Title: Strong Gravity Lecture

Speakers: William East

Collection: Strong Gravity 2023/24

Date: April 18, 2024 - 10:15 AM

URL: https://pirsa.org/24040015

Initial data / constraint equations

For Conformal Rescaling: $g_{ab} = \omega^2 \tilde{g}_{ab}$

Difference in Christ. Symbols $C^c_{ab} = \Gamma^c_{ab} - \tilde{\Gamma}^c_{ab} = \omega^{-1}\left(\delta^c_a \tilde{\nabla}_b \omega + \delta^c_a \tilde{\nabla}_a \omega - \tilde{g}_{ab} \tilde{\nabla}^c \omega \right)$

Decomposition of extrinsic curvature:

$$K^{ij} = \underbrace{A^{ij}}_{\text{trace-free}} + \frac{1}{3} K \gamma_{ij} \quad , \quad A^{ij} = \psi^{-10} \hat{A}^{ij}$$

$\tilde{g}_{ab} \tilde{D}^i \omega)$

$$D_j K^{ij} = D_j A^{ij} + \frac{1}{3} D^i K$$

$$D_j A^{ij} = \tilde{D}_j A^{ij} + C^i_{jk} A^{kj} + C^j_{jk} A^{ik}$$

$$= \tilde{D}_j A^{ij} + 10 A^{ij} \tilde{D}_j \ln\psi - 2 \cancelto{0}{(\tilde{D}^i \ln\psi) \tilde{\gamma}_{jk} A^{jk}}$$

$$= \psi^{-10} \tilde{D}_j (\psi^{10} A^{ij}) = \psi^{-10} \tilde{D}_j \hat{A}^{ij}$$

Mon. const $$\tilde{D}_j \hat{A}^{ij} - \frac{2}{3} \psi^6 \tilde{D}^i K = 8\pi \psi^{10} \rho^i$$

Further decompose $\hat{A}^{ij} = (\tilde{\mathbb{L}} X)^{ij} + \hat{A}^{ij}_{TT}$ (Generalization of Helmholtz Thm.)

longitudinal part

transverse part (trace-free)

TT means $\tilde{D}_j \left(\hat{A}^{ij}_{TT}\right) = 0 = \tilde{\gamma}_{ij} \hat{A}^{ij}$

Conformal Killing operator $(\tilde{\mathbb{L}} X)^{ij} = \tilde{D}^i X^j + \tilde{D}^j X^i - \frac{2}{3}\left(\tilde{D}_k X^k\right) \tilde{\gamma}^{ij}$

$$\vec{F} = \nabla\phi + \nabla\times\vec{A}$$

$$\tilde{\Delta}_L X^i = \tilde{D}_j (\tilde{\mathcal{L}}X)^{ij} = \tilde{D}_j \hat{A}^{ij}$$ (by transverse property)

CTT

Hamiltonian const. $$\tilde{D}_i\tilde{D}^i\psi - \frac{1}{8}\tilde{R}\psi + \frac{1}{8}\left[(\tilde{\mathcal{L}}X)_{ij} + \hat{A}_{ij}^{TT}\right]^2\psi^{-7} - \frac{1}{12}K^2\psi^5 + 2\pi\tilde{E}\psi^{-n+5} = 0$$

Mom. const. $$(\tilde{\Delta}_L X)^i - \frac{2}{3}\psi^6\tilde{D}^i K = 8\pi\tilde{p}^i\psi^{10-m}$$

$$\tilde{E} = \psi^n E \quad , \quad \tilde{p}^i = \psi^m p^i$$

Choose $m=10$, mom. const. independent of ψ for $K=$ constant

$\psi = \bar{\psi} + \varepsilon$, $|\varepsilon| << |\bar{\psi}|$

$$\tilde{D}_i \tilde{D}^i \varepsilon = \underbrace{\left[\frac{1}{8}\tilde{R} + \frac{7}{8} \hat{A}^{ij}\hat{A}_{ij} + \frac{5}{2}k^2 + 2\pi (n-5)\tilde{E} \right]}_{C} \varepsilon$$

If $C \geq 0$, $\varepsilon = 0$ on domain by maximum principle

Here want $n \geq 5$ for local uniqueness

$n=8$:

Dominant energy condition

$$-T^a{}_b n^b \leq 0$$

$\hat{n}$ future pointing, causal

$$-E^2 + p_i p^i \leq 0$$

$$p_i p^i \leq E^2$$

ip'le

$$\tilde{E}^2 \geq \tilde{\gamma}_{ij} \tilde{p}^i \tilde{p}^j$$

$$E^2 = \psi^{-16} \tilde{E}^2 \geq \psi^{-16} \tilde{\gamma}_{ij} \tilde{p}^i \tilde{p}^j = \gamma_{ij} p^i p^j$$

Recap of CTT constraint eqns

γ_{ij} , K^{ij}

$6 + 6 = 12$

Free data: $\tilde{\gamma}_{ij}$, K , $\hat{A}^{ij}_{TT}$, $\tilde{E}$, $\tilde{p}^i$

$5 \quad 1 \quad 2 = 8$

(solve for)

Constrained data: ψ, X^i

$1 + 3 = 4$

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \qquad K_{ij} = \frac{1}{3} K \gamma_{ij} + \psi^{-10}\left(\hat{A}^{ij}_{TT} + (\tilde{\mathbb{L}} X)^{ij} \right)$$

$$E = \psi^{-8} \tilde{E} \ , \quad p^i = \psi^{-10} \tilde{p}^i$$

Simple Example

Vaccum: $\tilde{E} = \tilde{p}^i = 0$

Maximal $K = 0$, $\hat{A}^{ij}_{TT} = 0$

Conformally flat: $\gamma_{ij} = f_{ij}$

BCs: $\psi \to 1$, $X^i = 0$ as $r \to \infty$

Eqns: $\partial_i \partial^i \psi + \frac{1}{8} (\mathcal{L}X)^{ij} (\mathcal{L}X)_{ij} \psi^{-7} = 0$, $\Delta_L X^i = 0$

Example (i) $\Sigma_0 = \mathbb{R}^3 \Rightarrow \gamma_{ij} = f_{ij}\ ,\ K_{ij} = 0$

Example (ii) $\Sigma_0 = \mathbb{R}^3 / B_R$

at $r = R$ $\quad D_i s^i = 0 = \tilde{D}_i\left(\psi^4 \tilde{s}^i\right)$

$$\frac{1}{r^2}\partial_r\left(\psi^4 r^2\right) = 0$$

$$\partial_r \psi + \frac{\psi}{2r} = 0 \quad \text{at } r = R$$

$$\psi \to 1 \quad \text{at } r \to \infty$$

$$\partial_i \partial^i \psi = 0$$

Unique solution

$$\psi = 1 + \frac{R}{r}$$

$$\gamma_{ij} = \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right)$$

$$M = 2R$$

Schwarzshild in isotropic coordinates

$=0$

Other formulations of const. eqns.

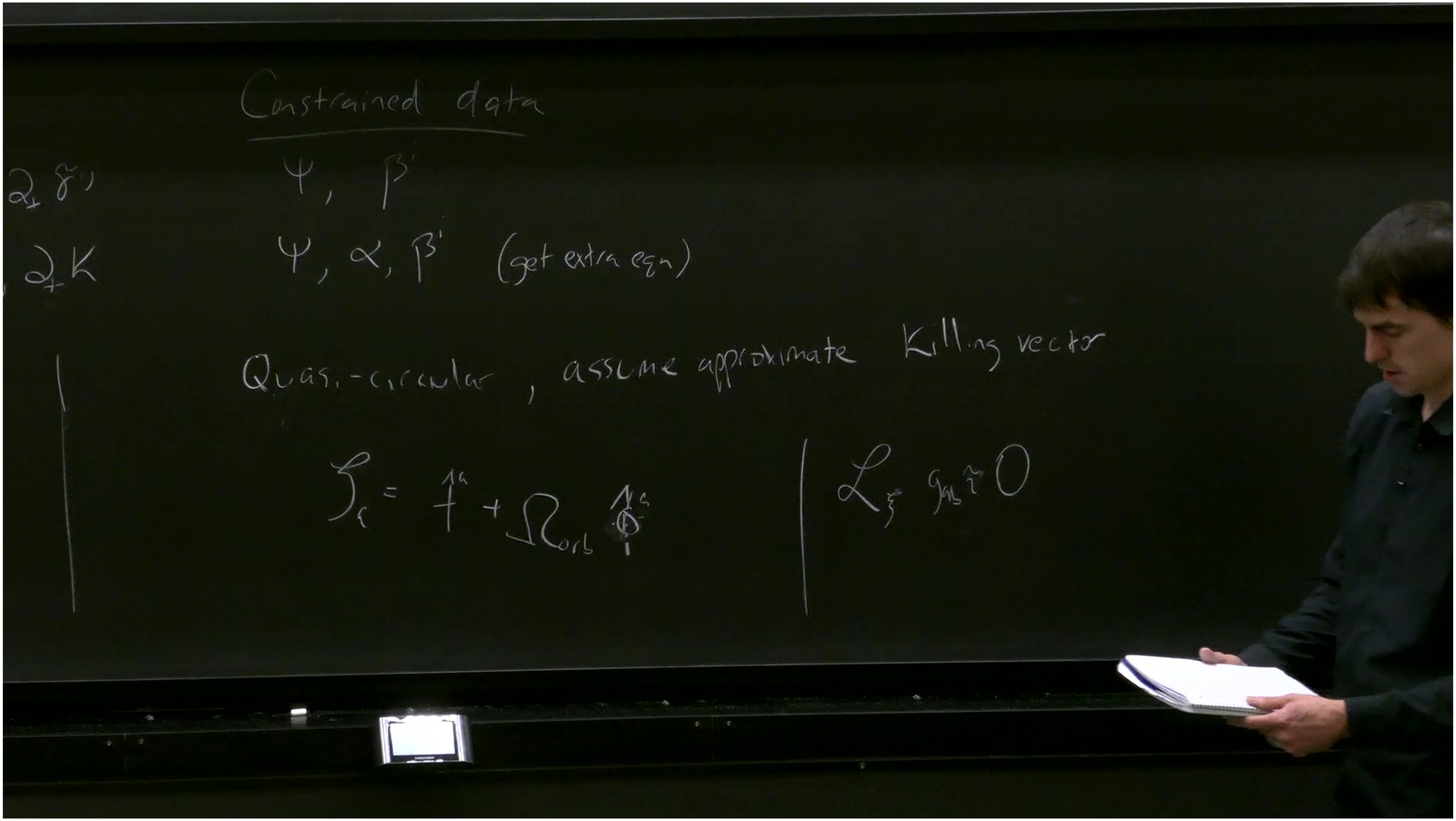
Free data

Conformal thin sandwich (CTS) : $\tilde{\gamma}_{ij}$, K, $\tilde{\alpha}$, $\partial_t \tilde{\gamma}^{ij}$

Extended CTS : $\tilde{\gamma}_{ij}$, K, $\partial_t \tilde{\gamma}^{ij}$, $\partial_t K$

Remarks on choosing free data

Superposition: $\tilde{\gamma}_{ij} = \gamma_{ij}^{(1)} + \gamma_{ij}^{(2)} - f_{ij}$


Constrained data
$\partial_t \gamma^{ij}$
Ψ, β^i
$\partial_t K$
Ψ, α, β^i (get extra eqn)
Quasi-circular, assume approximate Killing vector
$\xi_t = t^a + \Omega_{orb} \phi^a$
$\mathcal{L}_\xi g_{ab} \approx 0$