Title: Strong Gravity Lecture

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Collection: Strong Gravity 2023/24

Date: April 11, 2024 - 10:15 AM

URL: https://pirsa.org/24040013

3+1 Decomposition of Spacetime

References: GR, Wald Chap 10.2
3+1 Formalism, Gourgoulhon arXiv: gr-qc/0703035

Hypersurfaces of Spacetime

4-dim manifold M

3-dim manifold $\hat{\Sigma}$

Embedding of $\hat{\Sigma}$ in M

$\Phi : \hat{\Sigma} \to M$ 1:1, Φ, Φ^{-1} continuous

Then $\Sigma = \Phi(\hat{\Sigma})$ is called a hypersurface

Level set: $\Sigma = \{ x^a \in M \mid F(x^a) = 0 \}$

$\partial_a F$ is normal to Σ

$\iff \partial_a F V^a = 0$ for any vector tangent to Σ

, Φ^{-1} continuous

$\partial_a F$ is timelike, call Σ spacelike

$\partial_a F$ is spacelike, call Σ timelike

$\partial_a F$ is null, call Σ null

Restrict to Σ spacelike

Choose coördinates: $x^a = (t, \underbrace{x^i}_{\text{3-index}})$ with x^i coördinate $\hat{\Sigma}$

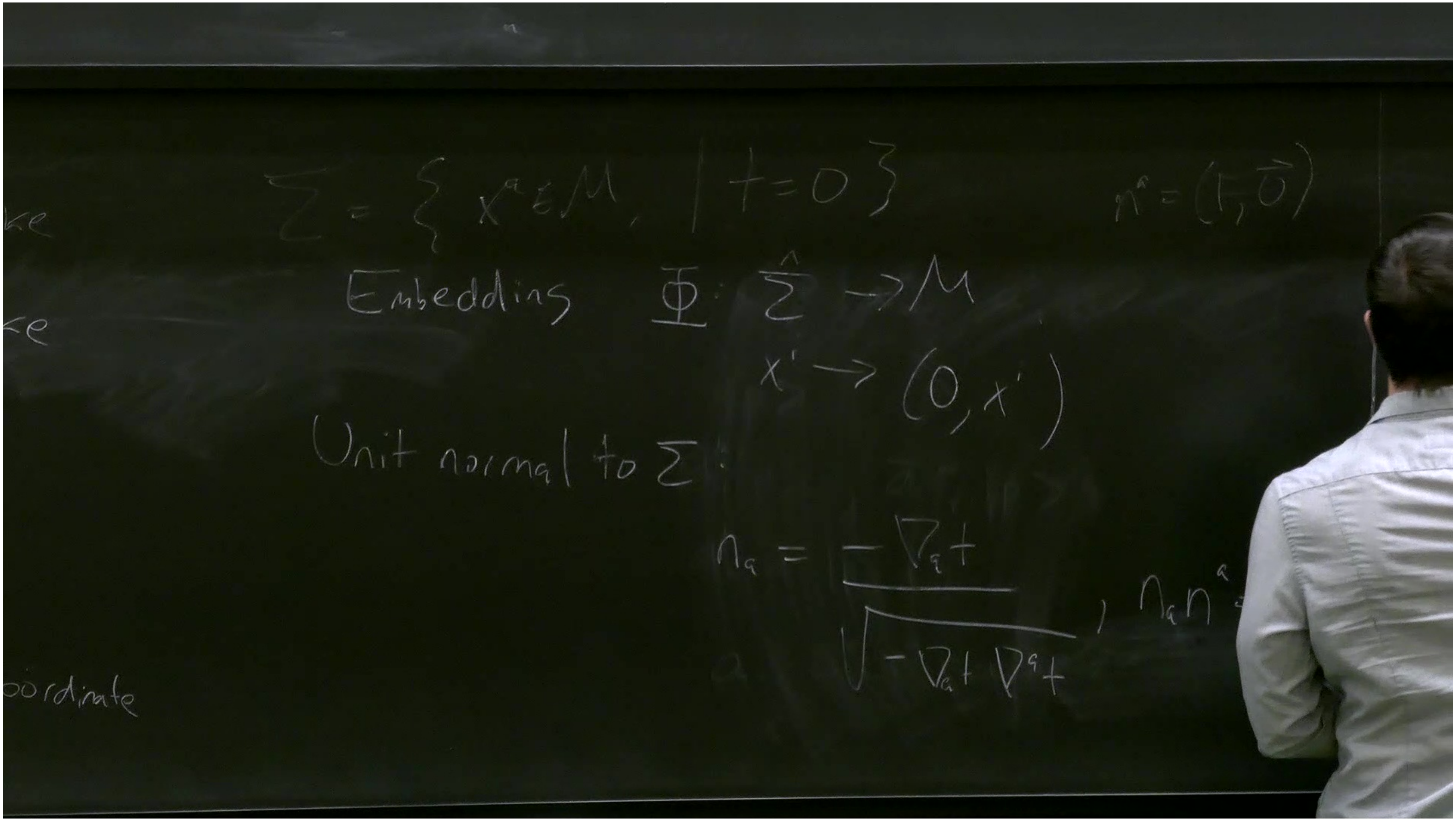
Then $\Sigma = \Phi(\hat{\Sigma})$ is called a hypersurface

Level set: $\Sigma = \{ x^a \in \mathcal{M} \mid F(x^a) = 0 \}$

$\partial_a F$ is normal to Σ

$\Leftrightarrow \underline{\partial_a F V^a} = 0$ for any vector tangent to Σ

inuous


$\Sigma = \{ x^a \in M \mid t = 0 \}$
$n^a = (1, \vec{0})$
Embedding $\Phi : \hat{\Sigma} \to M$
$x^i \to (0, x^i)$
Unit normal to Σ:
$n_a = \frac{-\nabla_a t}{\sqrt{-\nabla_a t \nabla^a t}}$, $n_a n^a$

$\}$ $\quad n^a = (1, \vec{0})$

$\rightarrow M$

$\rightarrow (0, x^i)$

$$\frac{-\nabla_a t}{\sqrt{-\nabla_a t \nabla^a t}}, \quad n_a n^a = -1$$

$\Phi : \hat{\Sigma} \rightarrow M$ push forward mapping

$$v^i = (0, v^i)$$

$\Phi^* \quad M \longrightarrow \hat{\Sigma}$ pull back

$(\upsilon_t, \upsilon_i) \qquad (\upsilon_i)$

In M

In $\hat{\Sigma}$

$(0, v^i) \cdot (v_+, v_i) = v^i v_i$

$v^i \cdot (v_i \quad)$

Caution

$M \qquad \hat{\Sigma}$

$(v^+, v^i) \not\to v^i$

Not true in general

For g_{ab} in M, induce

For g_{ab} in $\mathcal{M}$, induced metric γ_{ij} on Σ

$\gamma_{ij} = g_{ij}$ (in adapted coördinates)

Doff hats $\Sigma \simeq \hat{\Sigma}$

Not true in general

$$V^a = \underbrace{-V^b n_b n^a}_{\text{orthogonal}} + \underbrace{\left(V^a + V^b n_b n^a\right)}_{\text{tangent}}$$

$$=: \gamma^a_b V^b$$

$$\gamma^a_b = \delta^a_b + n_b n^a$$

$$n^a \gamma^b_a V_b = n^a V_a - n_b V^b = 0$$

$= 0$

$M \to \Sigma$

vectors $V^a \to \gamma^i_b V^b$

$\Sigma \to M$

$U_i = \gamma^i_a U_i$

$\gamma_{ab} = \gamma^i_a \gamma^j_b \gamma_{ij} = g_{ab} + n_a n_b$

3-index Σ

Intrinsic curvature same $(g_{ab}, M) \to (\gamma_{ij}, \Sigma)$

Introduce covariant deriv. $D_i \gamma_{jk} = 0$ (also assume torsion free)

defined ${}^{(3)}\Gamma^i_{jk}$

Also introduce 3-Dim Riemann tensor

$$(D_i D_j - D_j D_i) V^k = {}^{(3)}R^k{}_{\ell ij} V^\ell$$

$\nabla_a f \, \nabla^a f$

3D Ricci tensor ${}^{(3)}R_{ij} = {}^{(3)}R^{k}{}_{ikj}$

" " scalar ${}^{(3)}R = \gamma^{ij}\, {}^{(3)}R_{ij}$

$$D_a T^{b_1 b_2 \dots}{}_{c_1 c_2 \dots} = \gamma^{b_1}{}_{d_1} \cdots \gamma^{e_1}{}_{c_1} \cdots \gamma^{f}{}_{a} \nabla_f T^{d_1 d_2 \dots}{}_{e_1 e_2 \dots}$$

$$D_a \gamma_{bc} = \gamma^{d}{}_{b}\, \gamma^{e}{}_{c}\, \gamma^{f}{}_{a} \nabla_f \left(\cancelto{0}{g_{de}} + \cancelto{0}{n_d n_e} \right)$$

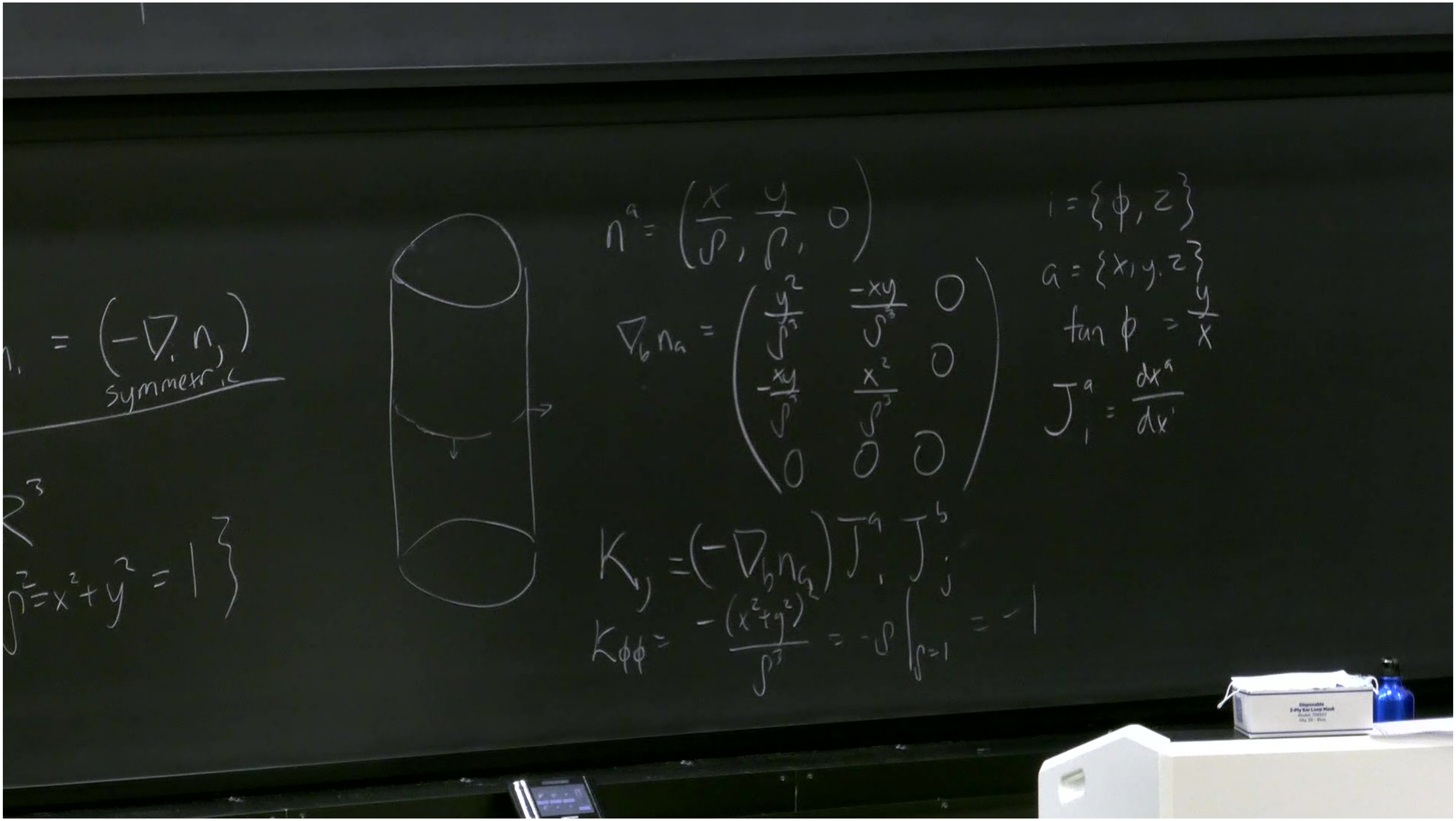
$$= 0$$

Extrinsic Curvature

$$K_{ij} = -\nabla_j n_i = \underbrace{\left(-\nabla_i n_j\right)}_{\text{symmetric}}$$

Example: Cylinder as hypersurface in $\mathbb{R}^3$

$$\Sigma = \left\{ (x,y,z) \in \mathbb{R}^3 \mid \rho^2 = x^2 + y^2 = 1 \right\}$$


$= (-\nabla_i n_j)$
symmetric
R^3
$\rho^2 = x^2 + y^2 = 1 \}$
$n^a = \left(\frac{x}{\rho}, \frac{y}{\rho}, 0\right)$
$\nabla_b n_a = \begin{pmatrix} \frac{y^2}{\rho^3} & \frac{-xy}{\rho^3} & 0 \\ \frac{-xy}{\rho^3} & \frac{x^2}{\rho^3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$K_{ij} = (-\nabla_b n_a) J^a_i J^b_j$
$K_{\phi\phi} = -\frac{(x^2+y^2)^2}{\rho^3} = -\rho\big|_{\rho=1} = -1$
$i = \{\phi, z\}$
$a = \{x, y, z\}$
$\tan\phi = \frac{y}{x}$
$J^a_i = \frac{dx^a}{dx^i}$

Choose coordinates $x = (t, x^i)$ with x^i coordinate

$\uparrow$ 3-index $\quad \hat{\Sigma}$

$$K_{ab} = \gamma_a^i \gamma_b^j K_{ij} = -\nabla_b n_a - \underbrace{(n^c \nabla_c n_a)}_{:= a^c \text{ acceleration}} n_b \qquad (**)$$

$n_b v^b = 0 \qquad K_{ab} v^b = -v^b \nabla_b n_a$

$v^b = \lambda n^b \qquad \Rightarrow v^b K_{ab} = 0$

Foliation

$\mathcal{M}$ is globally

$$\mathcal{M} = \{\Sigma_t\}_{t\in\mathbb{R}}$$

$$\Sigma_t = \{x^a \in \mathcal{M} : f(x^a) = t\}$$

slice or leaf

f smooth and $\nabla_a f \neq 0$

Foliation

$\mathcal{M}$ is globally hyperbolic

$$\mathcal{M} = \{\Sigma_t\}_{t\in\mathbb{R}}$$

$$n_a = -K\nabla_a t \quad , \quad n_a n^a = -1$$

$$\Sigma_t = \{x^a \in \mathcal{M} : \hat{f}(x^a) = t\}$$

$$n_a = (-\alpha, \vec{0})$$

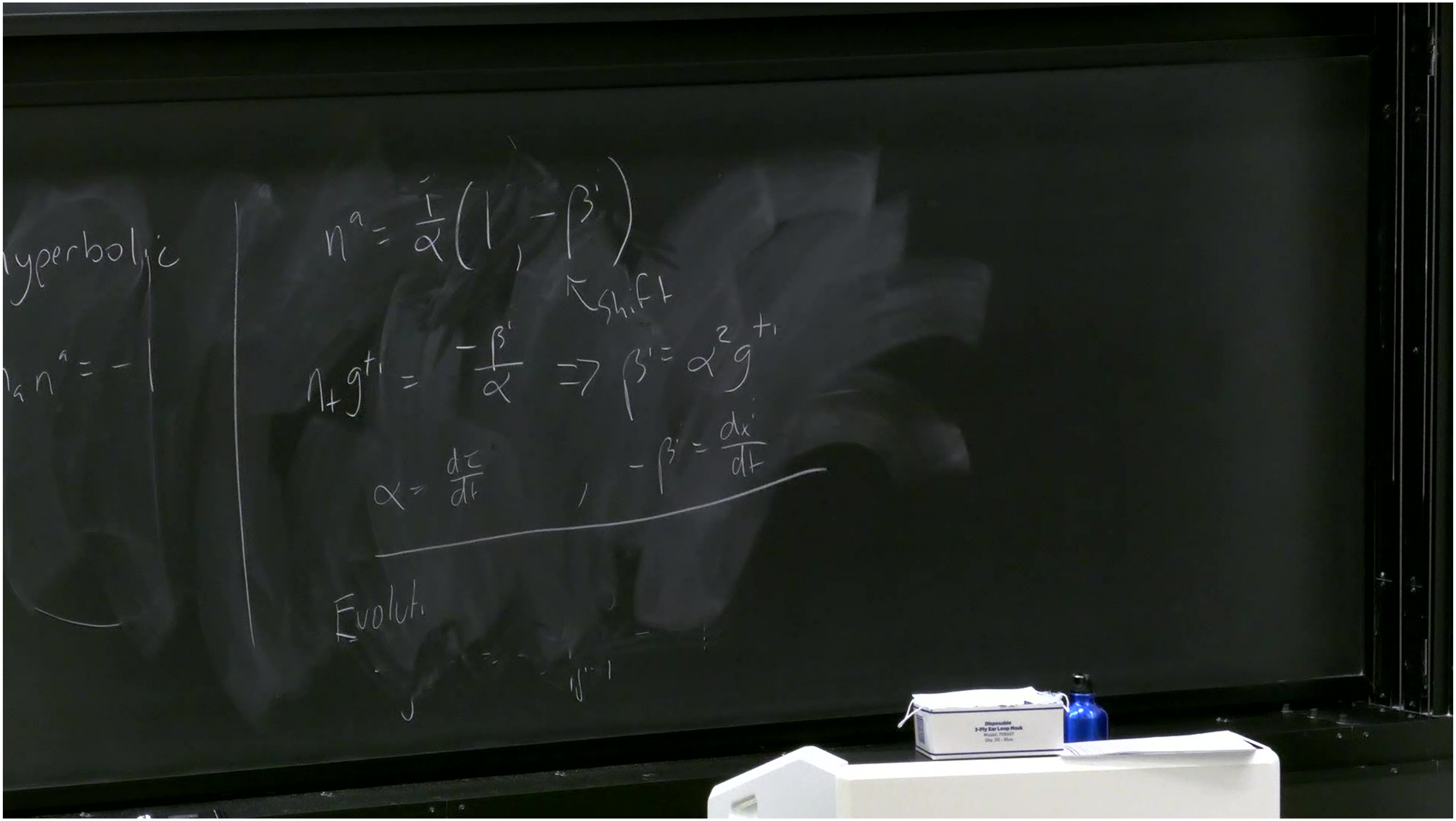
lapse

slice or leaf

$$-1 = g^{tt}(n_t)^2 = g^{tt}\alpha^2$$

$\hat{f}$ smooth and $\nabla_a \hat{f} \neq 0$

$$\alpha = \sqrt{\frac{-1}{g^{tt}}}$$


yperbolic
$n_a n^a = -1$
$n^a = \frac{1}{\alpha}(1, -\beta^i)$
shift
$n_t g^{ti} = -\frac{\beta^i}{\alpha} \Rightarrow \beta^i = \alpha^2 g^{ti}$
$\alpha = \frac{d\tau}{dt}, \quad -\beta^i = \frac{dx^i}{dt}$
Evoluti

Evolution operator:

$$m_a = \alpha n_a$$

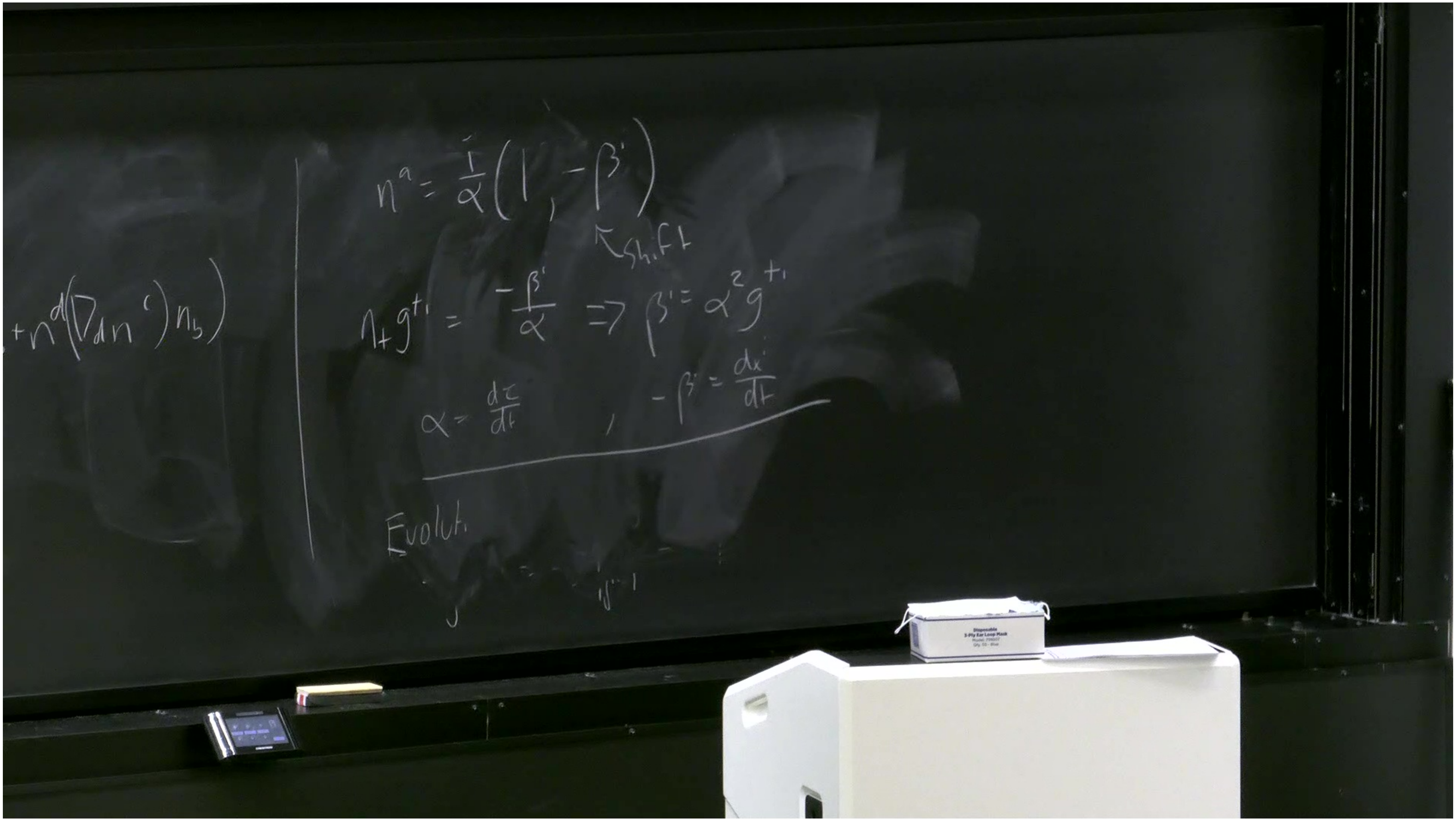
$$m^a \nabla_a t = 1$$

$$t(p + \delta t \, \vec{m}) = t(p) + \delta t$$

$$\mathcal{L}_{\vec{n}}\, \gamma_{ab} = n^c \nabla_c \gamma_{ab} + \gamma_{cb} \nabla_c n^c + \gamma_{ac} \nabla_b n^c$$

$$= n^c \nabla_c (n_a n_b) - \gamma_{cb} \left(K_a{}^c + n^d (\nabla_d n^c) n_a \right) - \gamma_{ac} \left(K_b{}^c + n^d (\nabla_d \ldots \right.$$

$$= -2K_{ab}$$


$+n^d(\nabla_d n^c) n_b)$
$n^a = \frac{1}{\alpha}(1, -\beta^i)$
shift
$n_t g^{ti} = -\frac{\beta^i}{\alpha} \Rightarrow \beta^i = \alpha^2 g^{ti}$
$\alpha = \frac{d\tau}{dt}$, $-\beta^i = \frac{dx^i}{dt}$
Evoluti