Title: Strong Gravity Lecture

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Collection: Strong Gravity 2023/24

Date: April 16, 2024 - 10:15 AM

URL: https://pirsa.org/24040014

Types of partial differential equations

Wave Eqn $\quad (-\partial_t^2 + \partial_x^2)\, u = 0 \quad$ hyperbolic PDE

Laplace Eqn $\quad (\partial_x^2 + \partial_y^2)\, u = 0 \quad$ elliptic PDE

Heat Eqn $\quad (\partial_t - \partial_x^2)\, u = 0 \quad$ parabolic PDE

PDE

General linear, second order PDE

$$a\partial_x^2 u + b\partial_x\partial_y u + c\partial_y^2 u + d\partial_x u + e\partial_y u + fu = 0$$

$b^2 - 4ac > 0$ hyperbolic , $(-\partial_x^2 + \partial_y^2)\tilde{u} + \text{l.o.t.} = 0$

$= 0$ parabolic $\partial_x^2 \tilde{u} + \text{l.o.t.} + \dots = 0$

< 0 elliptic $(\partial_x^2 + \partial_y^2)\tilde{u} + \text{l.o.t.} + \dots = 0$

Consider : $\partial_t \bar{u} + \underline{A} \partial_x \bar{u} + \underline{B} \bar{u} = 0 \quad (*)$

vector

matrix

$\underline{A}$ has all real eigenvalues $\Rightarrow$ $(*)$ is weakly hyperbolic

$\underline{A}$ also has complete set of eigenvectors $\Rightarrow$ strongly hyperbolic

Well-posedness : $\|u(t)\| \leq K e^{\alpha t} \|\bar{u}(t=0)\|$

$\Updownarrow$

Strong hyperbolicity

K, α, are independent of $u(t=0)$

Wave Eqn

$$\vec{u} = \begin{pmatrix} u \\ \partial_t u \\ \partial_x u \end{pmatrix}, \qquad \partial_t \vec{u} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{} \partial_x \vec{u} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{u}$$

Eigenvalues: $0, \pm 1$, eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\partial_t u + \partial_x u = \text{constant}$

$\partial_t u - \partial_x u = \text{constant}$

Laplace Eqn

$$\underline{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Eigenvalues: $0, \pm i$

$\sim e^{\omega t + i\omega x}$.

Strong hyperbolicity

t_α, are independent of $u(t=0)$

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix}$$

Eigenvalue, 1 (multiplicity)

Eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Has solutions of the form $\begin{pmatrix} u \\ v \end{pmatrix} \sim \begin{pmatrix} \frac{1}{2} i\omega t \\ 1 \end{pmatrix} e^{i\omega(t+x)}$

Initial data in GR (Chap. 8 arXiv gr-qc/0703035)

Two goals

1) Choose γ_{ij}, K_{ij} s.t.

$${}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi E \qquad \text{Hamiltonian const.}$$

$$D_j K^j{}_i - D_i K = 8\pi p_i \qquad \text{Momentum Const}$$

$E = n^a n^b T_{ab}$

$p_i = \gamma_i{}^a n^b T_{ab}$

2) Choose γ_{ij}, K_{ij} to correspond to our physical system as closely as possible

const.

st

m

Divide DOF into two groups

Free data + constrained data

Constraint Eqns 4

Degrees of freedom: 6+6=12

Specify: 8 DOFs

Conformal transformation (see also Chap 6, Appendix G, Carroll)

$$g_{ab} = \omega^2 \tilde{g}_{ab}$$

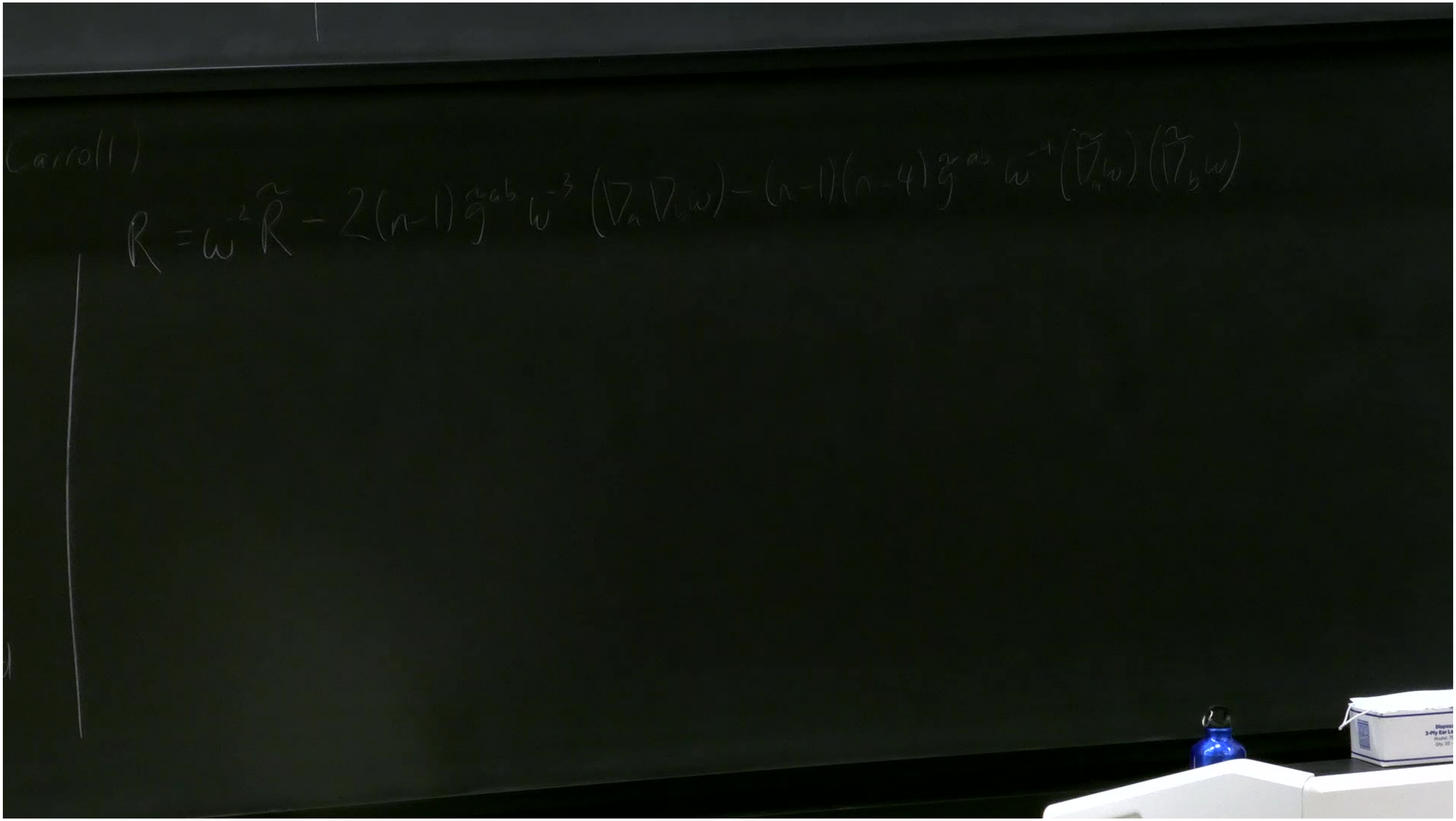
n-dim

Difference in Christ. Symbols

$$C^c_{ab} = \Gamma^c_{ab} - \tilde{\Gamma}^c_{ab} = \omega^{-1}\left(\delta^c_a \tilde{\nabla}_b \omega + \delta^c_b \tilde{\nabla}_a \omega - \tilde{g}_{ab} \tilde{\nabla}^c \omega\right)$$

Scalar: $\nabla_a \phi = \tilde{\nabla}_a \phi = \partial_a \phi$

Vector deriv: $\nabla_a V_b = \tilde{\nabla}_a V_b - \left(\delta^c_a \delta^d_b + \delta^d_a \delta^c_b - \tilde{g}_{ab} \tilde{g}^{cd}\right) \omega^{-1} (\tilde{\nabla}_c \omega) V_d$


Carroll)
$R = \omega^2 \tilde{R} - 2(n-1)\tilde{g}^{ab}\omega^{-3}(\tilde{\nabla}_a \tilde{\nabla}_b \omega) - (n-1)(n-4)\tilde{g}^{ab}\omega^{-4}(\tilde{\nabla}_a \omega)(\tilde{\nabla}_b \omega)$

Conformal Transverse Traceless decomposition

$$\gamma_{ij} = \underbrace{\psi^4}_{\text{constrained}} \; \underbrace{\tilde{\gamma}_{ij}}_{\text{free-data}}$$

$$^{(3)}R = \tilde{R}\psi^{-4} - 4\tilde{\gamma}^{ij}\psi^{-6}\left(\cancel{2\tilde{D}_i\psi\tilde{D}_j\psi} + 2\psi\tilde{D}_i\tilde{D}_j\psi\right) + 2\tilde{\gamma}^{ij}\psi^{-8}\left(\cancel{(2\psi)^2}\tilde{D}_i\psi\tilde{D}_j\psi\right.$$

$$= \tilde{R}\psi^{-4} - 8\psi^{-5}\tilde{D}_i\tilde{D}^i\psi$$

$$\ldots \tilde{D}_i \psi \tilde{D}_j \psi \Big)$$

$$K^{ij} = A^{ij} + \frac{1}{3} K \gamma^{ij}$$

A^{ij}: trace-free

K: free-data

$$A^{ij} = \psi^{-10} \hat{A}^{ij}$$

Hamiltonian Const: $\tilde{D}_i \tilde{D}^i \psi - \frac{1}{8} \tilde{R} \psi + \frac{1}{8} \hat{A}^{ij} \hat{A}_{ij} - \frac{1}{12} K^2 \psi^5 + 2\pi E \psi^5 = 0$