Title: Quantum Gravity Lecture

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Collection: Quantum Gravity 2023/24

Date: April 05, 2024 - 9:00 AM

URL: https://pirsa.org/24040022

Recap

- sympl. $\omega = \frac{1}{2}\omega_{IJ}\, dz^I \wedge dz^J$
  1. CLOSED $d\omega = 0$
  2. NON DEG $\ker(\omega^\flat) = 0$
     $\mapsto$ P 2n-dim
- Poisson $\{\cdot,\cdot\} = \Pi^{IJ}\partial_I \wedge \partial_J : C^\infty(P)\times C^\infty(P) \to C^\infty(P)$
  - A. SKEW
  - B. BI DERIV.
  - C. JACOBI

  Rmk: Π can be degenerate

- SYMPL $\Rightarrow$ POISSON (1) $\Rightarrow$ (C)
- Poisson $\not\Rightarrow$ sympl, unless Π non deg.

Rmk

ω degenerate: "pre sympl"

DIFFERENT from Poisson

- SYMPLECTOMORPH

$\phi \in \mathrm{Diff}(P) \qquad \phi^* \omega = \omega$

$X \in \mathcal{X}^1(P) \qquad L_X \omega = 0$

- HAMILTONIAN V.F. (HVF)

s.t. $\exists f_X \in C^\infty(P) : i_X \omega = -df_X$

"Hamiltonian generator of X"

$X = \{f_X, \cdot\}$

- HVF $\Rightarrow$ SYMPL. $L_X \omega = i_X \underbrace{d\omega}_{=0} + d \underbrace{i_X \omega}_{-df} = \underbrace{-d^2 f}_{=0}$

Rmk

X HVF $\Rightarrow f_X$ unique up to constants

$i_X \omega = -df_1 = -df_2 \Rightarrow d(f_1 - f_2) = 0$ ☒

PROP If $H^1(P) = 0$, then sympl $\Rightarrow$ HVF

Pf: $H^1(P) := \dfrac{\text{closed 1-forms}}{\text{exact 1-forms}}$

if $H^1(P) = 0$, then closed $\Rightarrow$ exact

$0 \underset{\text{sympl}}{=} L_X \omega = i_X \underbrace{d\omega}_{=0} + d i_X \omega \Rightarrow i_X \omega$ closed, therefore exact if $H^1(P) = 0$

☒

PROP If $\exists$ sympl. pot θ $(\omega = d\theta)$ such that $\underline{L_X \theta = 0}$, then

- X is HVF
- $f_X = i_X \theta$ is generator

Pf $0 = L_X\theta = i_X d\theta + d i_X \theta$
$= i_X \omega + d f_X$ $\blacksquare$

Rmk

- X HVF if f_X exists
- $f \Rightarrow X_f$ HVF
$$X_f = \{f, \cdot\}$$

PROP $\{\cdot,\cdot\} = \omega^{-1}$

- $\omega(X_f, X_g) = \{f, g\}$
- $[X_f, X_g] = X_{\{f,g\}}$

Pf: Exercise.

$$\left(C^\infty(P), \{\cdot,\cdot\} \right) \xrightarrow[X_\bullet]{} \left(\mathfrak{X}^1(P), [\cdot,\cdot]_{TP} \right)$$

Lie algebra homomorph.

(*)

Example

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}\,\delta_{ij}\,\dot{q}^i\dot{q}^j - A_i(q)\,\dot{q}^i - V(q)$$

fixed 1-form on Q

fixed function on Q

① $p_i = \dfrac{\partial L}{\partial \dot{q}^i} = \delta_{ij}\,\dot{q}^j - A_i(q)$, $\omega = \sum dp_i \wedge dq^i$

$$\mathcal{H}(p,q) = \frac{1}{2}\,\delta^{ij}\,(p_i + A_i(q))(p_j + A_j(q)) + V(q)$$

$$X_{\mathcal{H}} = \mathrm{HVF} = \underbrace{(X_{\mathcal{H}})^i_q}_{\dot{q}^i}\,\frac{\partial}{\partial q^i} + \underbrace{(X_{\mathcal{H}})_{p_i}}_{\dot{p}_i}\,\frac{\partial}{\partial p_i}$$

② ϕ :

(q)

ed

nction $-\,?$

$\sum dp_i \wedge dq^i$

$(q)) + V(q)$

$(X_{\mathcal{H}})_{p_i} \frac{\partial}{\partial p_i}$

$\dot{p}_i$

② $\phi : (\tilde{q}, \tilde{p}) \longrightarrow (q, p) = (\tilde{q}, \tilde{p} - A)$

$\tilde{\omega} = \phi^* \omega = d\tilde{p}_i \wedge d\tilde{q}^i - \frac{1}{2} B_{ij}(q)\, d\tilde{q}^i \wedge d\tilde{q}^j$

$\tilde{\mathcal{H}} = \phi^* \mathcal{H} = \frac{1}{2} \delta^{ij} \tilde{p}_i \tilde{p}_j + V(\tilde{q})$

$\tilde{X}_{\tilde{\mathcal{H}}} = \phi_*^{-1} X_{\mathcal{H}} =$ encode some dynamics

$(\tilde{p} \sim \dot{q})$

$B_{ij} = \partial_i A_j - \partial_j A_i$

$B = dA$

$d\tilde{\omega} = 0$

$\frac{d}{dt} := X_{\mathcal{H}} = \{\mathcal{H}, \cdot\}$

$\frac{d}{dt} \mathcal{H} = \{\mathcal{H}, \mathcal{H}\} = 0$

$B_{ij} = \partial_i A_j - \partial_j A_i$. HVF encode syms.

$d\tilde{q}^j$

$B = dA$

$d\tilde{\omega} = 0$

eom $\dot{H} = 0$

d

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SYMMETRIES

Def: Kin. sym. $\Leftrightarrow$ HVF ($\to L_X \omega = 0$)

Dyn. sym: HVF that preserves $\mathcal{H}$:

$0 = L_X \mathcal{H} = \{f_X, \mathcal{H}\} = \frac{d}{dt} f_X$

DEF: (Lie algebra action)

$\mathfrak{g}$ (finite dim), real, Lie alg $[\cdot,\cdot]$

$\rho: \mathfrak{g} \longrightarrow \mathfrak{X}^1(P)$ Lie algebra homomorph

$\rho([\xi,\eta]_{\mathfrak{g}}) = [\rho(\xi), \rho(\eta)]_{\pi}$ $\xi,\eta \in \mathfrak{g}$

$\rho(a\xi + \eta) = a\rho(\xi) + \rho(\eta)$, $a \in \mathbb{R}$

Ex 1

$P = T^*\mathbb{R}^n \ni (p,q)$

$\mathfrak{g}$ = translations $= (\mathbb{R}^n, +)$ Abelian

$\rho(\xi)\big|_{(q,p)} = \xi^i \dfrac{\partial}{\partial q^i}$

$[\rho(\xi), \rho(\eta)] = 0$ b.c. ξ^i is a constant

Ex 2 $\mathfrak{g}$ = rotations $= (\mathbb{R}^n, \times)$ $[\xi, \eta]^k = (\xi \times \eta)^k = \epsilon^k{}_{ij}\xi^i\eta^j$

$\rho(\xi)\big|_{(q,p)} = (\xi \times q)^i \dfrac{\partial}{\partial q^i} + (\xi \times p)_i \dfrac{\partial}{\partial p_i}$, check $[\rho(\xi), \rho(\eta)]$

$= \rho(\xi \times \eta)$

$\{H, H\} = 0$ sym. $\rho(a\xi + \eta) = a\rho(\xi) + \rho(\eta)$

$(*)$

$$\begin{tikzcd}[column sep=small]
 & (\mathfrak{g}, [\cdot,\cdot]) \arrow[dl, dashed, "?"'] \arrow[dr, "\rho"] & \\
(C^\infty(P), \{\cdot,\cdot\}) \arrow[rr, "X_\cdot = \{\cdot,\ \}"'] & & (\mathfrak{X}^1(P), [\cdot,\cdot]_{TP})
\end{tikzcd}$$

Lie algebra homomorph.

$(*)$

DEF [Comomentum map]

(P, ω) sympl.

$(\mathfrak{g}, \rho)$ action on P

ρ admits a comomentum map if $\exists\, \check{J} : \mathfrak{g} \to C^\infty(P)$ s.t. (the comom. map)

$$\boxed{i_{\rho(\xi)}\,\omega = -d\check{J}(\xi)} \quad (*)$$

Rmk: $\rho(\xi)$ is linear in $\xi \Rightarrow \check{J}(\xi)$ is linear in ξ

given $\check{J}$ there exists a $J \in C^\infty(P, \mathfrak{g}^*)$: $\left(\check{J}(\xi)\right)(z) = \langle J(z), \xi \rangle$

momentum map

$$i_{\rho(\xi)}\,\omega = -d\langle J, \xi\rangle$$

natural pairing between $\mathfrak{g}^*, \mathfrak{g}$

$\langle\ ,\ \rangle : \mathfrak{g}^* \times \mathfrak{g} \longrightarrow \mathbb{R}$

$$i_{\rho(\cdot)}\,\omega = -\langle dJ, \cdot\rangle$$

$$\boxed{i_\rho\,\omega = -dJ} \qquad (*)$$

sym.

$\rho(a\xi+\eta) = a\rho(\xi)+\rho(\eta)$, $a\in\mathbb{R}$

$$\begin{tikzcd}
& \xi \arrow[dl, "\check{J}"'] \arrow[dr, "\rho"] & \\
\check{J}(\xi) \arrow[rr, "X_\bullet"'] & & X_{\check{J}(\xi)} \underset{(*)}{=} \rho(\xi)
\end{tikzcd}$$

$$\begin{tikzcd}
& (\mathfrak{g}, [\cdot,\cdot]) \arrow[dl, dashed, "\check{J}"'] \arrow[dr, "\rho"] & \\
(C^\infty(P), \{\cdot,\cdot\}) \arrow[rr, "X_\bullet = \{\bullet, \}"'] & & (\mathfrak{X}^1(P), [\cdot,\cdot]_{TP})
\end{tikzcd}$$

Lie algebra homomorph.
(*)

$i_{\rho(\xi)}\omega = -d\langle J, \xi \rangle$ ← natural pairing between $\mathfrak{g}^*, \mathfrak{g}$

$\langle \, , \, \rangle : \mathfrak{g}^* \times \mathfrak{g} \longrightarrow \mathbb{R}$

$i_{\rho(\cdot)}\omega = -\langle dJ, \cdot \rangle$

$\boxed{i_{\rho}\,\omega = -dJ}$ $(*)$

$\rho(\xi) =$ HVF of $\check{J}(\xi)$ $\forall \xi \in \mathfrak{g}$