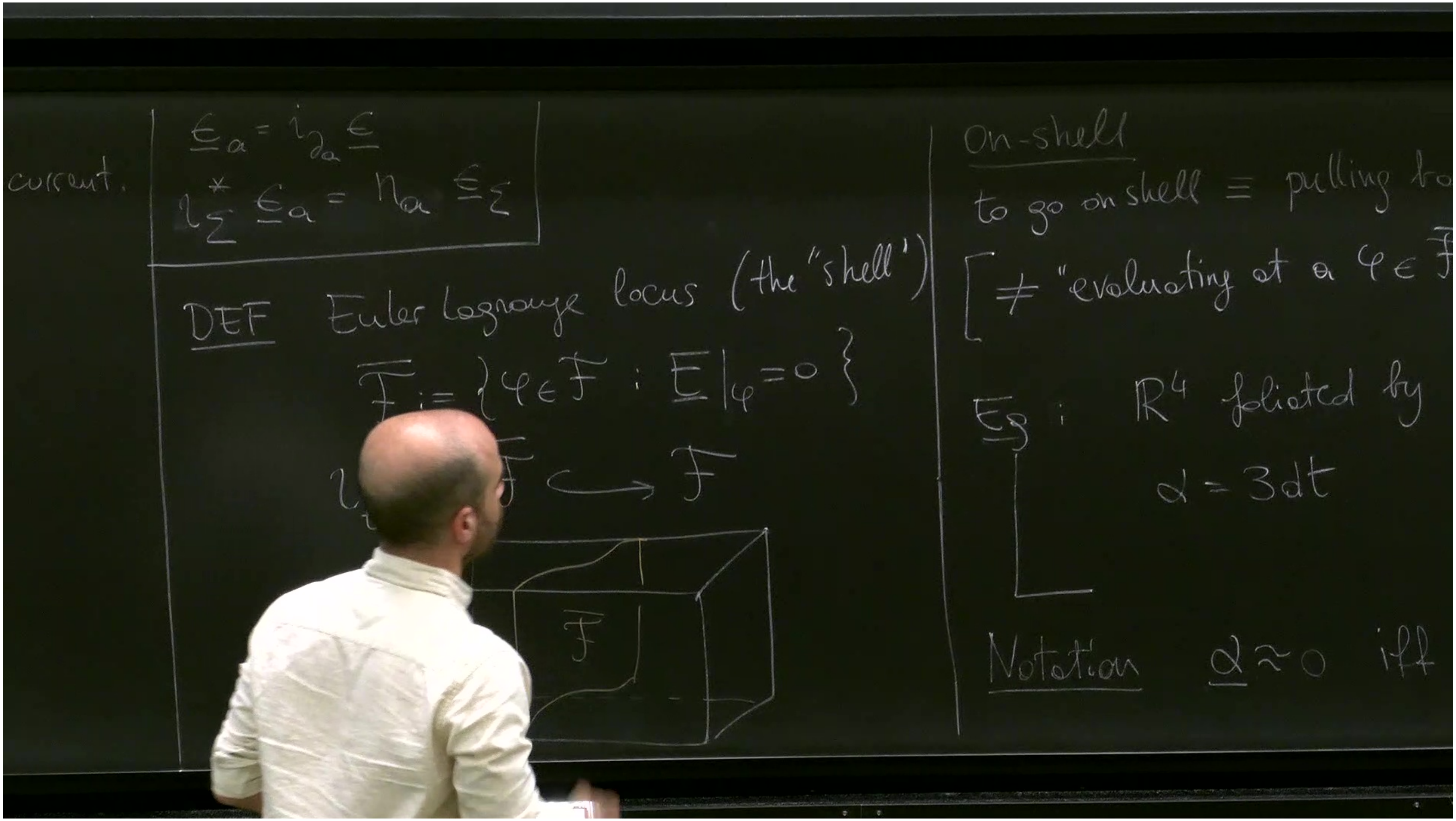
[$\neq$ "evaluating at a $\varphi \in \mathcal{F}$]

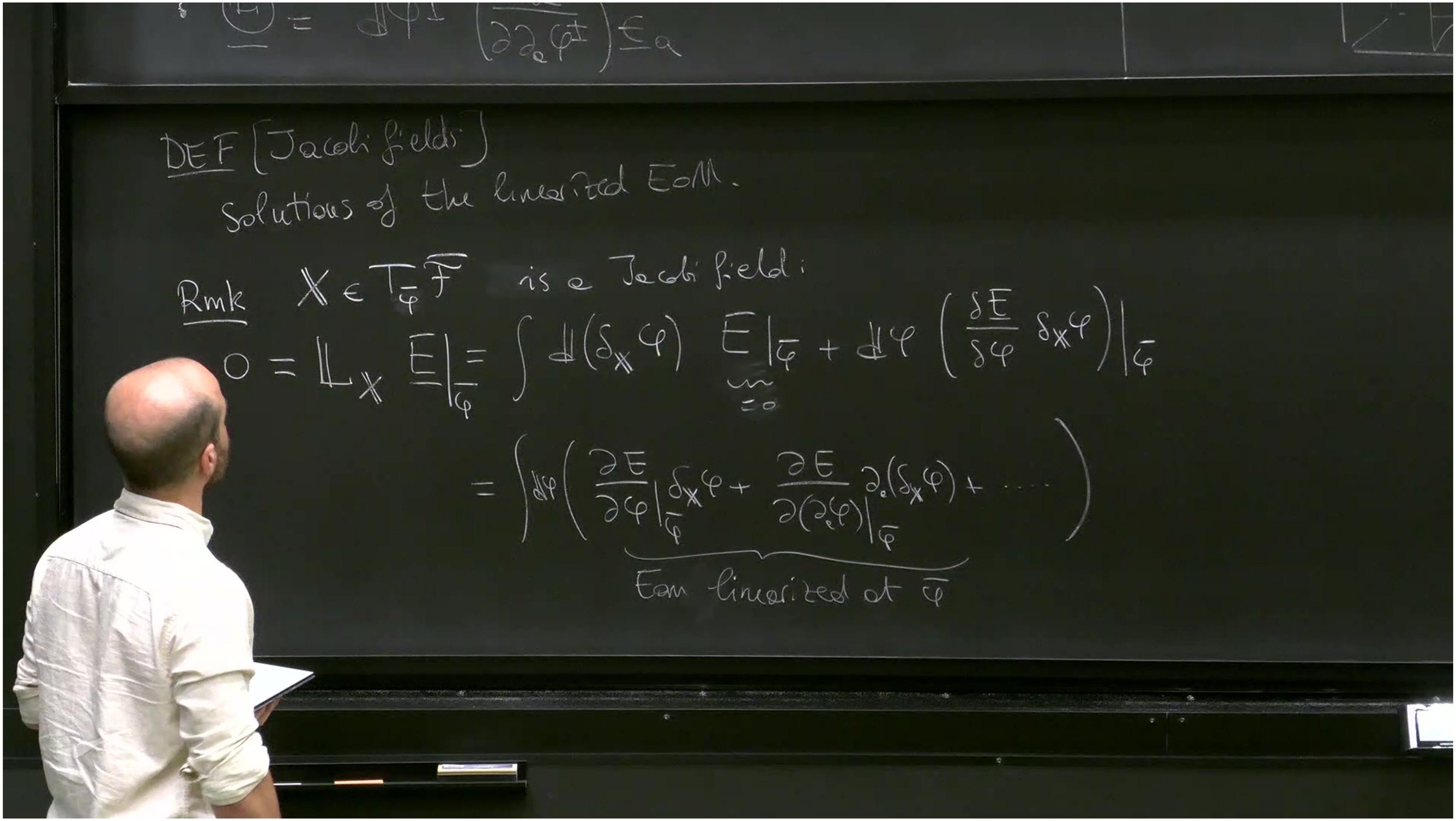
$= 0$ }

Eg : $\mathbb{R}^4$ foliated by const. time slices, $\Sigma_0 = \{t=0\}$


On-shell
to go on shell $\equiv$ pulling back to $\mathcal{F}$
[$\neq$ "evaluating at a $\varphi \in \mathcal{F}$]
Eg: $\mathbb{R}^4$ foliated by const. time slices, $\Sigma_0 = \{t=0\}$
$\alpha = 3dt$
$\alpha|_{\Sigma_0} = 3dt$
$\iota^*_{\Sigma_0}\alpha = 0$


On-shell
to go on shell ≡ pulling back to $\mathcal{F}$
[≠ "evaluating at a $\varphi \in \mathcal{F}$]
locus
Eg: $\mathbb{R}^4$ foliated by const. time slices, $\Sigma_0 = \{t=0\}$
$\alpha = 3dt$
$\alpha|_{\Sigma_0} = 3dt$
$\iota^*_{\Sigma_0} \alpha = 0$
Notation $\underline{\alpha} \approx 0$ iff $\iota^*_{EL} \underline{\alpha} = 0$


current.
$\underline{\epsilon}_a = i_{\partial_a} \underline{\epsilon}$
$\iota_\Sigma^* \underline{\epsilon}_a = n_a \underline{\epsilon}_\Sigma$
DEF Euler Lagrange locus (the "shell")
$\overline{\mathcal{F}} := \{ \varphi \in \mathcal{F} : \underline{E}|_\varphi = 0 \}$
$\overline{\mathcal{F}} \hookrightarrow \mathcal{F}$
$\overline{\mathcal{F}}$
On-shell
to go on shell ≡ pulling to
≠ "evaluating at a $\varphi \in \overline{\mathcal{F}}$
$\underline{E}_g$: $\mathbb{R}^4$ foliated by
$\alpha = 3dt$
Notation $\underline{\alpha} \approx 0$ iff


DEF [Jacobi fields]
Solutions of the linearized EoM.
Rmk $\mathbb{X} \in T_{\bar\varphi}\mathcal{F}$ is a Jacobi field:
$0 = \mathbb{L}_{\mathbb{X}} \mathbb{E}|_{\bar\varphi} = \int \mathbb{d}(\delta_{\mathbb{X}}\varphi)\ \underbrace{\mathbb{E}|_{\bar\varphi}}_{=0} + \mathbb{d}\varphi \left(\frac{\delta E}{\delta\varphi}\delta_{\mathbb{X}}\varphi\right)\Big|_{\bar\varphi}$
$= \int \mathbb{d}\varphi \left(\underbrace{\frac{\partial E}{\partial\varphi}\Big|_{\bar\varphi}\delta_{\mathbb{X}}\varphi + \frac{\partial E}{\partial(\partial_a\varphi)}\Big|_{\bar\varphi}\partial_a(\delta_{\mathbb{X}}\varphi) + \dots}_{\text{EoM linearized at } \bar\varphi} \right)$

$$\underline{\Theta} = \mathbb{d}\varphi^I \left(\frac{\quad}{\partial \partial_a \varphi^I} \right) \epsilon_a$$

DEF [Jacobi fields]

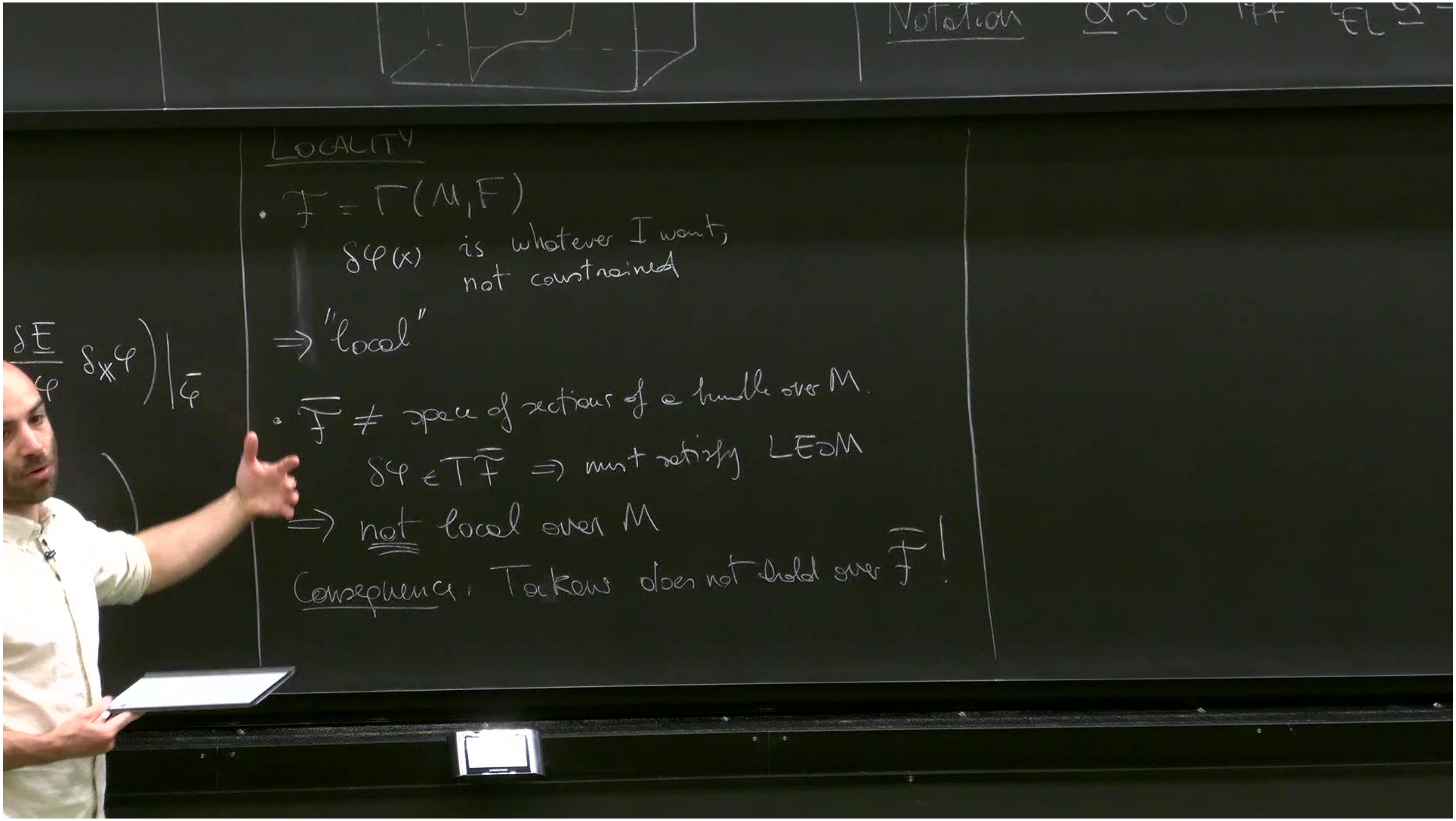
Solutions of the linearized EoM.

Rmk $\mathbb{X} \in T_{\bar\varphi}\bar{\mathcal{F}}$ is a Jacobi field:

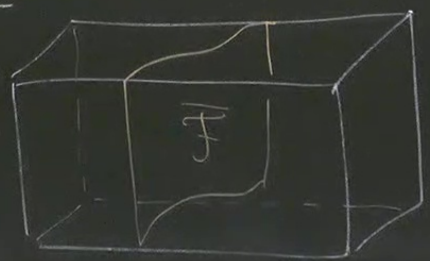
$$0 = \mathbb{L}_{\mathbb{X}} \underline{E}\Big|_{\bar\varphi} = \int \mathbb{d}(\delta_{\mathbb{X}}\varphi)\ \underbrace{E\big|_{\bar\varphi}}_{=0} + \mathbb{d}\varphi \left(\frac{\delta E}{\delta \varphi} \delta_{\mathbb{X}}\varphi \right)\Big|_{\bar\varphi}$$

$$= \int \mathbb{d}\varphi \Big(\underbrace{\frac{\partial E}{\partial \varphi}\Big|_{\bar\varphi} \delta_{\mathbb{X}}\varphi + \frac{\partial E}{\partial(\partial_a \varphi)}\Big|_{\bar\varphi} \partial_a(\delta_{\mathbb{X}}\varphi) + \dots}_{\text{EoM linearized at } \bar\varphi} \Big)$$

Assumption:
converse is also true


Notation
LOCALITY
• F = Γ(M, F)
δφ(x) is whatever I want,
not constrained
⇒ "local"
• F̄ ≠ space of sections of a bundle over M.
δφ ∈ TF̄ ⇒ must satisfy LEOM
⇒ not local over M
Consequence: Takens does not hold over F̄!

$\iota_{EL} : \overline{\mathcal{F}} \hookrightarrow \mathcal{F}$



$\alpha = 3dt$ $\qquad \alpha|_{\Sigma_0} = 3dt$

$\iota^*_{\Sigma_0} \alpha = 0$

Notation $\underline{\alpha} \approx 0$ iff $\iota^*_{EL} \underline{\alpha} = 0$

$)|_{\overline{\varphi}}$

LOCALITY

- $\mathcal{F} = \Gamma(M, F)$
  $\delta\varphi(x)$ is whatever I want, not constrained

$\Rightarrow$ "local"

- $\boxed{\overline{\mathcal{F}}} \neq$ space of sections of a bundle over M.
  $\delta\varphi \in T\overline{\mathcal{F}} \Rightarrow$ must satisfy LEOM

$\Rightarrow$ not local over M

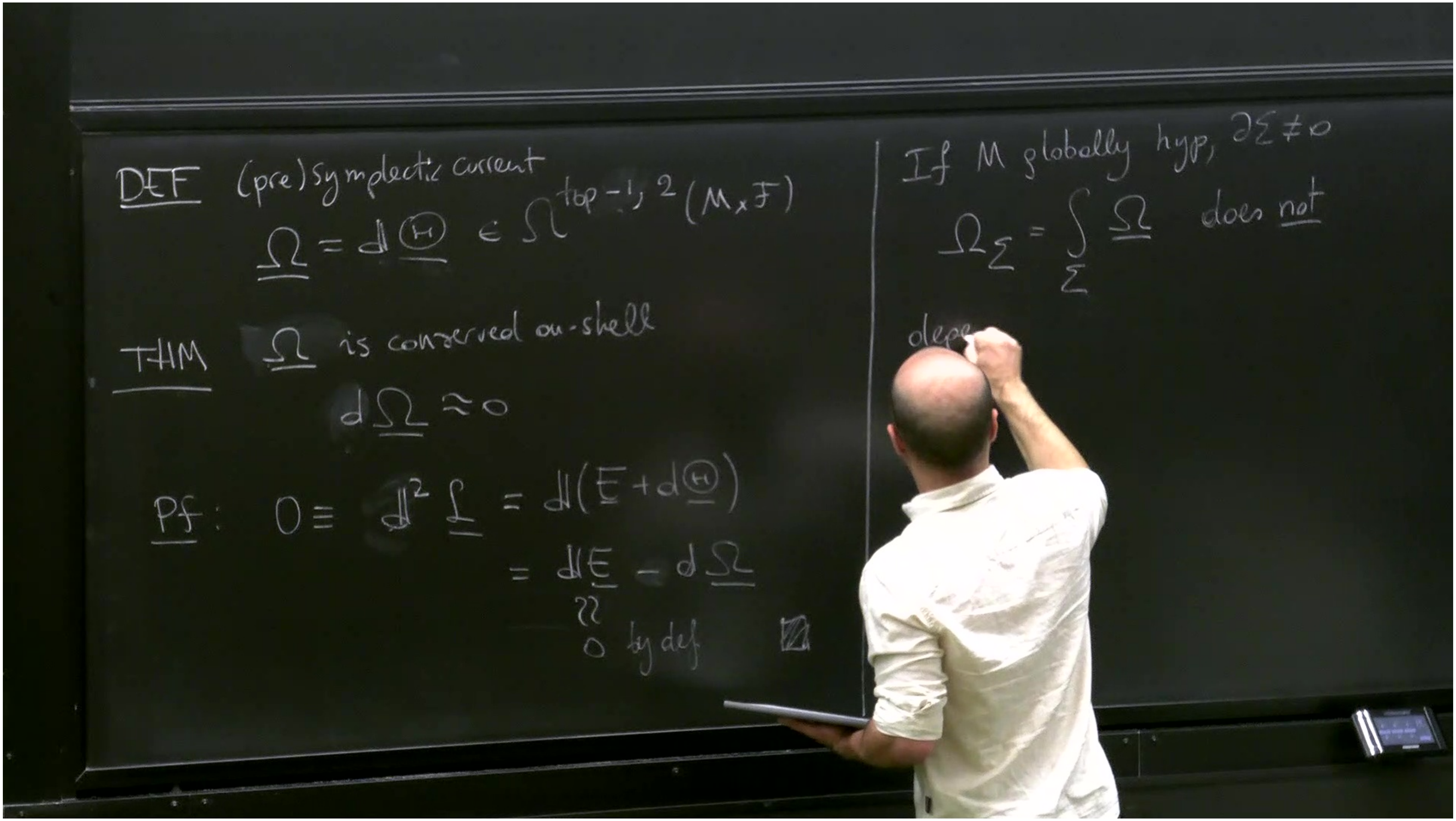
Consequence: Takens does not hold over $\overline{\mathcal{F}}$!

$\overline{\mathcal{F}}$ can still be isomorphic to a local space, just not one over M.

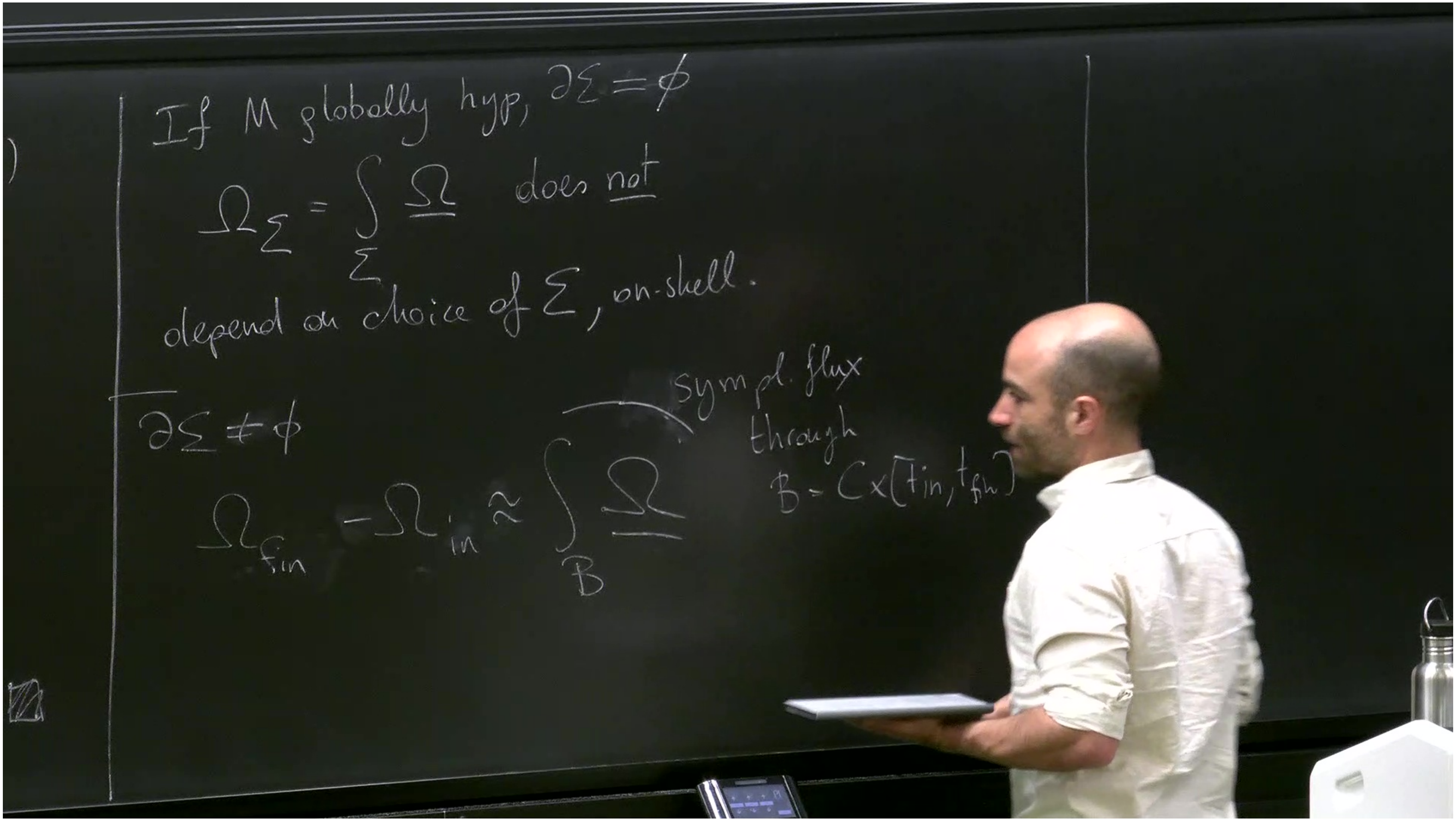
Usually: one over Σ.

If 1-to-1 correspondence btw (unconstrained) initial conds at Σ and $\overline{\varphi} \in \overline{\mathcal{F}}$

"canonical ph. sp."



DEF (pre)symplectic current
$\underline{\Omega} = d\underline{\Theta} \in \Omega^{top-1,2}(M \times \mathcal{F})$
THM $\underline{\Omega}$ is conserved on-shell
$d\underline{\Omega} \approx 0$
Pf: $0 \equiv d^2 \underline{L} = d(\underline{E} + d\underline{\Theta})$
$= d\underline{E} - d\underline{\Omega}$
?? 0 by def
If M globally hyp, $\partial\Sigma \neq 0$
$\Omega_\Sigma = \int_\Sigma \underline{\Omega}$ does not
depe


If M globally hyp, $\partial\Sigma = \phi$
$\Omega_\Sigma = \int_\Sigma \underline{\Omega}$ does not
depend on choice of Σ, on-shell.
$\partial\Sigma \neq \phi$
$\Omega_{fin} - \Omega_{in} \approx \int_B \underline{\Omega}$
sympl. flux through
$B = C \times [t_{in}, t_{fin}]$

If M globally hyp, $\partial\Sigma = \emptyset$

$$\Omega_\Sigma = \int_\Sigma \underline{\Omega} \quad \text{does } \underline{\text{not}}$$

depend on choice of Σ, on-shell.

$\partial\Sigma \neq \emptyset$

$$\Omega_{fin} - \Omega_{in} \approx \overbrace{\int_B \underline{\Omega}}^{\text{sympl. flux through}}$$

$B = C \times [t_{in}, t_{fin}]$

$$\Rightarrow \left(\mathcal{F}, \imath^*_{EL}\Omega_\Sigma\right)$$

If M globally hyp, $\partial\Sigma = \phi$

$$\Omega_\Sigma = \int_\Sigma \underline{\Omega} \quad \text{does } \underline{\text{not}}$$

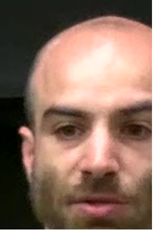
depend on choice of Σ, on-shell.

$\partial\Sigma \neq \phi$

$$\Omega_{fin} - \Omega_{in} \approx \overbrace{\int_B \underline{\Omega}}^{\text{sympl. flux}}$$

sympl. flux through $B = C \times [t_{in}, t_{fin}]$

$$\Rightarrow \left(\mathcal{F}, \iota^*_{EL}\Omega_\Sigma\right) \equiv \text{"}\underline{\text{covariant ph. sp.}}\text{"}$$



EXAMPLES

1) Particle

$$\underline{L} = \left(\frac{1}{2}\dot{q}^2 - V(q)\right) dt$$

$$\mathbb{d} L = \mathbb{d}q^{(t)}\,(\text{eom}) + \underbrace{dt\,\partial_t}_{\equiv d}\Big(\underbrace{\mathbb{d}q\ \dot{q}}_{\underline{\Theta}}\Big) \qquad \underline{\Theta} \in \Omega^{\overset{0}{\text{top}-1},1}(\overset{\mathbb{R}}{\downarrow}{M} \times F)$$

$$\underline{\Omega}(t) = \mathbb{d}\dot{q} \wedge \mathbb{d}q \overset{!}{=} \partial_t\, \mathbb{d}q^{(t)} \wedge \mathbb{d}q^{(t)}$$

$$\Sigma = \{t_0\} \hookrightarrow M = \mathbb{R}$$

$$\underline{\Omega}_{\Sigma} = \mathbb{d}\dot{q}(t_0) \wedge \mathbb{d}q(t_0)$$

At $t = t_0$, $\begin{cases} x \equiv q(t_0) \\ p = \dot{q}(t_0) \end{cases}$

can be specified freely & independently

$$(x, p) \in \mathcal{P} = C^{\infty}(\underset{\{t_0\}}{\Sigma}, \mathbb{R}^2) = \mathbb{R}^2$$

EXAMPLES

1) Particle $\qquad \mathcal{L} = \left(\frac{1}{2}\dot{q}^2 - V(q)\right) dt$

$$\mathbb{d}\mathcal{L} = \mathbb{d}q(t)\,(\text{eom}) + \underbrace{dt\,\partial_t}_{\equiv d}\Big(\underbrace{\mathbb{d}q\;\dot{q}}_{\Theta}\Big) \qquad \Theta \in \Omega^{\overset{0}{\text{top}-1},1}(\overset{\mathbb{R}}{M}\times F)$$

$$\Omega(t) = \mathbb{d}\dot{q}\wedge\mathbb{d}q = \partial_t\,\mathbb{d}q(t)\wedge\mathbb{d}q(t)$$

$$\Sigma = \{t_0\} \hookrightarrow M = \mathbb{R}$$

$$\Omega_\Sigma = \mathbb{d}\dot{q}(t_0)\wedge\mathbb{d}q(t_0)$$

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finite dim

EXAMPLES

1) Particle $\quad L = \left(\frac{1}{2}\dot q^2 - V(q)\right) dt$

$$\mathbb{d}L = \mathbb{d}q^{(t)}\left(\text{eom}\right) + \underbrace{dt\,\partial_t}_{\equiv d}\Big(\underbrace{\mathbb{d}q\ \dot q}_{\Theta}\Big) \qquad \Theta \in \Omega^{\overset{0}{\text{top}-1},1}(\overset{\mathbb{R}}{\overset{\downarrow}{M}}\times F)$$

$$\Omega(t) = \mathbb{d}\dot q \wedge \mathbb{d}q \overset{!}{=} \partial_t\, \mathbb{d}q^{(t)} \wedge \mathbb{d}q^{(t)}$$

$$\Sigma = \{t_0\} \hookrightarrow M = \mathbb{R}$$

$$\Omega_\Sigma = \mathbb{d}\dot q(t_0) \wedge \mathbb{d}q(t_0)$$

At $t = t_0$, $\begin{cases} x \equiv q(t_0) \\ p = \dot q(t_0) \end{cases}$

can be specified freely & independently

$$(x,p) \in \mathcal{P} = C^\infty(\underset{\{t_0\}}{\Sigma}, \mathbb{R}^2) = \mathbb{R}^2$$

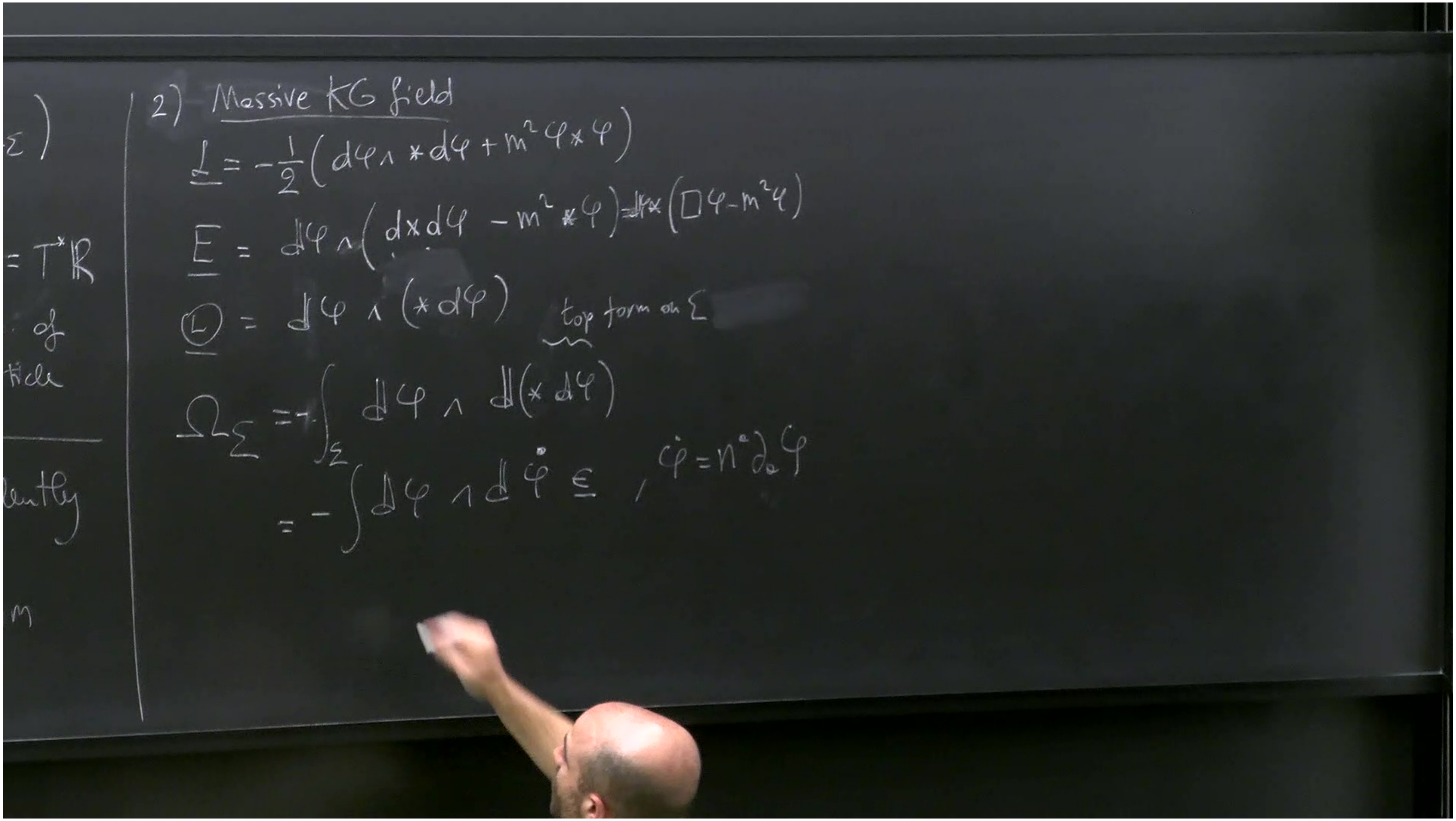
finite dim

$(\mathcal{F}, \iota_\Sigma^* \Omega_\Sigma)$

$\|$

$(\mathcal{P}, \omega) = T^*\mathbb{R}$

phase sp. of point particle



2) Massive KG field
$\underline{L} = -\frac{1}{2}\left(d\varphi \wedge * d\varphi + m^2 \varphi * \varphi\right)$
$\underline{E} = d\varphi \wedge \left(d * d\varphi - m^2 * \varphi\right) = *\left(\Box\varphi - m^2\varphi\right)$
$\underline{\Theta} = d\varphi \wedge (* d\varphi)$
top form on Σ
$\Omega_\Sigma = -\int_\Sigma d\varphi \wedge d(* d\varphi)$
$= -\int d\varphi \wedge d\dot{\varphi}\, \underline{\epsilon}\ ,\quad \dot{\varphi} = n^a \partial_a \varphi$
$= T^*\mathbb{R}$
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2) Massive KG field

$$\underline{L} = -\frac{1}{2}\left(d\varphi \wedge * d\varphi + m^2 \varphi * \varphi\right)$$

$$\underline{E} = d\varphi \wedge \left(d * d\varphi - m^2 * \varphi\right) = \cancel{d} * \left(\Box\varphi - m^2\varphi\right)$$

$$\underline{\Theta} = d\varphi \wedge (* d\varphi)$$ top form on Σ

$$\Omega_\Sigma = -\int_\Sigma d\varphi \wedge d(* d\varphi)$$

$$= -\int_\Sigma d\varphi \wedge d\dot{\varphi}\, \underline{\epsilon}\,, \qquad \dot{\varphi} = n^a \partial_a \varphi$$

$$(\mathcal{P}, \omega)\,, \quad \omega = \int_\Sigma d\phi \wedge d\pi$$

$$\phi(x \in \Sigma) = \varphi(x \in \Sigma)$$

$$\pi(\quad) = (n^a \partial_a \varphi)(x \in \Sigma)$$

$\epsilon)$

$= T^*\mathbb{R}$

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2) Massive KG field

$$\underline{L} = -\frac{1}{2}\left(d\varphi \wedge * d\varphi + m^2 \varphi * \varphi\right)$$

$$\underline{E} = \mathbb{d}\varphi \wedge \left(d * d\varphi - m^2 * \varphi\right) \cancel{\mathbb{d}} * \left(\Box\varphi - m^2\varphi\right)$$

$$\underline{\Theta} = \mathbb{d}\varphi \wedge (* d\varphi)$$ $\underbrace{\text{top}}$ form on Σ

$$\Omega = \int \mathbb{d}\varphi \wedge \mathbb{d}(\nabla^a\varphi)\, n_a$$

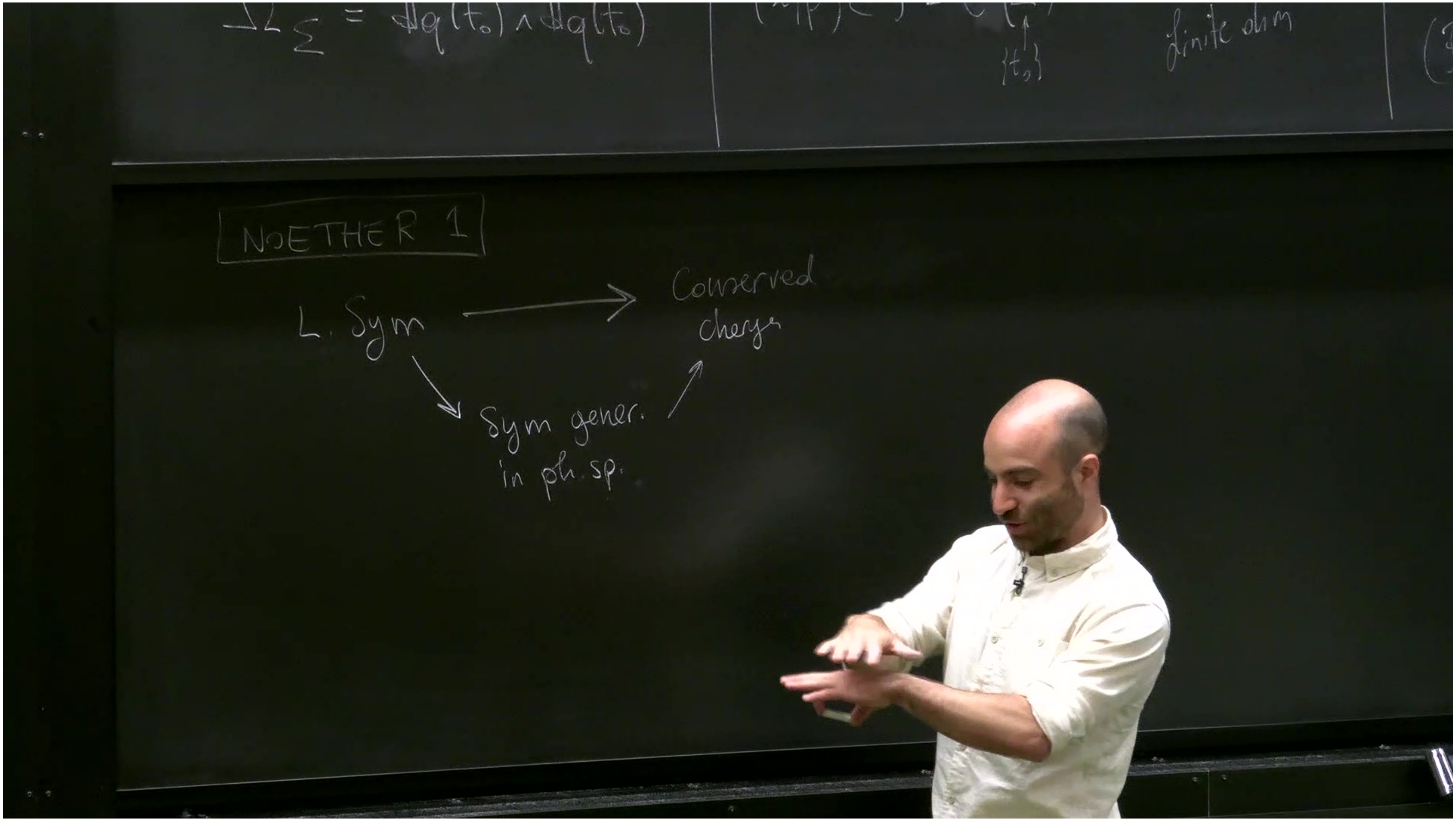
$$\Omega_\Sigma = -\int_\Sigma \mathbb{d}\varphi \wedge \mathbb{d}(* d\varphi)$$

$$= -\int_\Sigma \mathbb{d}\varphi \wedge \mathbb{d}\dot{\varphi}\, \underline{\epsilon}\ , \quad \dot{\varphi} = n^a\partial_a\varphi$$

$$\phi(x\in\Sigma) = \varphi(x\in\Sigma)$$

$$\pi(\quad) = (n^a\partial_a\varphi)(x\in\Sigma)$$

$$(\mathcal{P}, \omega), \quad \omega = \int_\Sigma \mathbb{d}\phi \wedge \mathbb{d}\pi$$


finite dim
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L. Sym
Conserved
charge
Sym gener.
in ph. sp.