Title: Quantum Gravity Lecture

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Collection: Quantum Gravity 2023/24

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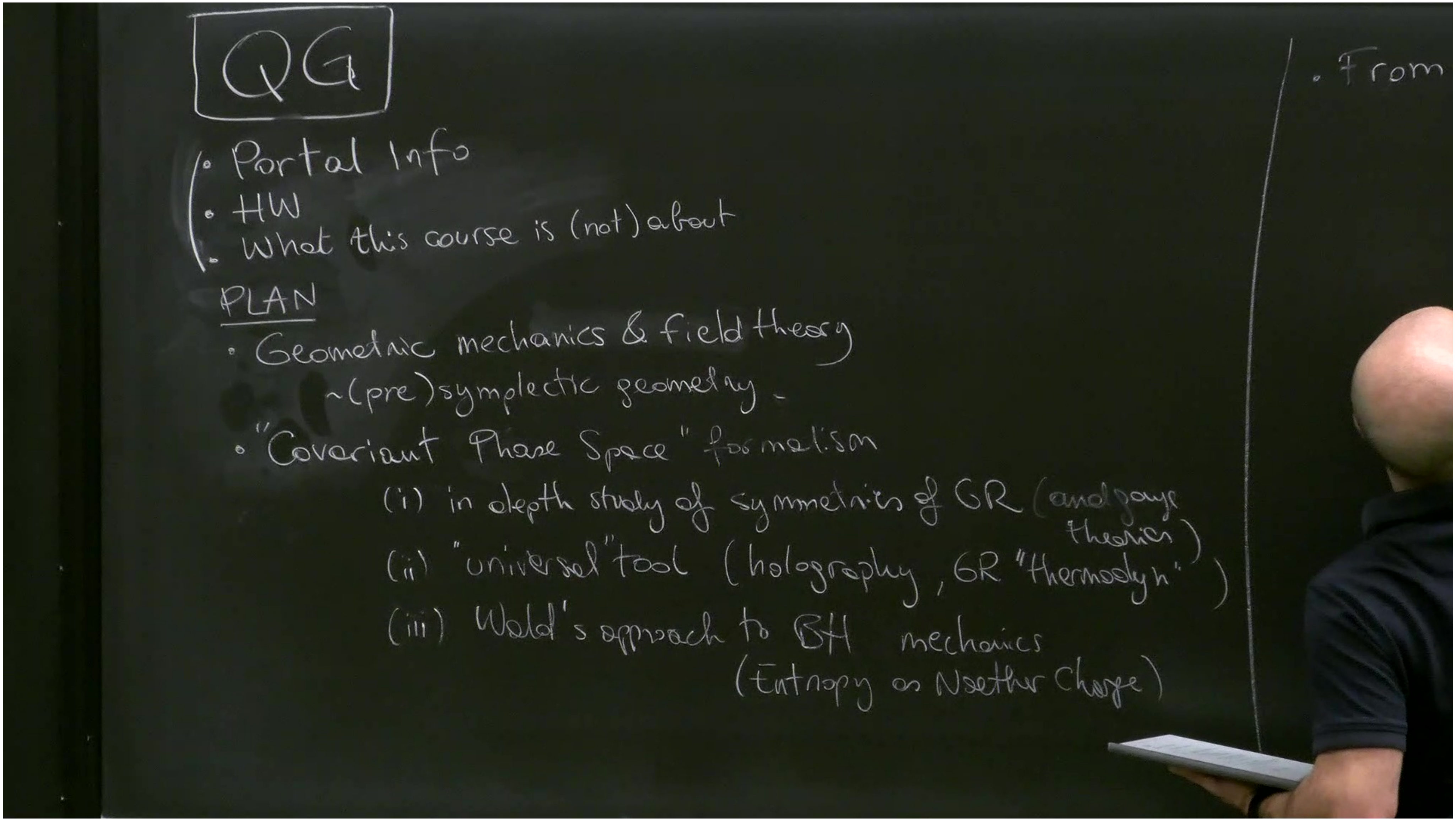
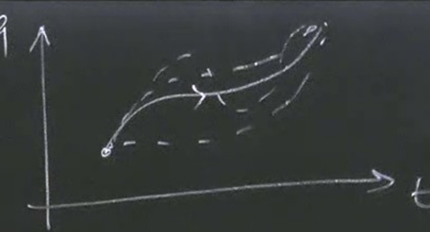

QG
• Portal Info
• HW
• What this course is (not) about
PLAN
• Geometric mechanics & field theory
~(pre)symplectic geometry
• "Covariant Phase Space" formalism
(i) in depth study of symmetries of GR (and gauge theories)
(ii) "universal" tool (holography, GR "thermodyn")
(iii) Wald's approach to BH mechanics
(Entropy as Noether Charge)
• From

- From CPS to canonical ph. sp.
  - (i) GR constraint algebra
  - (ii) reconstruct GR from its sym.
- Glimpses into QG
  - (i) WdW eq.
  - (ii) Ashtekar's variables

1) GEOM MECHANICS
(symplectic geom.)



action principle.

$$S[q^i(t)] = \int_{t_0}^{t_1} dt \; \mathcal{L}(q(t), \dot{q}(t))$$

"history"

$$q(t) : \mathbb{R} \longrightarrow Q$$

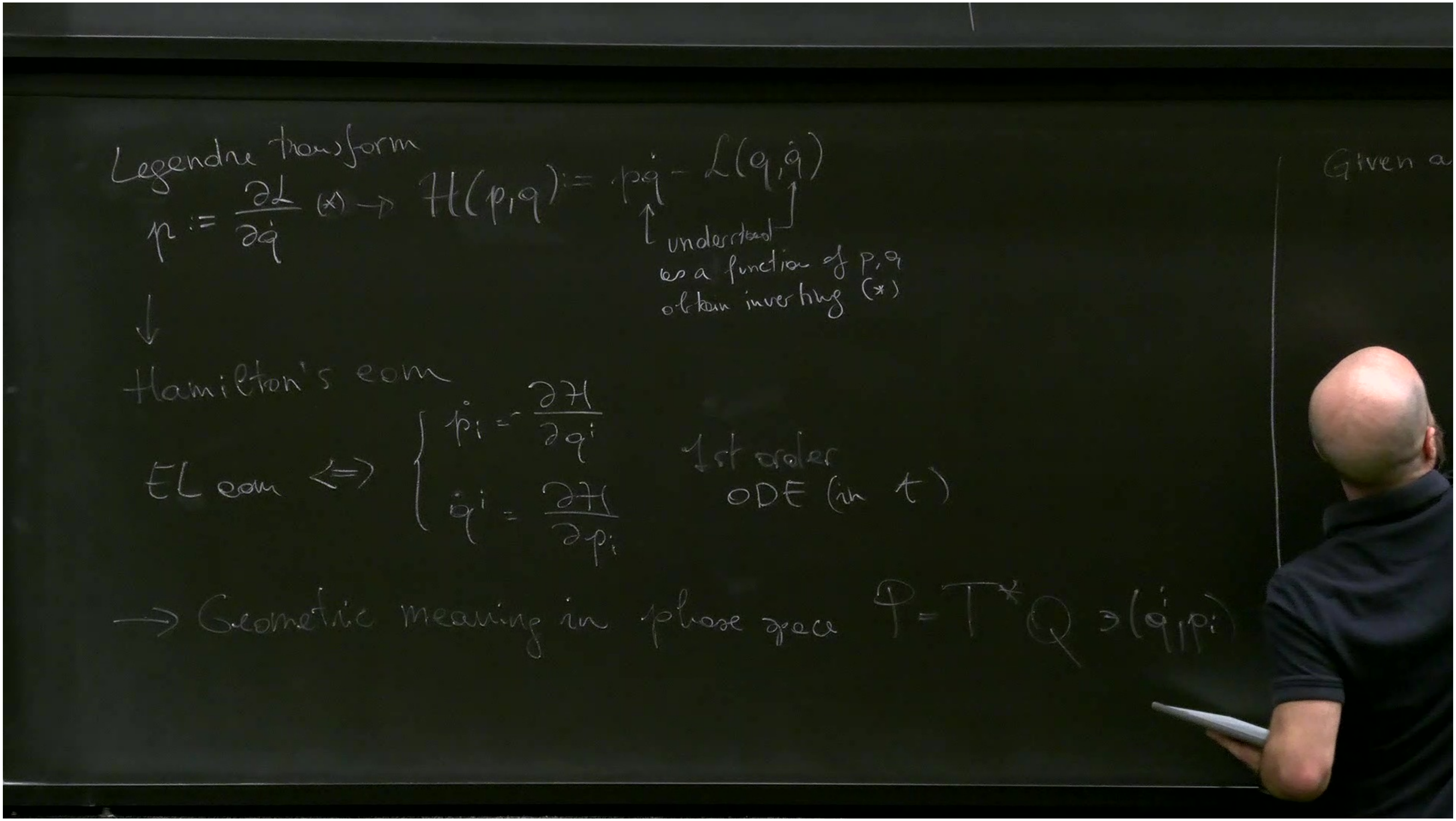
time — configuration space

$$\mathcal{L}(q, v) : TQ \longrightarrow \mathbb{R}$$

canonical momentum p

$$\delta S[q(t)] = \int_{t_0}^{t_1} dt \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{d}{dt} \delta q \right) = \int_{t_0}^{t_1} dt \overbrace{\left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right)}^{\text{Euler-Lagrange eom}} \delta q + \left[\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \right]_{t_0}^{t_1}$$

Phys. traj = stationary history of S @ fixed $\delta q(t_1) = \delta q(t_0) = 0$.
$\delta S = 0$


Legendre transform
$p := \frac{\partial L}{\partial \dot q}$ (*) $\to$ $\mathcal{H}(p,q) := p\dot q - \mathcal{L}(q,\dot q)$
understood as a function of p, q obtain inverting (*)
Hamilton's eom
EL eom $\Longleftrightarrow$ $\begin{cases} \dot p_i = -\frac{\partial \mathcal{H}}{\partial q^i} \\ \dot q^i = \frac{\partial \mathcal{H}}{\partial p_i} \end{cases}$
1st order ODE (in t)
$\to$ Geometric meaning in phase space $P = T^*Q \ni (q^i, p_i)$
Given a

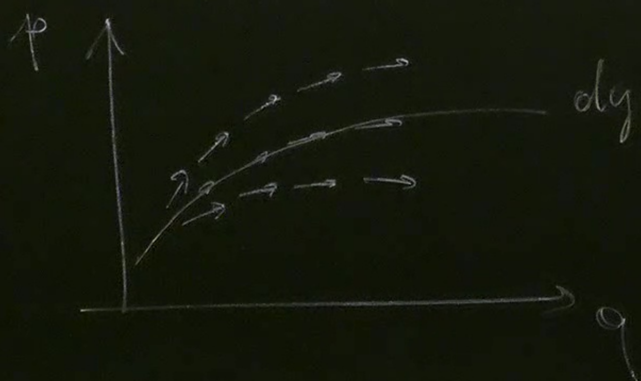
$\partial J = 0$

(q^i, p_i)

Given a Hamiltonian $\mathcal{H}(p,q)$

$X_{\mathcal{H}} \in \mathfrak{X}^1(P)$ vector field on P

$$X_{\mathcal{H}} = \begin{pmatrix} -\partial H/\partial q \\ \partial H/\partial p \end{pmatrix} = \sum_i -\frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}$$



dynamics = integral curves of $X_{\mathcal{H}}$

$\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i}$

integral curves of $X_{\mathcal{H}}$

Poisson bracket

$$\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P)$$

$$(f, g) \mapsto \{f,g\} = \sum \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}$$

$$X_{\mathcal{H}} = \{\mathcal{H}, \cdot\} \quad [\text{sign}]$$

Symplectic Geom

DEF [symplectic mfd]

(P, ω), P 2n-dim mfd

$\omega \in \Omega^2(P)$ s.t.

1. CLOSED
2. NON DEGENERATE

① Closed $d\omega = 0$

② Non-degenerate

$$\omega^\flat : TP \longrightarrow T^*P$$

$$(z, X) \longmapsto \omega^\flat(z,X) = (z, i_X\omega(z))$$

$$\left[\begin{array}{l} \equiv X \lrcorner\, \omega(z) \equiv \omega(X)(z) \\ \equiv X^I \omega_{IJ}\, dz^J \end{array} \right]$$

$\boxed{\text{injective}}$

in finite dim $\Rightarrow$ surjectivity $\Rightarrow$ bijectivity i.e. invertibili[ty]

$(\omega$ ... $)$ ... well defined.

... to Poisson ...

$= 0$

$$TP \longrightarrow T^*P$$

$$(z, X) \longmapsto \omega^\flat(z, X) = (z, i_X \omega(z))$$

$$\left[\begin{aligned} &\equiv X \lrcorner\, \omega(z) \equiv \omega(X)(z) \\ &\equiv X^I \omega_{IJ}\, dz^J \end{aligned} \right]$$

$\boxed{\text{ive}}$

$\Rightarrow$ surjectivity $\Rightarrow$ bijectivity i.e. invertibility

$(\omega_{IJ})^{-1}$ is well defined.

(related to Poisson bracket
- see below)

② NON-DEGENERATE

THM Q n-dim mfd.
Then, $P = T^*Q$ is canonically symplectic.

Pf: $T^*_q Q$ vector sp. w/ basis $\{dq^i\}$
denote p_i the coords on $T^*_q Q$

$\rightarrow T^*_q Q \ni \alpha = \sum_i p_i(\alpha)\, dq^i$

$\rightarrow z^I = (q^i, p_i)$ coords on P

$\rightarrow \theta(q,p) := \sum_i p_i\, dq^i \in \Omega^1(P)$

$\omega := d\theta = \sum_i dp_i \wedge dq^i$

SYMPLECTIC POTENTIAL
(related to $\delta S = \cdots + p\,\underline{\underline{\delta q}}$)

② NON-DEGENERATE

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$\rightarrow \theta(q,p) := \sum_i p_i\, dq^i \in \Omega^1(P)$ SYMPLECTIC POTENTIAL (related to $\delta S = \cdots + p\,\underline{\underline{\delta q}}$)

$\omega := d\theta = \sum_i dp_i \wedge dq^i$ SYMPLECTIC

1. Closed $d\omega = 0$
2. Non-degenerate

$$\omega^\flat : TP \longrightarrow T^*P$$

$$(z, X) \longmapsto \omega^\flat(z, X) = (z, i_X \omega(z))$$

$$\left[\equiv X \lrcorner\, \omega(z) \equiv \omega(X)(z) \equiv X^I \omega_{IJ}\, dz^J \right]$$

$\omega^\flat$ $\uparrow$ $\boxed{\text{injective}}$

in finite dim $\Rightarrow$ surjectivity $\Rightarrow$ bijectivity i.e. invertibility

$\left(\omega_{IJ}(z)\right)^{-1}$ is well defined.

(related to Poisson bra[...]

- see below

THM (Darboux)

(P,ω) symplectic, locally looks like T^*Q.

Namely: $\forall z \in P$, $\exists$ open neighb and coords $U \simeq \mathbb{R}^{2n} \ni (q^1, \dots, q^n, p_1, \dots, p_n) = z^I$

s.t. $\omega|_U = \frac{1}{2} \omega_{IJ}\, dz^I \wedge dz^J = \sum_i dp_i \wedge dq^i$.

DEF: $\phi \in \mathrm{Diff}(P)$, $\phi^*\omega = \omega$ is a SYMPLECTOMORPHISM.

$X \in \mathfrak{X}^1(P)$, $L_X \omega = 0$ — " —

Rmk: $\exists\ \infty$ many symplectomorph (infinite dim grp)

ω non deg. $\Rightarrow$ $\Pi \in \mathfrak{X}^2(P)$

$$\Pi(z) = \Pi^{IJ}(z) \frac{\partial}{\partial z^I} \wedge \frac{\partial}{\partial z^J}$$

$$= \left(\omega_{IJ}(z)\right)^{-1} \frac{\partial}{\partial z^I} \wedge \frac{\partial}{\partial z^J}$$

Darboux coords.

$$\{f, g\}_\Pi := \Pi^{IJ}(z) \frac{\partial f}{\partial z^I} \frac{\partial g}{\partial z^J} = \sum \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}$$

PROP ω sympl $\to \{\cdot,\cdot\}$ which is

- skew
- (bi)derivation
- Jacobi. $\leftarrow$ (from $d\omega = 0$)

Inspired by $X_{\mathcal{H}} = \{\mathcal{H}, \cdot\} \in \mathfrak{X}^1(P)$

$\rightarrow$ $X_f = \{f, \cdot\}$ is the "Hamiltonian v.f. of $f \in C^\infty(P)$"

DEF: $X \in \mathfrak{X}^1(P)$ s.t. $\exists\, f_X \in C^\infty(M)$; $\iota_X \omega = -df_X$

is called HAMILTONIAN.

Rmk. $X^I \omega_{IJ} = -\partial_J f$ $\quad \Pi = \omega^{-1}$

$X^I = -\Pi^{IJ} \partial_J f$

$X^I \partial_I \equiv \{f, \cdot\}$

$-df_X$

HVF are symplectom.

$$L_X \omega = \iota_X d\omega + d\iota_X \omega$$

$$= 0 - d^2 f_X = 0$$