Title: Quantum Gravity Lecture

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Collection: Quantum Gravity 2023/24

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SUMMARY

$$\begin{tikzcd}
 & (\mathfrak{g}, [\cdot,\cdot]) \arrow[dl, dashed, "\substack{\text{if } \rho \text{ admits an} \\ \text{(equiv.) comom map} \\ \check{J}}"'] \arrow[d, "\text{action } \rho"] \\
(C^\infty(P), \{\cdot,\cdot\}) \arrow[r, "X_\bullet = \{\bullet, \}"'] & (\mathfrak{X}^1(P), [\cdot,\cdot]_{TP})
\end{tikzcd}$$

$$i_{\rho(\xi)}\,\omega = -\,d\check{J}(\xi)$$
$$\rho(\xi) = \{\check{J}(\xi), \cdot\}$$

- A mom. map $\exists$ iff $\check{\rho} : \mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]} \to H^1(P)$ vanishes identically.

- Equiv: $\{\check{J}(\xi), \check{J}(\eta)\} = \check{J}([\xi,\eta])$
$$\{\check{J}(\xi), \check{J}(\eta)\} \equiv L_{\rho(\xi)}\check{J}(\eta)$$

See lecture notes for SYMPL. REDUCTION

CHAPTER 2

In field theory, two rele

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In field theory, two relevant manifolds:

(i) Spacetime M, $\dim < \infty$

(ii) Field Space $\mathcal{F}$, $\dim = \infty$

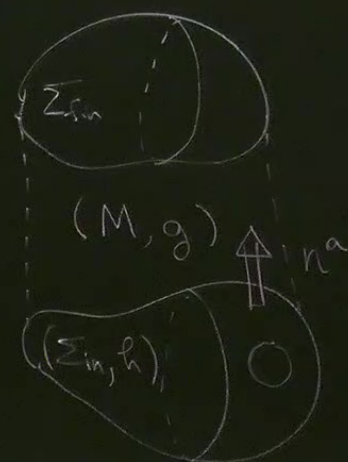
plays the role of P (morally)

Need tools to do geom. on $\mathcal{F}$.

(i) Spacetime

Globally Hyp. $M = \Sigma \times \mathbb{R}$, $\partial\Sigma = \phi$

$M = \Sigma \times [t_{in}, t_{fin}]$

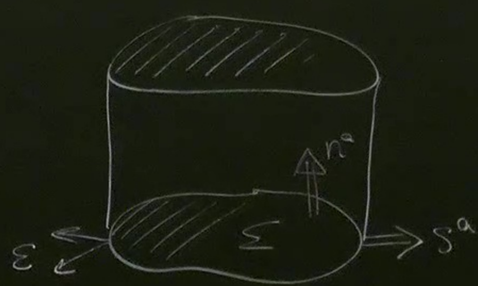


Σ , $\partial\Sigma = \phi$

In some cases we will consider Σ with boun

$\partial\Sigma \neq \phi$

In some cases
we will consider Σ with boundaries $\partial\Sigma \neq \emptyset$



$$\partial\Sigma = C$$
$$\partial M = \Sigma_{in} \times \Sigma_{fin} \times \left(\underbrace{C \times [t_{in}, t_{fin}]}_{B}\right)$$

$$\varepsilon^{ab} = \frac{n^a s^b - n^b s^a}{\sqrt{1 + (n s)^2}}$$

$N \hookrightarrow M$ submanifold of dim n.

$\Omega^k(N)$

$\Omega^{top-k}(N) = \Omega^{n-k}(N)$

$\times \Sigma_{fin} \times (\underbrace{C \times [t_{in}, t_{fin}]}_{B})$

(ii) Field Space

$\mathcal{F} = \Gamma(M, F)$ $\qquad \pi : F \to M$ fiber bundle over M

Ex:

1) real Scalar field $\qquad F = M \times \mathbb{R} \longrightarrow M$, $\Gamma(M,F) = C^\infty_{(\text{non-deg})}(M, \mathbb{R})$

2) GR $\qquad F = T^*M \odot_M T^*M$, $\Gamma(M,F) =$ symmetric covariant 2-tensors

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3) YM : $A \in \Omega^1(M, \mathfrak{g})$ $\quad \mathcal{F} = \mathrm{Conn}$ (it can be done)

4) Particles as 0+1d field theory $\quad \vec{q}(t) \in \mathcal{F} = C(M, \mathbb{R}^3)$

$M = \text{time} = \mathbb{R} \quad F = M \times \mathbb{R}^{3N}$,

5) Parametrized $\quad F = M \times \mathbb{R}^4$

$M =$ "parameter time" $= \mathbb{R}$

Doing geometry on M & F

$\boxed{M} \ni x$

- vector fields

$$X = \sum_i X^i(x) \frac{\partial}{\partial x^i}$$

$$X(f) = \lim_{\varepsilon \to 0} \frac{f(x^i + \varepsilon X^i) - f(x^i)}{\varepsilon} = \sum_i X^i \frac{\partial f}{\partial x^i}$$

$\boxed{F} \ni \varphi$

- vector fields

$$\mathbb{X} \text{ "=" } \int dx \; \delta_{\mathbb{X}} \varphi(x) \frac{\delta}{\delta \varphi(x)}$$

Interested in local function(al)s over F

$$S(\varphi) = \int dx \; \mathcal{L}(\varphi(x), \partial \varphi(x), x)$$

$$\mathbb{X}(S) = \int dx \; \frac{\delta S}{\delta \varphi} (\delta_{\mathbb{X}} \varphi) = \int dx \left(\frac{\partial \mathcal{L}}{\partial \varphi} \right.$$

1) Scalar field $F =$

2) GR $F =$

3) YM $A \in \Omega^1(M, \mathfrak{g})$

4) Particles as 0+1d field theory
$M = \text{time} = \mathbb{R}$ F

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not the EL derivative

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$$\mathbb{X}(S) = \int dx \, \frac{\delta S}{\delta \varphi} (\delta_{\mathbb{X}} \varphi) = \int dx \left(\frac{\partial \mathcal{L}}{\partial \varphi(x)} \delta_{\mathbb{X}} \varphi(x) + \frac{\partial \mathcal{L}}{\partial \partial_i \varphi(x)} \boxed{\partial_i} \delta_{\mathbb{X}} \varphi(x) \right) = \lim_{\epsilon \to 0} \frac{S(\varphi + \epsilon \delta_{\mathbb{X}} \varphi) - S(\varphi)}{\epsilon}$$

differential operator

Doing geometry on M & F

$\boxed{M} \ni x$

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$$X = \sum_i X^i(x) \frac{\partial}{\partial x^i}$$

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- differential

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

$\boxed{F} \ni \varphi$

not the EL derivative

- vector fields

$$\mathbb{X} = \int dx \; \delta_{\mathbb{X}} \varphi(x) \, " \frac{\delta}{\delta \varphi(x)} "$$

Interested in local function(al)s over F

$$S(\varphi) = \int dx \; \mathcal{L}(\varphi(x), \partial\varphi(x), x)$$

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differential operator

- diff forms

$$\mathbb{d} S = \int dx \; \frac{\delta S}{\delta \varphi} (\mathbb{d} \varphi)$$

2) GR

$\boxed{F} \ni \varphi$

not the EL derivative

• vector fields

$$\mathbb{X} = \int dx \; \delta_{\mathbb{X}} \varphi(x) \frac{\delta}{\delta \varphi(x)}$$

Interested in local function(al)s over F

$$S(\varphi) = \int dx \; \mathcal{L}(\varphi(x), \partial\varphi(x), x)$$

$$\mathbb{X}(S) = \int dx \; \frac{\delta S}{\delta \varphi}(\delta_{\mathbb{X}} \varphi) = \int dx \left(\frac{\partial \mathcal{L}}{\partial \varphi(x)} \delta_{\mathbb{X}} \varphi(x) + \frac{\partial \mathcal{L}}{\partial \partial_i \varphi(x)} \boxed{\partial_i} \delta_{\mathbb{X}} \varphi(x) \right) = \lim_{\varepsilon \to 0}$$

differential operator

• diff forms

$$\mathbb{d} S = \int dx \; \frac{\delta S}{\delta \varphi}(\mathbb{d}\varphi)$$

3) YM

4) Particles

M

5) Paramet

M

S(

Cartan's calculus

M

- d
- i_X
- $L_X = i_X d + d i_X$

$\mathbb{F}$

- $\mathbb{d}$ $\cancel{(= \delta)}$
- $\mathbb{i}_{\mathbb{X}}$
- $\mathbb{L}_{\mathbb{X}} = \mathbb{i}_{\mathbb{X}} \mathbb{d} + \mathbb{d} \mathbb{i}_{\mathbb{X}}$

$$\mathbb{L}_{\mathbb{X}} S = \mathbb{i}_{\mathbb{X}} \mathbb{d} S = \mathbb{i}_{\mathbb{X}} \int dx \, \frac{\delta S}{\delta \varphi} (\mathbb{d} \varphi) = \int dx \, \frac{\delta S}{\delta \varphi} \underbrace{(\mathbb{i}_{\mathbb{X}} \mathbb{d} \varphi)}_{\delta_{\mathbb{X}} \varphi}$$

Mixed forms

$\alpha \in \Omega^{p,q}(\underline{M} \times \underline{\underline{F}})$

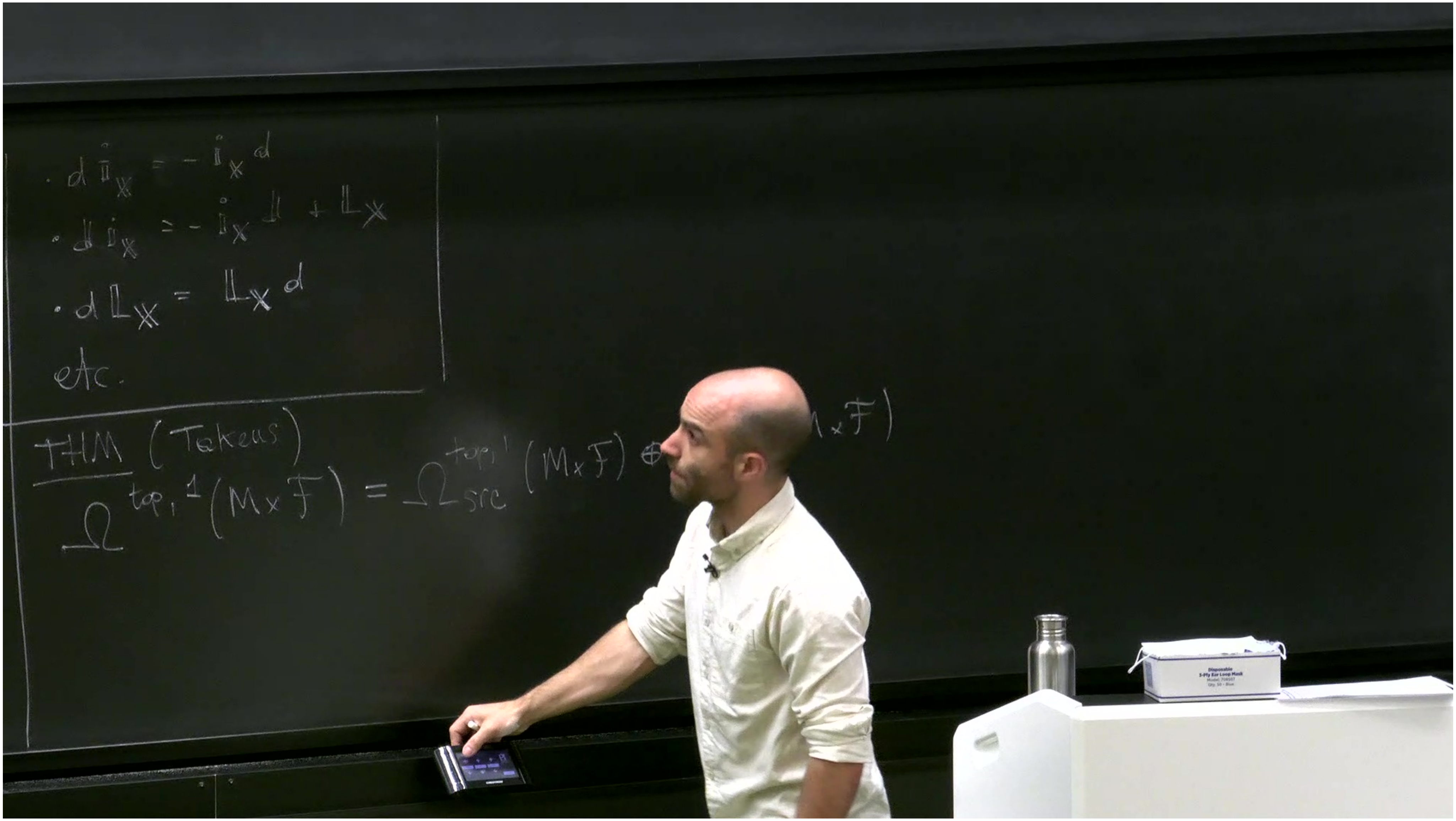
$\alpha(x) = \partial_a \varphi(x) \boxed{dx^a} \wedge \boxed{\mathbb{d} \varphi(x)} \in \Omega^{1,1}(M \times F)$

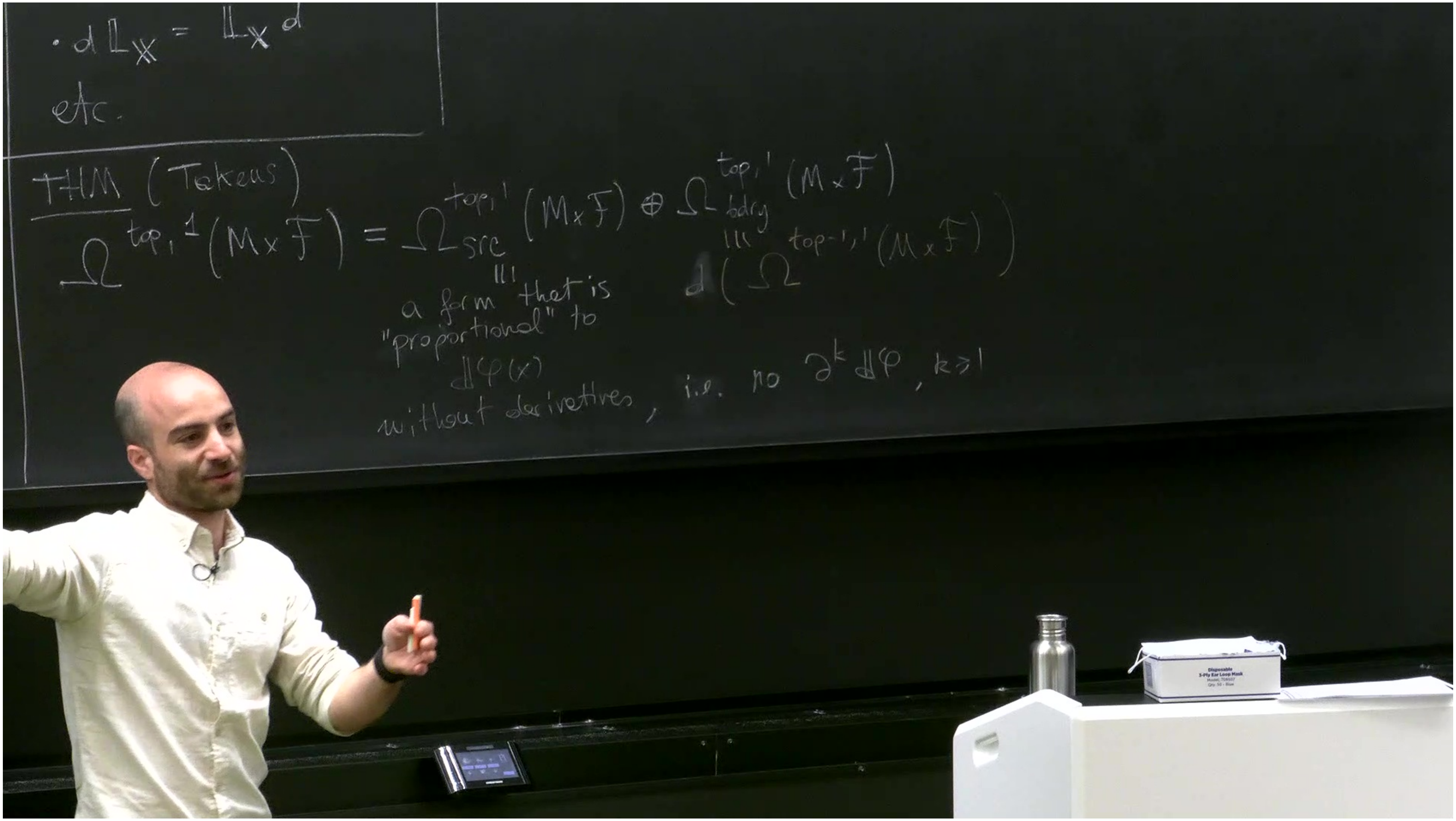
$\hookrightarrow \alpha = d\varphi \wedge \mathbb{d}\varphi$

- $\boxed{d\,\mathbb{d} + \mathbb{d}\,d = 0} \longleftrightarrow \begin{array}{l} \mathbb{D} = d + \mathbb{d} \\ \mathbb{D}^2 = 0 \end{array}$

- one $\wedge$ for all

$\alpha_1 \wedge \alpha_2 = (-1)^{(p_1+q_1)\cdot(p_2+q_2)} \alpha_2 \wedge \alpha_1$


$\cdot\ d\, \mathring{\mathbb{i}}_{\mathbb{X}} = -\, \mathbb{i}_{\mathbb{X}}\, d$
$\cdot\ d\, \mathbb{i}_{\mathbb{X}} = -\, \mathring{\mathbb{i}}_{\mathbb{X}}\, d + \mathbb{L}_{\mathbb{X}}$
$\cdot\ d\, \mathbb{L}_{\mathbb{X}} = \mathbb{L}_{\mathbb{X}}\, d$
etc.
THM (Takens)
$\Omega^{top,1}(M\times F) = \Omega^{top,1}_{src}(M\times F) \oplus$
$(M\times F)$


· $d \mathbb{L}_{\mathbb{X}} = \mathbb{L}_{\mathbb{X}} d$
etc.
THM (Takens)
$\Omega^{top,1}(M \times F) = \Omega^{top,1}_{src}(M \times F) \oplus \Omega^{top,1}_{bdry}(M \times F)$
$\Omega^{top,1}_{bdry}(M \times F) = d(\Omega^{top-1,1}(M \times F))$
a form that is "proportional" to $d\varphi(x)$ without derivatives, i.e. no $\partial^k d\varphi$, $k \geq 1$

$a_1 \wedge a_2 = (-1) \quad a_2 \wedge a_1$

$d\varphi(x$
without de

Lagrangian $\quad \mathcal{L} = \mathcal{L}(x, \varphi, \partial\varphi) \in \Omega^{top, 0}(M \times \mathcal{F})$, $\quad S = \int \mathcal{L}$

$$\mathbb{d}\mathcal{L} = E_I \, \mathbb{d}\varphi^I + d \, \Theta$$

$=$ ↑ Takens

E_I ↑ Euler-Lagrange : $\dfrac{\partial \mathcal{L}}{\partial \varphi} - \partial_a \dfrac{\partial \mathcal{L}}{\partial \partial_a \varphi}$

$$\Theta \in \Omega^{top-1, 1}(M \times \mathcal{F})$$

↑ universal current
(pre) symplectic potential $\underbrace{\text{current}}_{\text{top-1 form on } M}$

Lagrangian $\quad \mathcal{L} = \mathcal{L}(x, \varphi, \partial\varphi) \in \Omega^{top,0}(M \times \mathcal{F})$, $\quad S = \int \mathcal{L}$

$$\mathbb{d}\mathcal{L} \underset{\text{Takens}}{=} E_I \, \mathbb{d}\varphi^I + d\,\Theta$$

↳ Euler-Lagrange: $\dfrac{\partial \mathcal{L}}{\partial \varphi} - \partial_a \dfrac{\partial \mathcal{L}}{\partial \partial_a \varphi}$

$$\Theta \in \Omega^{top-1,1}(M \times \mathcal{F})$$

↳ universal current

(pre) symplectic potential $\underbrace{\text{current}}_{\text{top-1 form on } M}$

↳ will give us the ph. sp. structure

$\Omega_\Sigma := \int_\Sigma \mathbb{d}\Theta$

will be a degenerate 2-form on $\mathcal{F}$