

Title: Quantum Gravity Lecture

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Collection: Quantum Gravity 2023/24

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Recap

• Sympl.

$$\omega = \frac{1}{2} \omega_{IJ} dz^I dz^J$$

- ① CLOSED $d\omega = 0$
- ② NON DEG $\ker(\omega^b) = 0$
 $\hookrightarrow P$ $2n$ -dim

• Poisson

$$\{\cdot, \cdot\} = \Pi^{IJ} \partial_I \wedge \partial_J : C^\infty(P) \times C^\infty(P) \rightarrow C^\infty(P)$$

- Ⓐ SKEW
- Ⓑ BIDERIV.
- Ⓒ JACOBI

Rmk
| Π can be degenerate

• SYMPL \Rightarrow POISSON

① \Rightarrow Ⓒ

• Poisson $\not\Rightarrow$ sympl, unless Π non deg.

Rmk
 ω degenerate: "pre-sympl"
 DIFFERENT
 from Poisson

(P) • SYMPLECTOMORPHISM
 $\phi \in \text{Diff}(P)$ $\phi^* \omega = \omega$
 $X \in \mathcal{X}^1(P)$ $L_X \omega = 0$

• HAMILTONIAN V.F. (HVF)

st. $\exists f_X \in C^\infty(P)$: $i_X \omega = -df_X$
 "Hamiltonian generator of X "

$X = \{f_X, \cdot\}$

• HVF \Rightarrow SYMPL. $L_X \omega = i_X d\omega + d i_X \omega = \underbrace{d^2 f}_=0$

Rmk
 X HVF $\Rightarrow f_X$ unique up to constants

$$i_X \omega = -df_1 \Rightarrow d(f_1 - f_2) = 0 \quad \square$$

$$= -df_2$$

PROP If $H^1(P) = 0$, then
 sympl \Rightarrow HVF

$$\text{Pf: } H^1(P) := \frac{\text{closed 1-forms}}{\text{exact 1-forms}}$$

if $H^1(P) = 0$, then closed \Rightarrow exact

$$0 = \underbrace{L_X \omega}_{\text{sympl}} = i_X \underbrace{d\omega}_=0 + d i_X \omega \Rightarrow i_X \omega \text{ closed, therefore exact if } H^1(P) = 0 \quad \square$$

PROP If \exists symplectic θ ($\omega = d\theta$)
such that $\underline{L_X \theta = 0}$, then

• X is HVF

• $f_x = i_x \theta$ is generator

Pf $0 = L_X \theta = i_X d\theta + d i_X \theta$
 $= i_X \omega + d f_x \quad \square$

Rmk • X HVF iff f_x exists

• $f \Rightarrow X_f$ HVF
 \uparrow
 $\{f, \cdot\}$

PROP $\langle \cdot, \cdot \rangle = \omega^{-1}$

• $\omega(X_f, X_g) = \{f, g\}$

• $[X_f, X_g] = X_{\{f, g\}}$

Pf: Exercise.

$$\left(C^\infty(P), \{ \cdot, \cdot \} \right) \xrightarrow{X_\bullet} \left(\mathfrak{X}^1(P), [\cdot, \cdot]_{TP} \right)$$

Lie algebra homomorph.
(*)

Example

$$L(q, \dot{q}) = \frac{1}{2} \delta_{ij} \dot{q}^i \dot{q}^j - \underbrace{A_i(q) \dot{q}^i}_{\substack{\text{fixed} \\ \text{1-form on } \mathcal{Q}}} - \underbrace{V(q)}_{\substack{\text{fixed} \\ \text{function on } \mathcal{Q}}}$$

① $p_i = \frac{\partial L}{\partial \dot{q}^i} = \delta_{ij} \dot{q}^j - A_i(q), \quad \omega = \sum dp_i + dq^i$

$$H(p, q) = \frac{1}{2} \delta^{ij} (p_i + A_i(q))(p_j + A_j(q)) + V(q)$$

$$X_H = +HVF = \underbrace{(X_H)_q}_q \frac{\partial}{\partial q^i} + \underbrace{(X_H)_{p_i}}_{\dot{p}_i} \frac{\partial}{\partial p_i}$$

② ϕ

(q)

action = \mathcal{A}

$$\int dp_i \wedge dq^i$$

$$(q) + V(q)$$

$$\underbrace{(X_{\mathcal{H}})}_{\dot{p}_i} p_i = \frac{\partial}{\partial p_i}$$

$$\textcircled{2} \phi : (\tilde{q}, \tilde{p}) \rightarrow (q, p) = (\tilde{q}, \tilde{p} - A) \quad B_{ij} = \partial_i A_j - \partial_j A_i$$

$$\tilde{\omega} = \phi^* \omega = d\tilde{p}_i \wedge d\tilde{q}^i - \frac{1}{2} B_{ij}(q) d\tilde{q}^i \wedge d\tilde{q}^j$$

$$\tilde{\mathcal{H}} = \phi^* \mathcal{H} = \frac{1}{2} \delta^{ij} \tilde{p}_i \tilde{p}_j + V(\tilde{q})$$

$$\downarrow \tilde{X}_{\tilde{\mathcal{H}}} = \phi_*^{-1} X_{\mathcal{H}} = \text{encode some dynamics}$$

$$(\tilde{p} \sim \dot{q})$$

$$\frac{d}{dt} = X_{\mathcal{H}} = \{ \mathcal{H}, \cdot \}$$

$$\frac{d}{dt} \mathcal{H} = \{ \mathcal{H}, \mathcal{H} \} = 0$$

$$B = dA$$

$$d\tilde{\omega} = 0$$

$B_{ij} = 2A_i - 2_j A_j$. HVF encode symms.

$d\tilde{q}^j$
 $B = dA$
 $d\tilde{\omega} = 0$

com $\tilde{H} = 0$
 d
 consd.
 m

SYMMETRIES

Def: Kin. sym \Leftrightarrow HVF ($\Rightarrow L_X \omega = 0$)
 Dyn. sym: HVF that preserves \tilde{H} :
 $0 = L_X \tilde{H} = \{f_X, \tilde{H}\} = \frac{d}{dt} f_X$

DEF: (Lie algebra action)
 \mathfrak{g} (finite dim), real, Lie alg $[\cdot, \cdot]$

$\rho: \mathfrak{g} \rightarrow \mathfrak{X}'(P)$ Lie algebra homomorph
 $\rho([\xi, \eta]_{\mathfrak{g}}) = [\rho(\xi), \rho(\eta)]_{\pi}$ $\xi, \eta \in \mathfrak{g}$
 $\rho(a\xi + \eta) = a\rho(\xi) + \rho(\eta)$, $a \in \mathbb{R}$

Ex 1

$$P = T^* \mathbb{R}^n \ni (p, q)$$

$\mathfrak{g} = \text{translations} = (\mathbb{R}^n, +)$ Abelian

$$\rho(\xi) \Big|_{(q, p)} = \xi^i \frac{\partial}{\partial q^i}$$

$$[\rho(\xi), \rho(\eta)] = 0 \text{ b.c. } \xi^i \text{ is a constant}$$

Ex 2 $\mathfrak{g} = \text{rotations} = (\mathbb{R}^n, \times)$

$$[\xi, \eta]^k = (\xi \times \eta)^k = \epsilon^{kij} \xi^i \eta^j$$

$$\rho(\xi) \Big|_{(q, p)} = (\xi \times q)^i \frac{\partial}{\partial q^i} + (\xi \times p)_i \frac{\partial}{\partial p_i}$$

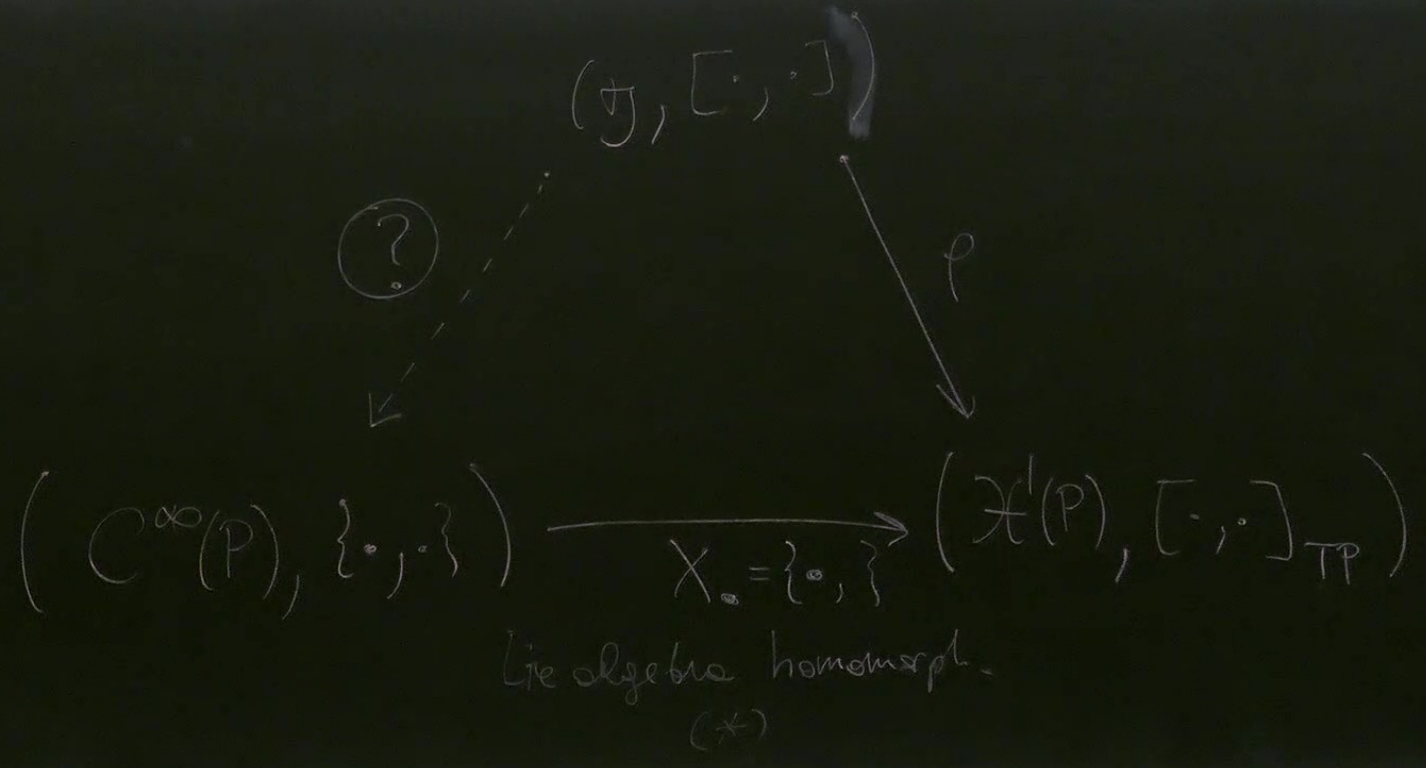
check $[\rho(\xi), \rho(\eta)] = \rho(\xi \times \eta)$

$$\{H, H\} = 0$$

sym.

$$p(a\delta + \eta) = ap(\delta) + p(\eta)$$

(*)



DEF [Comomentum map]

(P, ω) sympl.

(g, P) action on P

\mathfrak{g} admits a comomentum map if

the comom. map
 $\exists \check{J}: \mathfrak{g} \rightarrow C^\infty(P)$ s.t.

$$\boxed{p(\xi) \omega = -d\check{J}(\xi)} \quad (*)$$

Rmk: $p(\xi)$ is linear in $\xi \Rightarrow \check{J}(\xi)$ is linear in ξ

given \check{J} there exists a

$$J \in C^\infty(P, \mathfrak{g}^*)$$

$$(\check{J}(\xi))(z) = \langle J(z), \xi \rangle$$

↑ momentum map

$$i_{p(\xi)} \omega = -d\langle J, \xi \rangle \quad \leftarrow \text{natural pairing between } \sigma^*, \sigma$$

$$i_{p(\cdot)} \omega = -\langle dJ, \cdot \rangle$$

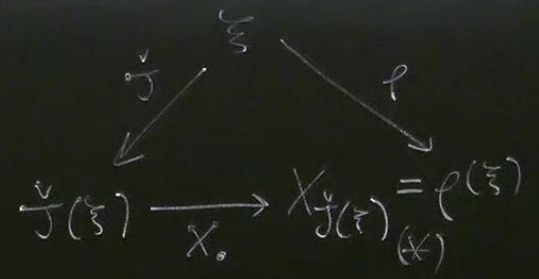
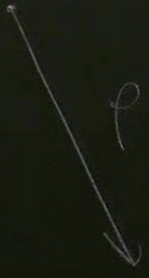
$$\boxed{i_p \omega = -dJ} \quad (*)$$

sym.

$$p(a\xi + \eta) = ap(\xi) + p(\eta), \quad a \in \mathbb{K}$$

$$(\mathfrak{g}, [\cdot, \cdot])$$

\mathfrak{g}



$$\left(C^\infty(P), \{\cdot, \cdot\} \right) \xrightarrow{X_\bullet = \{\cdot, \cdot\}} \left(\mathfrak{X}(P), [\cdot, \cdot]_{TP} \right)$$

Lie algebra homomorph.
(*)

$$i_{p(\xi)} \omega = -d\langle J, \xi \rangle \quad \leftarrow \text{natural pairing between } \sigma_j^*, \sigma_j$$

$$\langle, \rangle : \sigma_j^* \times \sigma_j \rightarrow \mathbb{R}$$

$$i_{p(\theta)} \omega = -\langle dJ, \cdot \rangle$$

$$\boxed{i_p \omega = -dJ} \quad (*)$$

$$p(\xi) = \text{HVF of } \check{J}(\xi) \quad \forall \xi \in \theta$$