Title: Quantum Gravity Lecture

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Collection: Quantum Gravity 2023/24

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(if $\exists$)

(co)momentum map $\check{J}$

$$
\begin{tikzcd}
 & (\mathfrak{g}, [\cdot,\cdot]) \arrow[dl, "\check{J}"'] \arrow[dr, "\rho \text{ action}"] & \\
(C^\infty(P), \{\cdot,\cdot\}) \arrow[rr, "X_\bullet = \{\bullet, \}"'] & & (\mathfrak{X}^1(P), [\cdot,\cdot]_{TP})
\end{tikzcd}
$$

$$i_{X_f} i_{X_g} \omega = \pm \{f, g\}$$

$$L_{X_f} \omega = 0$$

$i_{\rho(\xi)} \omega$

$J \in C^\infty$

$\check{J}(\xi)$

1. When do m
2. When is $\check{J}$

ion

), $[\cdot,\cdot]_{TP}$)

$$i_{\rho(\xi)}\omega = -d\check{J}(\xi)$$

comomentum map (if $\exists$)

$$J \in C^\infty(P, \mathfrak{g}^*)$$

$$\check{J}(\xi)(x) = \langle J(x), \xi \rangle$$

momentum map.

① When do momentum maps exist?

② When is $\check{J}$ a homomorph of Lie alg?

① $\exists$ of J

Lemma Assu

The action o

$[\xi,\eta] \in \mathfrak{g}$ is

of generator

Pf: $i_{\rho([\xi,\eta])}$

① $\exists$ of J

$L_{\rho(\xi)}\omega = 0 \quad \forall \xi.$

Lemma Assume

The action of a commutator $[\xi,\eta]\in\mathfrak{g}$ is Hamiltonian of generator $f = -i_{\rho(\xi)} i_{\rho(\eta)}\omega$

$\mathfrak{g}$?

Pf: $i_{\rho([\xi,\eta])}\omega = i_{[\rho(\xi),\rho(\eta)]}\omega = \Big(L_{\rho(\xi)} i_{\rho(\eta)} - i_{\rho(\eta)} \overbrace{L_{\rho(\xi)}}^{=0}\Big)\omega$

$= d\, i_{\rho(\xi)} i_{\rho(\eta)}\omega + i_{\rho(\xi)}\Big(\underbrace{d\, i_{\rho(\eta)}\omega + i_{\rho(\eta)} \underset{=0}{d\omega}}_{L_{\rho(\eta)}\omega = 0}\Big)$

$= d\big(i_{\rho(\xi)} i_{\rho(\eta)}\omega\big)$

☒

Pf

(ii) $\Rightarrow$ (i) obvious

(i) $\Rightarrow$ (ii) because of linearity

$$i_{\rho(\xi)}\,\omega = -df_\xi$$

$\rho(\xi)$ linear in ξ $\Rightarrow$ f_ξ is also linear (up to constants we don't care about)

$$= \langle J, \xi \rangle$$

(iii) first check $\tilde{\rho}$ is well defined.

This is thanks to previous lemma:

$$i_{\rho([\xi,\eta])}\,\omega = d\left(i_{\rho(\xi)}\, i_{\rho(\eta)}\,\omega\right)$$

$$\Rightarrow \left[i_{\rho([\xi,\eta])}\,\omega\right] = 0$$

PROP

(P,ω) sympl.

$(\mathfrak{g},\rho) \circlearrowright P$ action

Assume $L_{\rho(\xi)}\omega = 0$, then the following 3 statements are equivalent

i) $\rho(\xi)$ is Hamiltonian $\forall \xi$

ii) ρ admits a momentum map $J : P \to \mathfrak{g}^*$

iii) the linear map $\tilde{\rho} : \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \longrightarrow H^1(P)$

$$[\xi] \longmapsto [i_{\rho(\xi)}\omega]$$

is identically zero.

Pf

(ii) $\Rightarrow$

(i) $\Rightarrow$

(iii) f

Pf

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$$\Rightarrow \left[i_{\rho([\xi,\eta])}\omega\right] = 0$$

So $\tilde{\rho}$ well defined.

If $\tilde{\rho} \equiv 0$ then all $\xi \in \mathfrak{g}$ is Hamiltonian.

PROP

(P,ω) sympl.

$(\mathfrak{g},\rho) \circlearrowright P$ action

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i) $\rho(\xi)$ is Hamiltonian $\forall \xi$

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iii) the linear map $\tilde{\rho}: \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \longrightarrow H^1(P) = \frac{\text{closed 1-forms}}{\text{exact 1-forms}}$

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is identically zero.

Pf

(ii) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii)

(iii) $\Rightarrow$ (i)

first

is we

This

So

If

e about)

Rmk

$H^1(P)=0 \Rightarrow$ A symplectic action ($L_{\rho(\xi)}\omega = 0$) is also Hamiltonian. Indeed

$$0 = L_{\rho(\xi)}\omega = i_{\rho(\xi)}\underbrace{d\omega}_{=0} + d(i_{\rho(\xi)}\omega)$$

closed $\Rightarrow$ exact bc $H^1(P)=0$

$i_{\rho(\xi)}\omega = df$

Rmk if $\mathfrak{g}$ semisimple, Whitehead's lemma says $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$

- then symplectic $\Rightarrow$ Ham.

Rmk: $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = H^1_{CE}(\mathfrak{g})$

otion. $\boxtimes$

Pf

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(i) $\Rightarrow$ (ii) because of linearity

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$\to \mathfrak{g}^*$

$\mathfrak{g} \longrightarrow H^1(P) = \dfrac{\text{closed 1-forms}}{\text{exact 1-forms}}$

$\longmapsto [i_{\rho(\xi)}\omega]$

Rmk

$H^1(P) =$

Rmk

Rmk:

Ex : $\mathfrak{g} = su(2)$ semisimple

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$$

$$\begin{cases} \tau_1 = [\tau_2, \tau_3] \\ \tau_2 = [\tau_3, \tau_1] \\ \tau_3 = [\tau_1, \tau_2] \end{cases}$$

$\mathfrak{g}$ Abelian, not semisimple

Ex if $\exists\ \theta$: $\omega = d\theta$, $L_{\rho(\xi)}\theta = 0$

then $\check{J}(\xi) = i_{\rho(\xi)}\theta$ $\square$

②

Lemm

$d(L$

Pf

② We ask, is $\check{J}([\xi,\eta]) \overset{?}{=} \{\check{J}(\xi), \check{J}(\eta)\}$?

$$\equiv L_{\rho(\xi)} \check{J}(\eta)$$

Lemma

$$d\left(L_{\rho(\xi)} \check{J}(\eta) - \check{J}([\xi,\eta])\right) = 0$$

Pf

$$d L_{\rho(\xi)} \check{J}(\eta) = L_{\rho(\xi)}\, d\check{J}(\eta)$$

$$= -L_{\rho(\xi)}\, i_{\rho(\eta)}\, \omega$$

$L_{\rho(\xi)}\omega = 0$

$$= -i_{[\rho(\xi),\rho(\eta)]}\,\omega + \text{zero}$$

$$= -i_{\rho([\xi,\eta])}\,\omega$$

$$= d\check{J}([\xi,\eta])$$

PROP

$$L_{\rho(\xi)} \check{J}(\eta) = \check{J}([\xi,\eta]) + \bar{\kappa}(\xi,\eta)$$

where $\bar{\kappa}$:

i) $\bar{\kappa}(\xi,\eta) + \bar{\kappa}(\eta,\xi) = 0$

ii) $d\bar{\kappa}(\xi,\eta) = 0$ (does not depend on P)

iii) $\bar{\kappa}([\xi_1,\xi_2],\xi_3) + \text{cycl} = 0$

] CE 2-Cocycle of $\mathfrak{g}$

A CE 2-cocycle is said exact (or trivial) when

$$\exists \lambda \in \mathfrak{g}^* : \bar{\kappa}(\xi,\eta) = \langle \lambda, [\xi,\eta] \rangle$$

$\Rightarrow$ if $\bar{\kappa}$ is trivial $\check{J}' = \check{J} - \lambda$ is such that

$\uparrow$ constant on P

$$L_{\rho(\xi)} \check{J}'(\eta) = \check{J}'([\xi,\eta])$$

DEF $\check{J}$ is said equivariant if $L_{\rho(\xi)}\check{J}(\eta) = \check{J}([\xi,\eta])$

$$\begin{array}{c} L_{\rho(\xi)}\check{J}(\eta) \\ \parallel \\ \{\check{J}(\xi), \check{J}(\eta)\} \end{array}$$

Rmk the diagram closes for $\check{J}$ equivariant.

Rmk $H^2_{CE}(\mathfrak{g}) = 0$ the diagr. can always be closed by shifting $J \mapsto J - \lambda$

Whitehead's seco

$\mathfrak{g}$ semisimple

(e.g. $su(2)$)

If $H^2_{CE}(\mathfrak{g}) \neq$

Then, I can

new algebra

$\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$

that realizes e

is said equivariant

$$L_{\rho(\xi)} \check{J}(\eta) = \check{J}([\xi,\eta])$$
$$\|$$
$$\{\check{J}(\xi), \check{J}(\eta)\}.$$

the diagram closes for $\check{J}$ equivariant.

$H^2_{CE}(g) = 0$ the diagr. can always be closed by shifting $J \mapsto J - \lambda$

Whitehead's second lemma:

g semisimple $\Rightarrow H^2_{CE}(g) = 0$

(e.g. $su(2)$).

If $H^2_{CE}(g) \neq 0$

Then, I can define a new algebra $\hat{g}$, "central extension" of g

$\hat{g} = g \oplus \mathbb{R}$ with a new Lie bracket that realizes equivariance.

...rsion" of $\mathfrak{g}$

...e bracket

<u>Ex</u> Chern-Simons
3d gravity
(CFTs)

$$[(\xi, r), (\eta, s)]^\wedge = ([\xi,\eta], K(\xi,\eta))$$

$$\underset{\mathfrak{g}\,\oplus\,\mathbb{R}}{(\xi, r)} \qquad \underset{\mathfrak{g}}{[\xi,\eta]} \oplus \underset{\mathbb{R}}{K(\xi,\eta)}$$

$$\hat{J}(\xi, r) = J(\xi) + r$$
$$\mathfrak{g}^* \oplus \mathbb{R}^*$$