Title: Strong Gravity Lecture

Speakers: William East

Collection: Strong Gravity 2023/24

Date: April 25, 2024 - 10:15 AM

URL: https://pirsa.org/24040017

Generation of GWs in the weak field regime

(4) $(x^0$

$g_{ab} = \eta_{ab}, \qquad T_{ab} \neq 0$

$$\Box \bar{h}_{ab} = -16\pi T_{ab}$$

Green's function $\Box G(x^a - \bar{x}^a) = \delta^4(x^a - \bar{x}^a)$

$$G(x^a - \bar{x}^a) = \frac{-1}{4\pi |x^i - \bar{x}^i|} \delta\left[|x^i - \bar{x}^i| - (x^0 - \bar{x}^0) \right] \Theta(x^0 - \bar{x}^0)$$

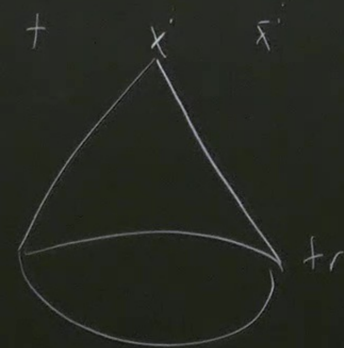
$$\bar{h}_{ab}(x^a) = -16\pi \int G(x - \bar{x}^a)\, T_{ab}(\bar{x}^a)\, d^4\bar{x}$$

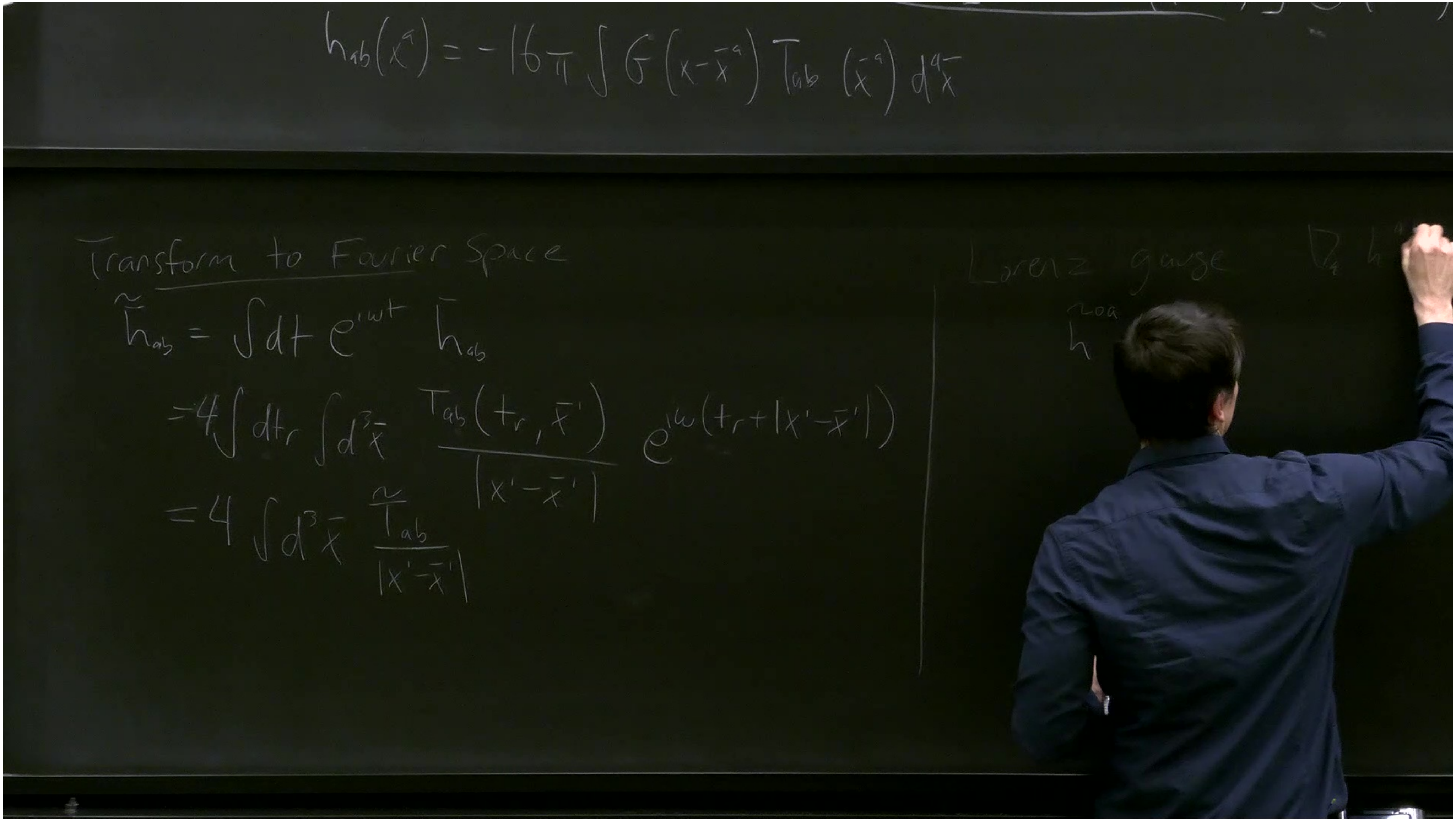
$$\theta(x^0-\bar{x}^0) = \begin{cases} 1 & x^0-\bar{x}^0 > 0 \\ 0 & x^0-\bar{x}^0 \le 0 \end{cases}$$

$(x^0-\bar{x}^0)$

$$\bar{h}_{ab} = 4\int \frac{T_{ab}(t-|x^i-\bar{x}^i|, \bar{x}^i)}{|x^i-\bar{x}^i|}\, d^3\bar{x}$$

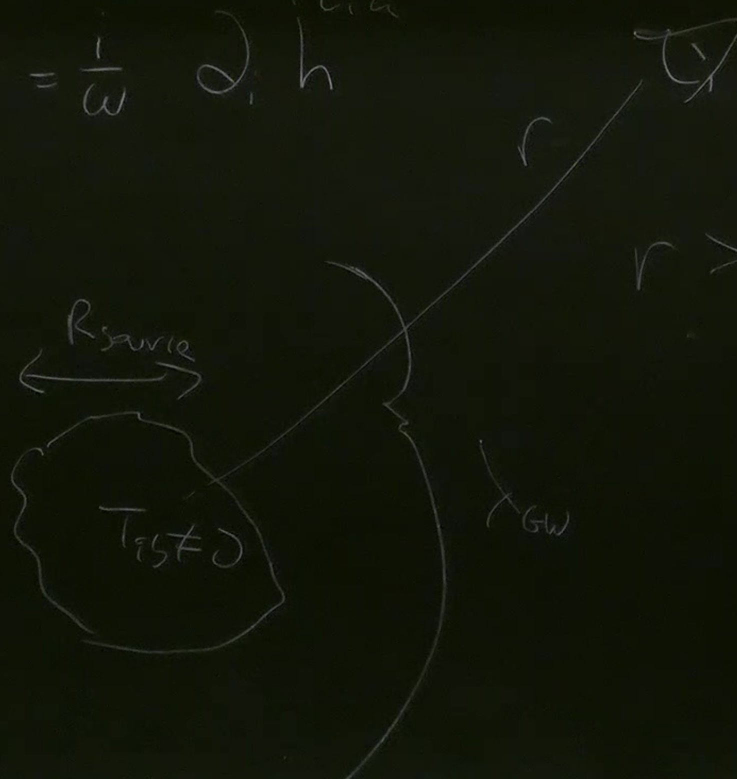
$t_r = t - |x^i - \bar{x}^i|$ (retarded time)




$h_{ab}(x^a) = -16\pi \int G(x - \bar{x}^a)\, T_{ab}(\bar{x}^a)\, d^4\bar{x}$
Transform to Fourier Space
$\tilde{h}_{ab} = \int dt\, e^{i\omega t}\, \bar{h}_{ab}$
$= 4\int dt_r \int d^3\bar{x}\, \frac{T_{ab}(t_r, \bar{x}')}{|x' - \bar{x}'|}\, e^{i\omega(t_r + |x' - \bar{x}'|)}$
$= 4 \int d^3\bar{x}\, \frac{\tilde{T}_{ab}}{|x' - \bar{x}'|}$
Lorenz gauge

Lorenz gauge $\nabla_a \bar{h}^{ab} = 0$

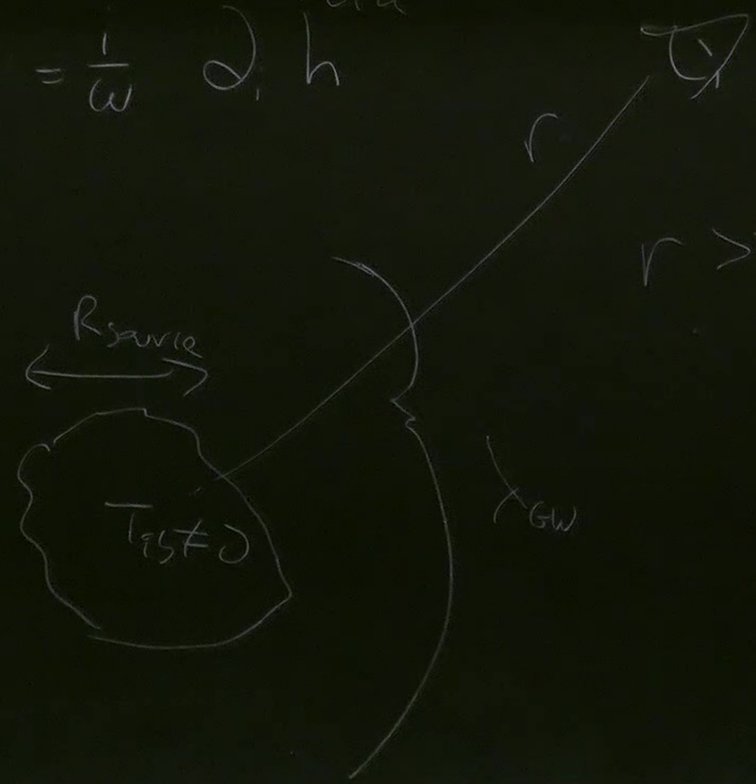
$$\tilde{h}^{0a} = \frac{1}{\omega}\partial_i \tilde{h}^{ia}$$



$$r \gg \lambda_{GW} = \frac{1}{\omega_{GW}} \gg R_{source}$$

Lorenz gauge $\nabla_a \tilde{h}^{ab} = 0$

$$\tilde{h}^{0a} = \frac{1}{\omega} \partial_i \tilde{h}^{ia}$$



$$r \gg \lambda_{GW} = \frac{1}{\omega_{GW}} \gg R_{source}$$

$$\frac{e^{i\omega|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \approx \frac{e^{i\omega r}}{r}$$

$$\gg \lambda_{GW} = \frac{1}{\omega_{GW}} \gg R_{source}$$

$$\int d^3\bar{x}\,\tilde{T}_{ij} = \int d^3\bar{x}\left[\partial_k\left(\bar{x}^i\,\tilde{T}^{kj}\right) - \partial_k\left(\tilde{T}^{kj}\right)\bar{x}^i\right]$$

$$\partial_k\left(\bar{x}^i\,\tilde{T}^{kj}\right) \to 0 \qquad \partial_k\left(\tilde{T}^{kj}\right) = i\omega\,\tilde{T}^{0j}$$

$$= \frac{i\omega}{2}\int\left(\tilde{T}^{0j}\bar{x}^i + \tilde{T}^{i0}\bar{x}^j\right)d^3\bar{x}$$

$$= \frac{i\omega}{2}\int\left[\partial_\ell\left(\bar{x}^i\bar{x}^j\,\tilde{T}^{0\ell}\right) - \bar{x}^i\bar{x}^j\left(\partial_\ell\,\tilde{T}^{0\ell}\right)\right]d^3\bar{x}$$

$$= \frac{-\omega^2}{2}\underbrace{\int \bar{x}^i\bar{x}^j\,\tilde{T}^{00}\,d^3x}_{\tilde{I}_{ij}}$$

$$\partial_k \left(\tilde{T}^{kj} \right) \bar{x}^{i} \Big]$$

$$= i\omega \, \tilde{T}^{0j}$$

$$\Big) d^3\bar{x}$$

$$\bar{x}^i \bar{x}^j \left(\partial_\ell \tilde{T}^{0\ell} \right) \Big] d^3\bar{x}$$

$$\nabla_a T^{ab} = 0$$

$$i\omega \, \tilde{T}^{0b} + \partial_\ell \tilde{T}^{\ell b} = 0$$

$$\tilde{\bar{h}}_{ij} = -2\omega^2 \frac{e^{i\omega r}}{r} \tilde{I}_{ij}$$

$$\bar{h}_{ij} = \frac{2}{r} \ddot{I}_{ij}(t-r)$$

Projection tensor $\quad P_{ij} = \delta_{ij} - \frac{x^i}{r}\frac{x^j}{r}$

$$\bar{h}^T_{ij} = \bar{h}^{kl} P_{ik} P_{jl}$$

$$h^{TT}_{ij} = \bar{h}^{kl} P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \bar{h}^{kl}$$

$$h_{ij}^{TT} = \frac{2}{r}\ddot{I}^{kl}\left[P_{ik}P_{jl} - \frac{1}{2}P_{kl}P_{ij}\right]$$

$$\tilde{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I$$

Lower-order multipole moments of $\rho = T^{00}$

Monopole: $M(t) = \int \rho \, d^3\bar{x}$

$$h \sim \cancel{\frac{G}{c^2}\frac{M(t)}{r}}$$

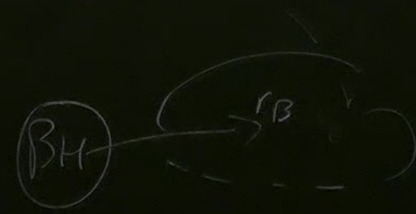
Dipole moment

$$\int d^3\bar{x}\, \rho\, \bar{x} = D_i(t)$$

$$h \sim \cancel{\frac{G}{c^3}\frac{\dot{D}_i}{r}}$$

$$\frac{e^{i\omega|\vec{x}' - \vec{x}'|}}{|\vec{x}' - \vec{x}'|} \approx \frac{e^{i\omega r}}{r}$$

$$r \gg \lambda_{GW} = \frac{1}{\omega_{GW}} \gg R_{\text{source}}$$

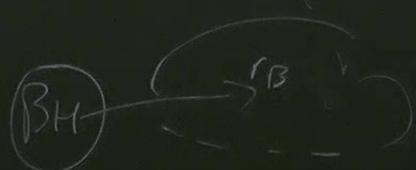


$$r_B = \frac{1}{\alpha\mu} \,, \quad \alpha := M_{BH} \cdot \mu$$

$$\omega_{GW} = 2\mu \qquad \alpha \ll 1$$

$$\frac{e^{i\omega|x'-\bar{x}'|}}{|x'-\bar{x}'|} \approx \frac{e^{i\omega r}}{r}$$

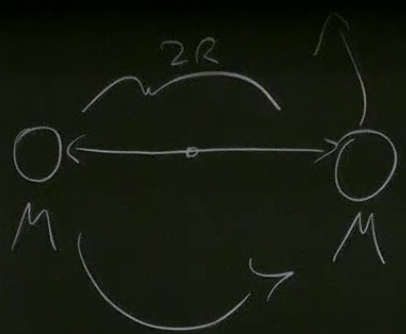
$$r >> \lambda_{GW} = \frac{1}{\omega_{GW}} >> R_{source}$$



$$r_B = \frac{1}{\alpha\mu}, \quad \alpha := M_{BH}\cdot\mu$$

$$\omega_{GW} = 2\mu \qquad \alpha << 1$$

$$\lambda_{GW} = \frac{1}{\omega_{GW}} = \frac{1}{2\mu} << r_B = \frac{1}{\alpha\mu}$$



$$\omega_{GW} = 2\omega = \frac{4\pi}{T} = \sqrt{\frac{M}{R^3}} = \frac{1}{R}\underbrace{\left(\frac{M}{R}\right)^{1/2}}_{\ll 1}$$

Quadrapole formula applies

Energy and angular momentum of GWs

$G^{(1)}_{ab}[$

$$g_{ab} = \hat{g}_{ab} + h^{(1)}_{ab} + h^{(2)}_{ab} \quad \leftarrow \text{second order}$$

$$R_{ab} = \cancel{R^{(0)}_{ab}} \to 0 + R^{(1)}_{ab} + R^{(2)}_{ab} \ldots$$

First order : $G^{(1)}_{ab}[h^{(1)}_{ab}] = 0$

Second order : $G^{1}_{ab}[h^{(2)}_{cd}] + G^{2}_{ab}[h^{(1)}_{cd}] = 0$

WS

cond order

$] = 0$

$$G^{(1)}_{ab}\left[h^{(2)}_{cd}\right] = 8\pi\, t_{ab} := -G^{(2)}_{ab}\left[h^{(1)}_{cd}\right]$$

$$\hat{\nabla}_a t^{ab} = 0$$

t_{ab} not gauge invariant

Get around by averaging over wavelength of GWs

$$\langle \partial_a X \rangle = 0$$

$$\langle X \partial_a Y \rangle = -\langle Y \partial_a X \rangle$$

$$\langle \partial_c (XY) \rangle = 0$$

$$t_{ab} = \frac{1}{32\pi}\left\langle \partial_a \bar{h}_{cd}\,\partial_b \bar{h}^{cd} - \frac{1}{2}\partial_a \bar{h}\,\partial_b \bar{h} - 2\left[\partial_{(a}\bar{h}_{b)c}\right]\partial_d \bar{h}^{cd}\right\rangle$$

In TT gauge: $$t_{ab} = \frac{1}{32\pi}\left\langle \left(\partial_a h_{cd}^{TT}\right)\left(\partial_b h^{cd}_{TT}\right)\right\rangle$$

(or $\partial_a \to \hat{\nabla}_a$ for non-flat background)

$$E_{GW} = \int_{\Sigma_t} t_{00}\, d^3x$$

For plane wave: $$E_{GW} = \frac{\omega^2}{32\pi}\left\langle h_+^2 + h_\times^2\right\rangle$$

$$\frac{dE_{GW}}{dt} = \int t_{0i}\, \underset{\text{unit normal}}{\hat{n}^i}\, dS = \frac{1}{5} \left\langle \left(\dddot{I}_{ij}\, \dddot{I}^{ij} \right) \right\rangle \Big|_{t=t_r}$$

$$\frac{dJ^i_{GW}}{dt} = \int \varepsilon^{ijk} (\hat{n}_j r)\, t_{0k}\, dS = \frac{2}{5} \varepsilon^{ijk} \left\langle \ddot{I}_{lj}\, \dddot{I}_k{}^{l} \right\rangle$$