Title: Strong Gravity Lecture

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Collection: Strong Gravity 2023/24

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In ergoregion

u^a

$g_{tt} > 0$

$u^\phi > 0$

$-1 = \chi^a \chi_a$

$\chi^a = \alpha\left(\hat{t}^a + \Omega \hat{\phi}^a\right)$

$\chi_a \hat{\phi}^a = 0$

ZAMO:

$g_{\phi t} + g_{\phi\phi}\Omega = 0$

$\Omega = \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}}$

As $r \to r_+$

$\Omega = \bar{a}/2r_+$

$\bar{\chi}^a = \hat{t}^a +$

For $\bar{a} \ll 1$

$\bar{a} = 1$

u^a

$u^\phi > 0$

$\left(\hat{t}^a + \Omega \hat{\phi}^a \right)$

$g_{\phi t} + g_{\phi\phi}\Omega = 0$

$\Omega = \frac{d\phi}{dt} = \frac{-g_{t\phi}}{g_{\phi\phi}}$

As $r \to r_+$

$\Omega = \bar{a}/2r_+ := \Omega_H$

$\chi^a = \hat{t}^a + \Omega_H \hat{\phi}^a$

For $\bar{a} \ll 1$, $r_+ \approx \left(2 - \frac{1}{2}\bar{a}^2\right)M$

$\Omega_H \approx \bar{a}/4M$

$\bar{a} = 1$ $\quad \Omega_H = \frac{1}{2M}$

$$= \frac{dx^a}{d\tau}$$

$\tilde{E} < 1$

bound orbits

Constants:

$$\tilde{E} = -\hat{t}^a u_a = -u_t = -\left(g_{tt} u^t + g_{t\phi} u^\phi\right)$$

$$= \left(1 - \frac{2Mr}{\Sigma}\right)\frac{dt}{d\tau} + \frac{2Mar\sin^2\theta}{\Sigma}\frac{d\phi}{d\tau}$$

$$\tilde{J} = \hat{\phi}^a u_a = u_\phi = g_{\phi\phi} u^\phi + g_{t\phi} u^t$$

$$= \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta \frac{d\phi}{d\tau} - \frac{2Mar\sin^2\theta}{\Sigma}$$

Geodesic Eqn: $U^a \nabla_a U^b = 0$, $U^a = \frac{dx^a}{d\tau}$

$U^a U_a = -1$

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0$$

$$U^b \nabla_b (K_a U^a) = \tfrac{1}{2} U^a U^b \nabla_{(b} K_{a)} + 0$$

$\tilde{E} < 1$

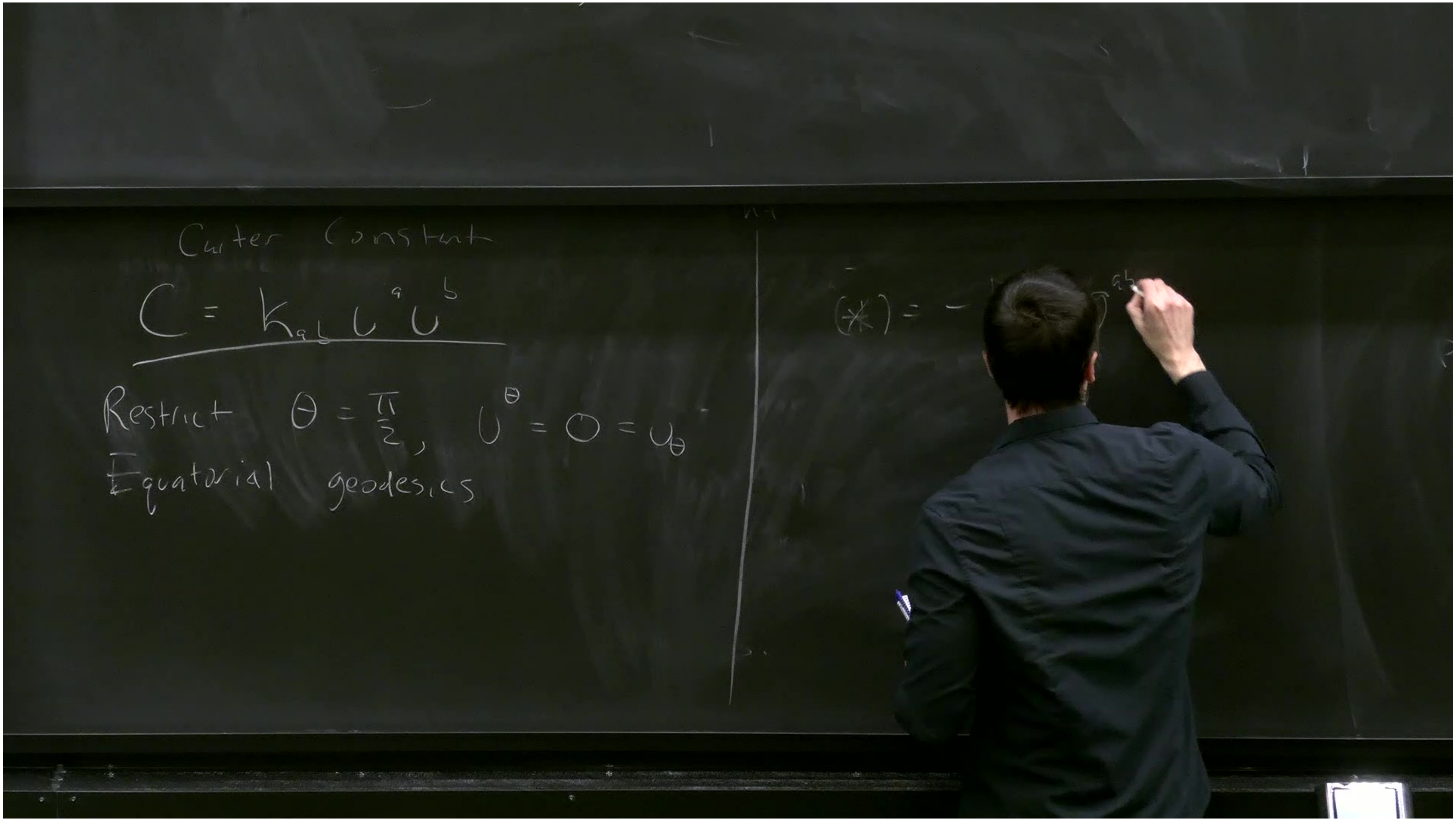
bound orbits

Consta

$\tilde{E} =$

$=$

$\tilde{J} =$


Carter Constant
C = K_ab u^a u^b
Restrict θ = π/2, u^θ = 0 = u_θ
Equatorial geodesics
(*) = −

$$= \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta \frac{d\phi}{d\tau}$$

$$(*) = -1 = g^{ab} u_a u_b = g^{tt} \tilde{E}^2 + g^{rr}(u_r)^2 + g^{\phi\phi}(\tilde{J})^2 - 2 g^{t\phi} \tilde{E}$$

$$g^{tt} = \frac{-((r^2+a^2)^2 - a^2\Delta)}{\Delta r^2}, \quad g^{rr} = \frac{\Delta}{r^2}, \quad g^{\phi\phi} = \frac{\Delta - a^2}{\Delta r^2}$$

$$-1 = \frac{r^2}{\Delta}(u^r)^2 + \frac{a^2\Delta - (r^2+a^2)^2}{\Delta r^2}\tilde{E}^2 + \frac{(\Delta - a^2)}{\Delta r^2}\tilde{J}^2 + \frac{4Ma}{\Delta r}\tilde{E}\tilde{J}$$

$$-(u^r)^2 = (1-\tilde{E}^2) - 2M\left(\frac{1}{r}\right) + \left[a^2(1-\tilde{E}^2) + \tilde{J}^2\right]\left(\frac{1}{r^2}\right) - 2M(a\tilde{E} - \tilde{J})^2 ($$

$$= 2V(\tilde{E}, \tilde{J}, r)$$

$$= \frac{(r^2+a^2) \quad - \Delta a \sin^2\theta}{\Sigma} \sin^2\theta \frac{d\phi}{d\tau} - \frac{\quad}{\Sigma} \frac{dt}{d\tau}$$

$$g^{ab} u_a u_b = g^{tt} \tilde{E}^2 + g^{rr} (u_r)^2 + g^{\phi\phi} (\tilde{J})^2 - 2 g^{t\phi} \tilde{E} \tilde{J}$$

$$g^{tt} = \frac{-\left((r^2+a^2)^2 - a^2 \Delta\right)}{\Delta r^2}, \quad g^{rr} = \frac{\Delta}{r^2}, \quad g^{\phi\phi} = \frac{\Delta - a^2}{\Delta r^2}, \quad g^{t\phi} = \frac{-2Ma}{\Delta r}$$

$$+ \frac{a^2 \Delta - (r^2+a^2)^2}{\Delta r^2} \tilde{E}^2 + \frac{(\Delta - a^2)}{\Delta r^2} \tilde{J}^2 + \frac{4Ma}{\Delta r} \tilde{E} \tilde{J}$$

$$- 2M\left(\frac{1}{r}\right) + \left[a^2 (1 - \tilde{E}^2) + \tilde{J}^2\right]\left(\frac{1}{r^2}\right) - 2M (a\tilde{E} - \tilde{J})^2 \left(\frac{1}{r^3}\right)$$

$r)$

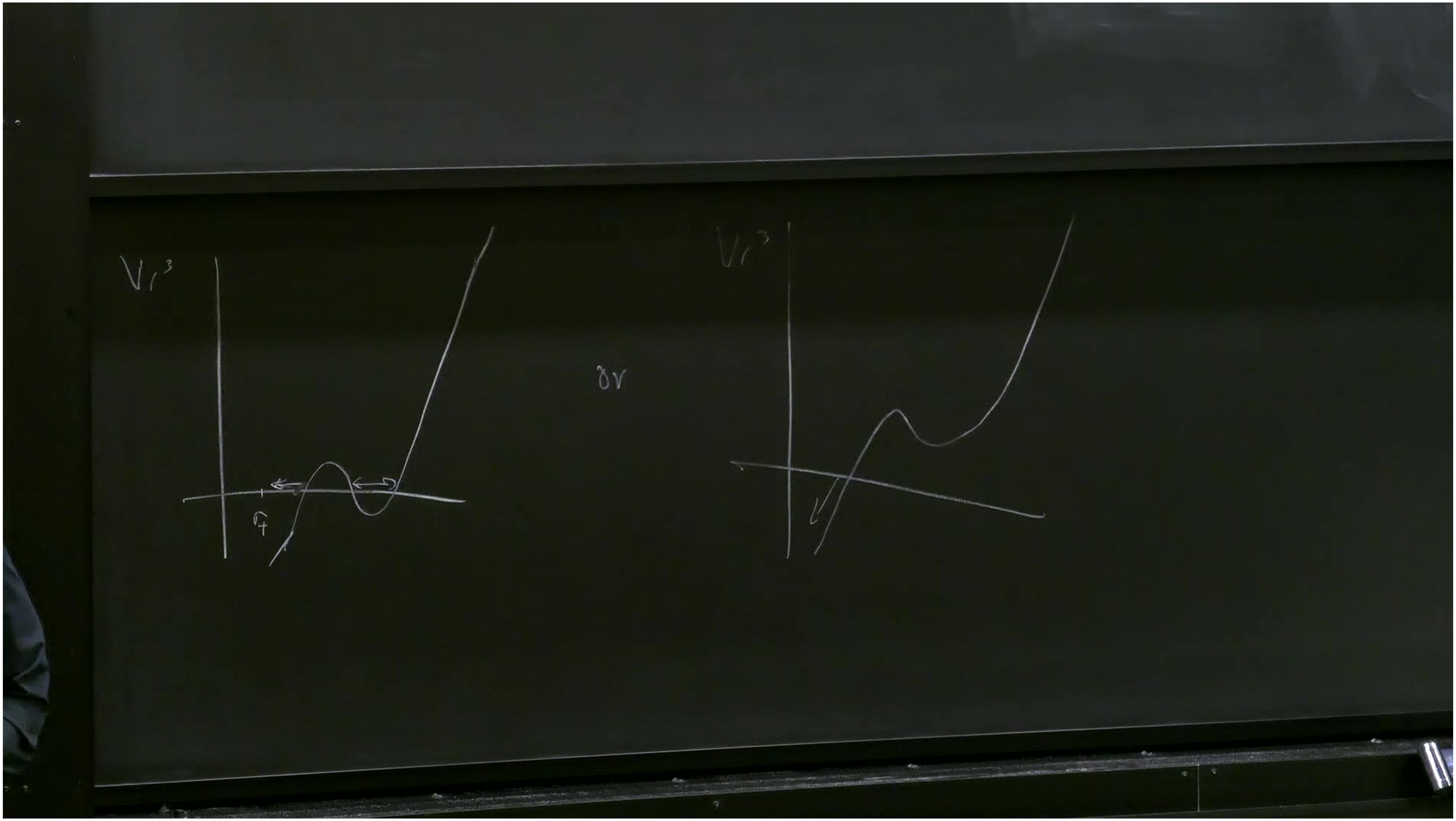
Need $V \leq 0$, turning points when $V = 0$ $\left(\Rightarrow \frac{dr}{d\tau} = 0\right)$

$\tilde{E} < 1$ for $r \to \infty$ $V(r) \to \frac{1}{2}(1 - \tilde{E}^2) > 0$

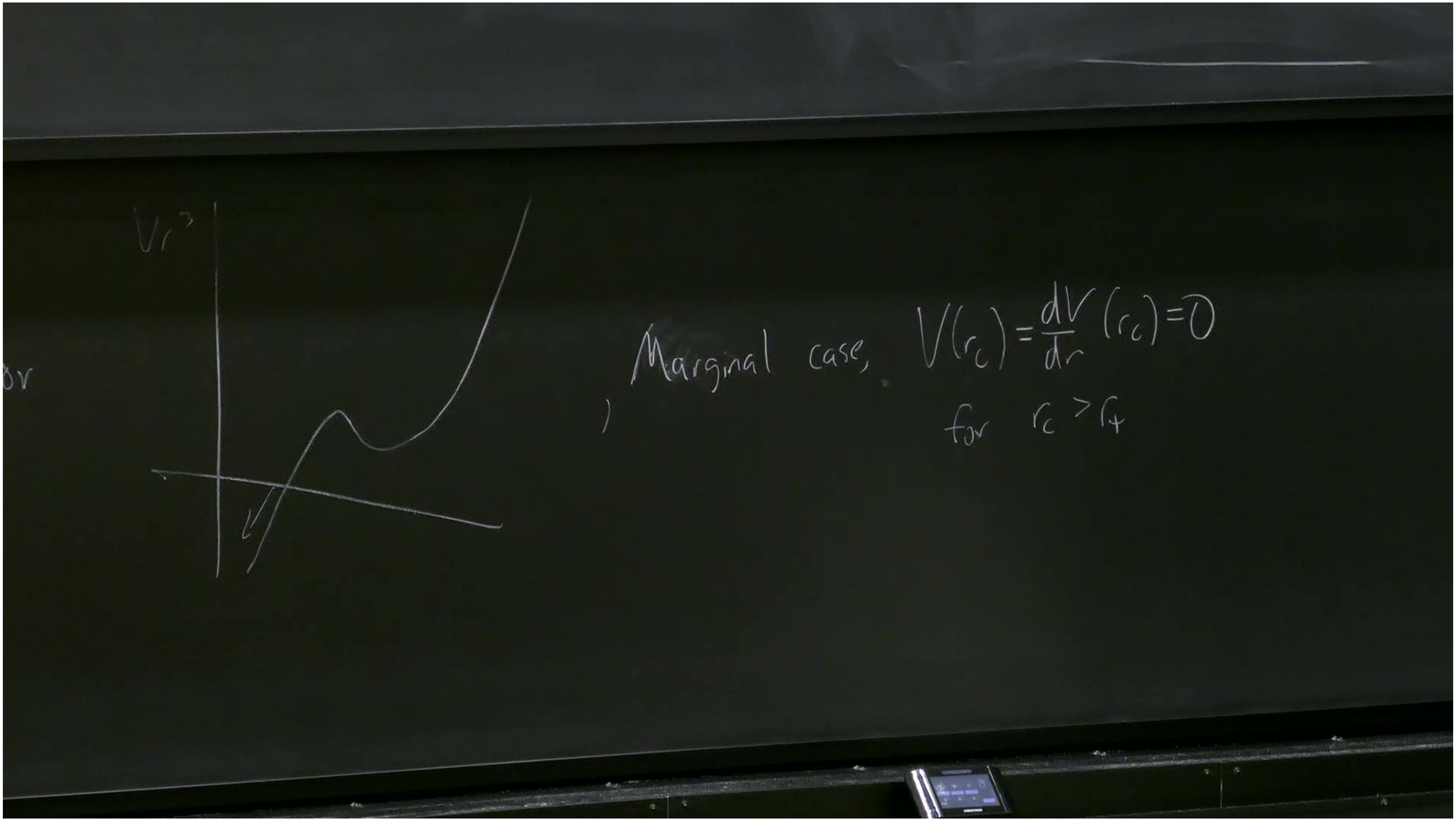
For $r = r_+$

$$2r_+^4 V(r_+) = -\left(2Mr_+\tilde{E}\right)^2 - \left(a\tilde{J}\right)^2 + 4Mar_+\tilde{E}\tilde{J}$$

$$= -\left(2Mr_+\tilde{E} - a\tilde{J}\right)^2 \leq 0$$



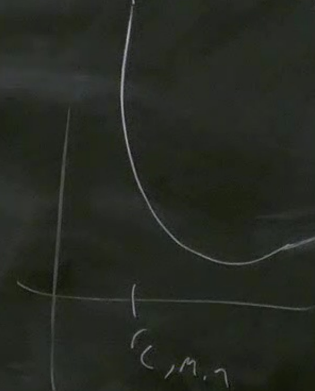
$V r^3$
r_+
or
$V r^3$

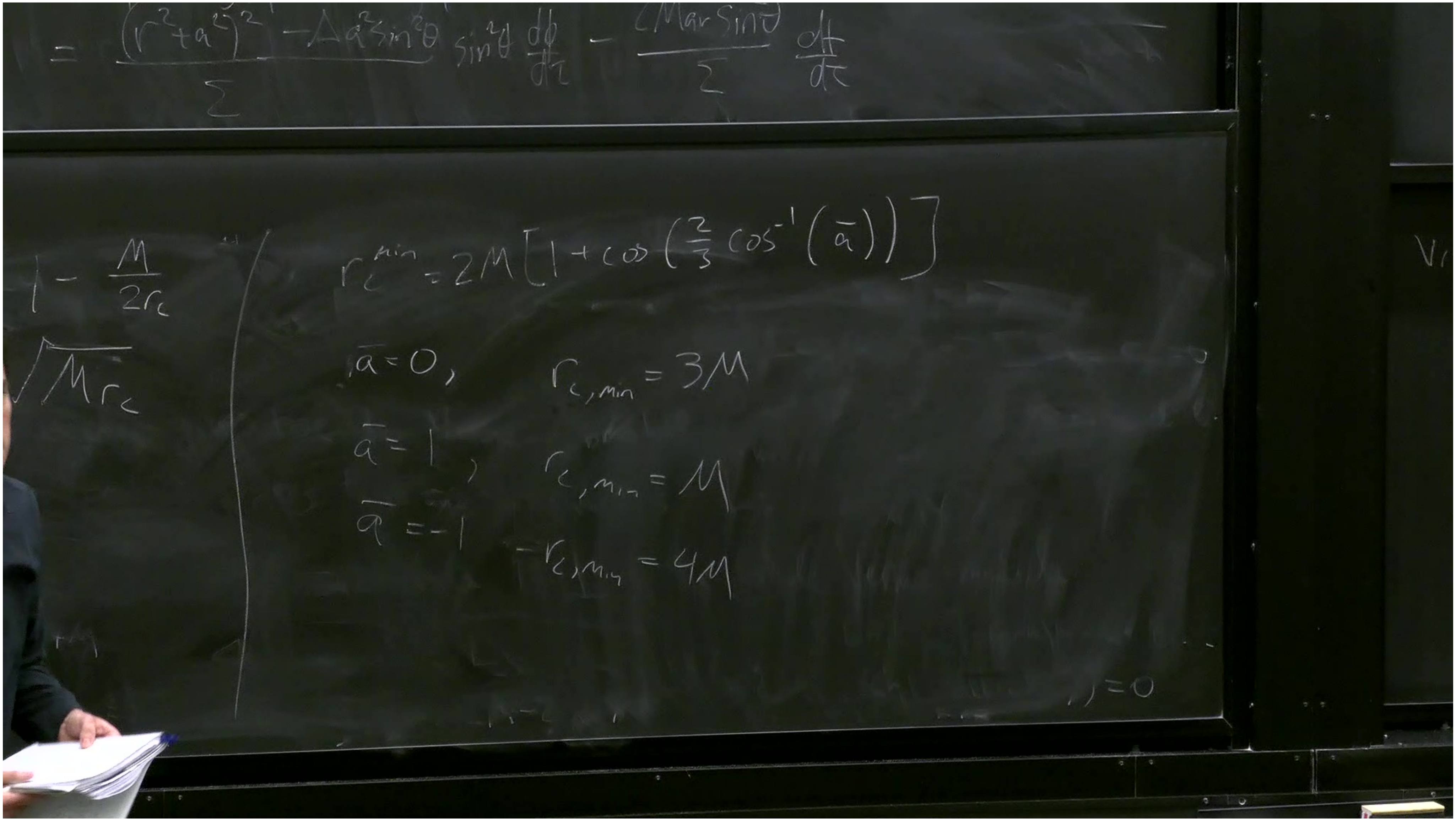

Vr^3
or
Marginal case, $V(r_c) = \frac{dV}{dr}(r_c) = 0$
for $r_c > r_+$

$$\tilde{E}_c = \frac{1 - \frac{2M}{r_c} + \frac{a}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$$

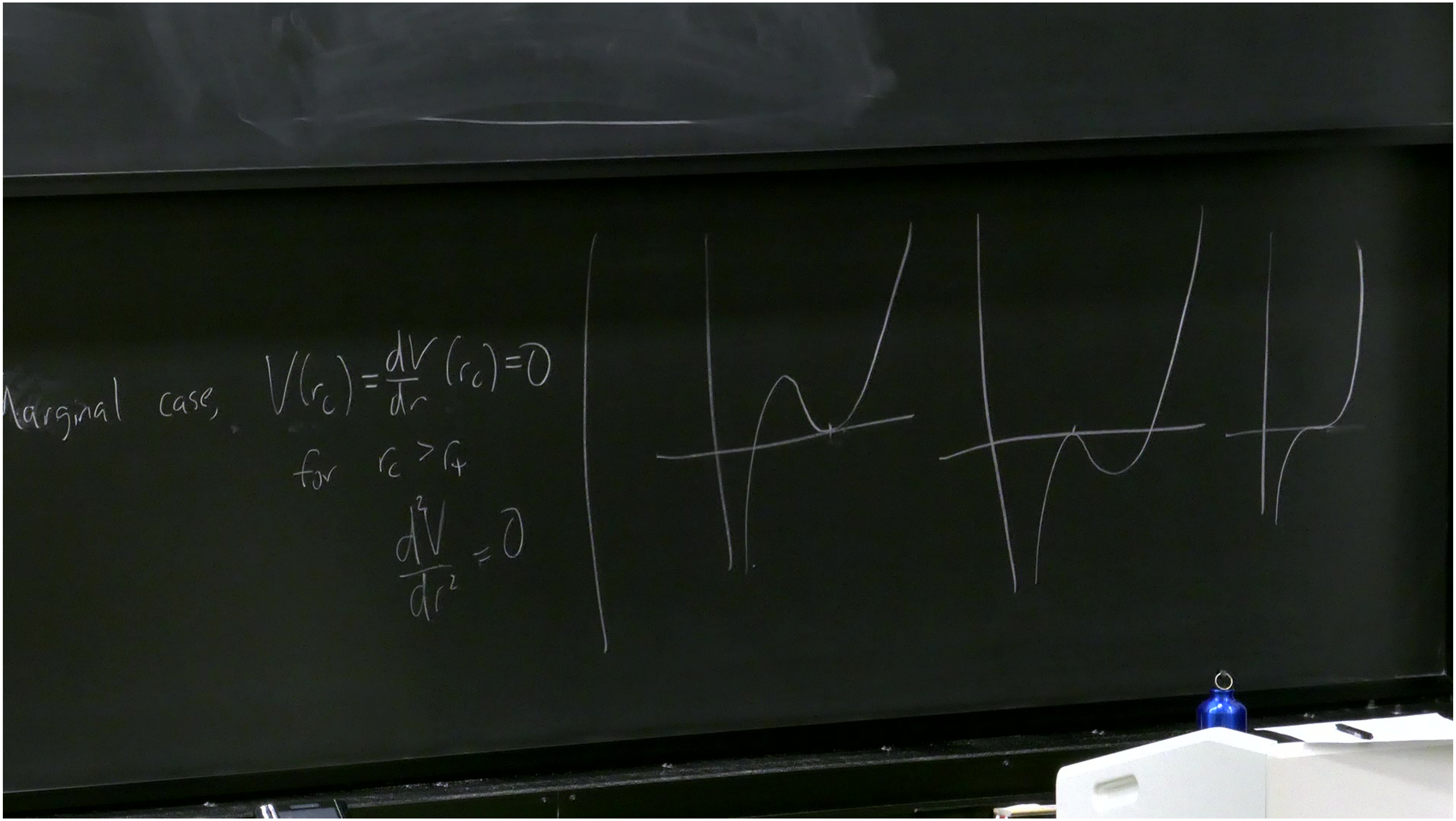
$$\tilde{J}_c = \frac{\sqrt{M r_c} - 2a\frac{M}{r_c} + \frac{a^2}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$$

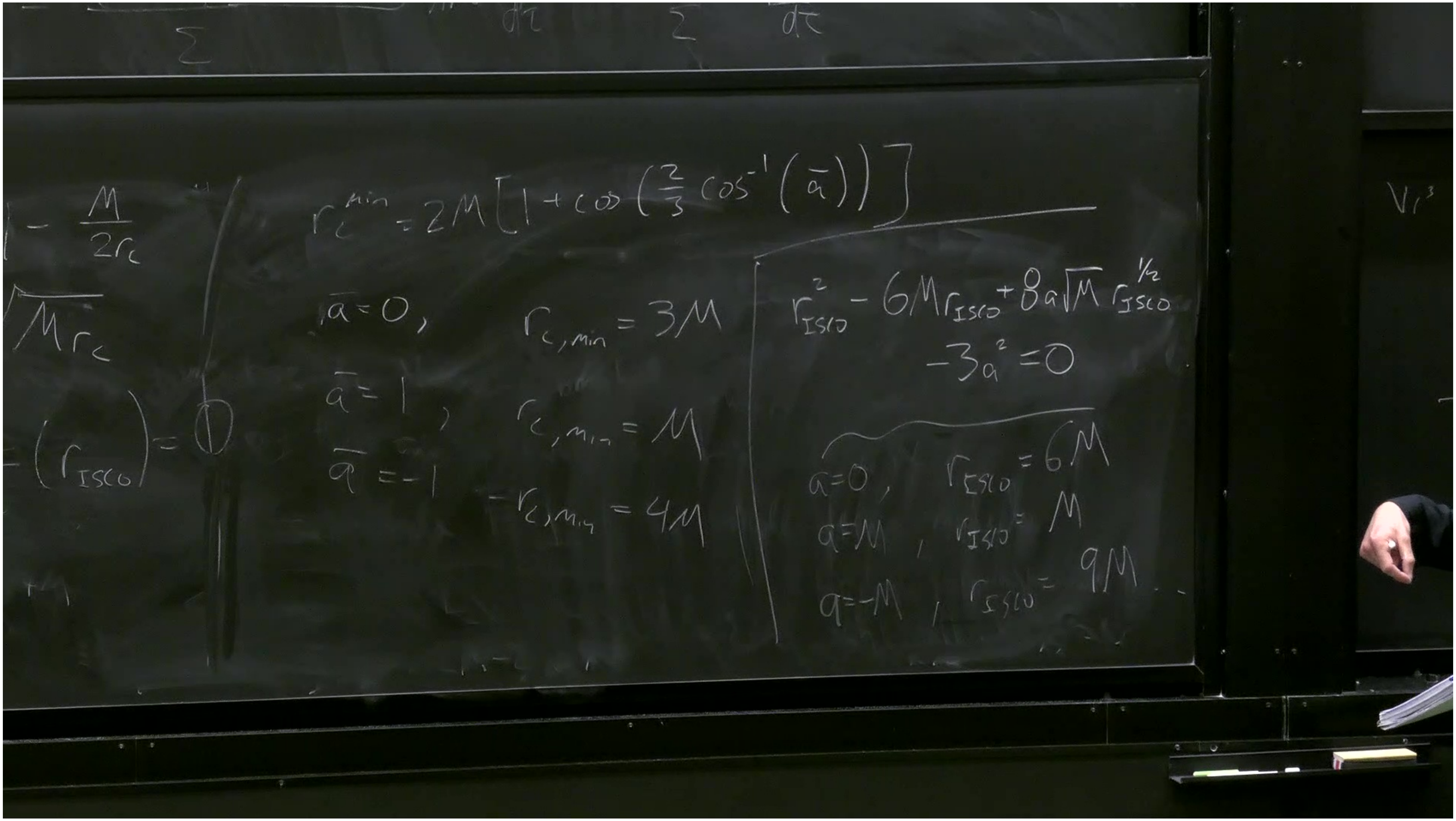
For $r_c >> M$





$= \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta \frac{d\phi}{d\tau} - \frac{2Mar\sin^2\theta}{\Sigma} \frac{dt}{d\tau}$
$1 - \frac{M}{2r_c}$
$\sqrt{Mr_c}$
$r_c^{min} = 2M\left[1 + \cos\left(\frac{2}{3}\cos^{-1}(\bar{a})\right)\right]$
$\bar{a} = 0, \quad r_{c,min} = 3M$
$\bar{a} = 1, \quad r_{c,min} = M$
$\bar{a} = -1 \quad r_{c,min} = 4M$


Marginal case, $V(r_c) = \frac{dV}{dr}(r_c) = 0$
for $r_c > r_+$
$\frac{d^2V}{dr^2} = 0$


$r_c^{min} = 2M\left[1 + \cos\left(\frac{2}{3}\cos^{-1}(\bar{a})\right)\right]$
$\bar{a} = 0, \quad r_{c,min} = 3M$
$\bar{a} = 1, \quad r_{c,min} = M$
$\bar{a} = -1, \quad r_{c,min} = 4M$
$r_{ISCO}^2 - 6M r_{ISCO} + 8a\sqrt{M}\, r_{ISCO}^{1/2} - 3a^2 = 0$
$a = 0, \quad r_{ISCO} = 6M$
$a = M, \quad r_{ISCO} = M$
$a = -M, \quad r_{ISCO} = 9M$

prograde / retro

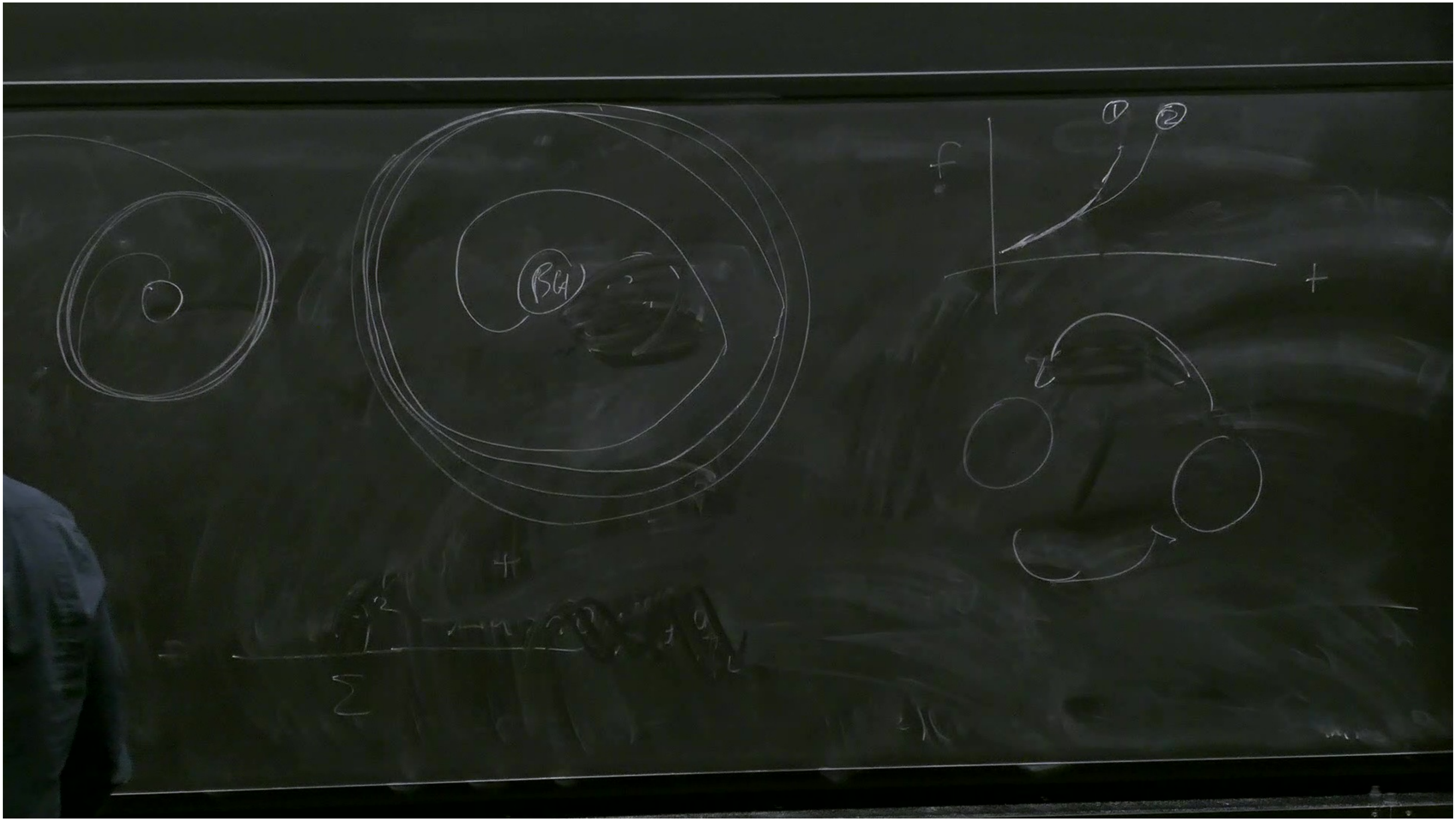
$$r_{ISCO}/M = 3 + B \mp \sqrt{(3-A)(3+A+2B)}$$

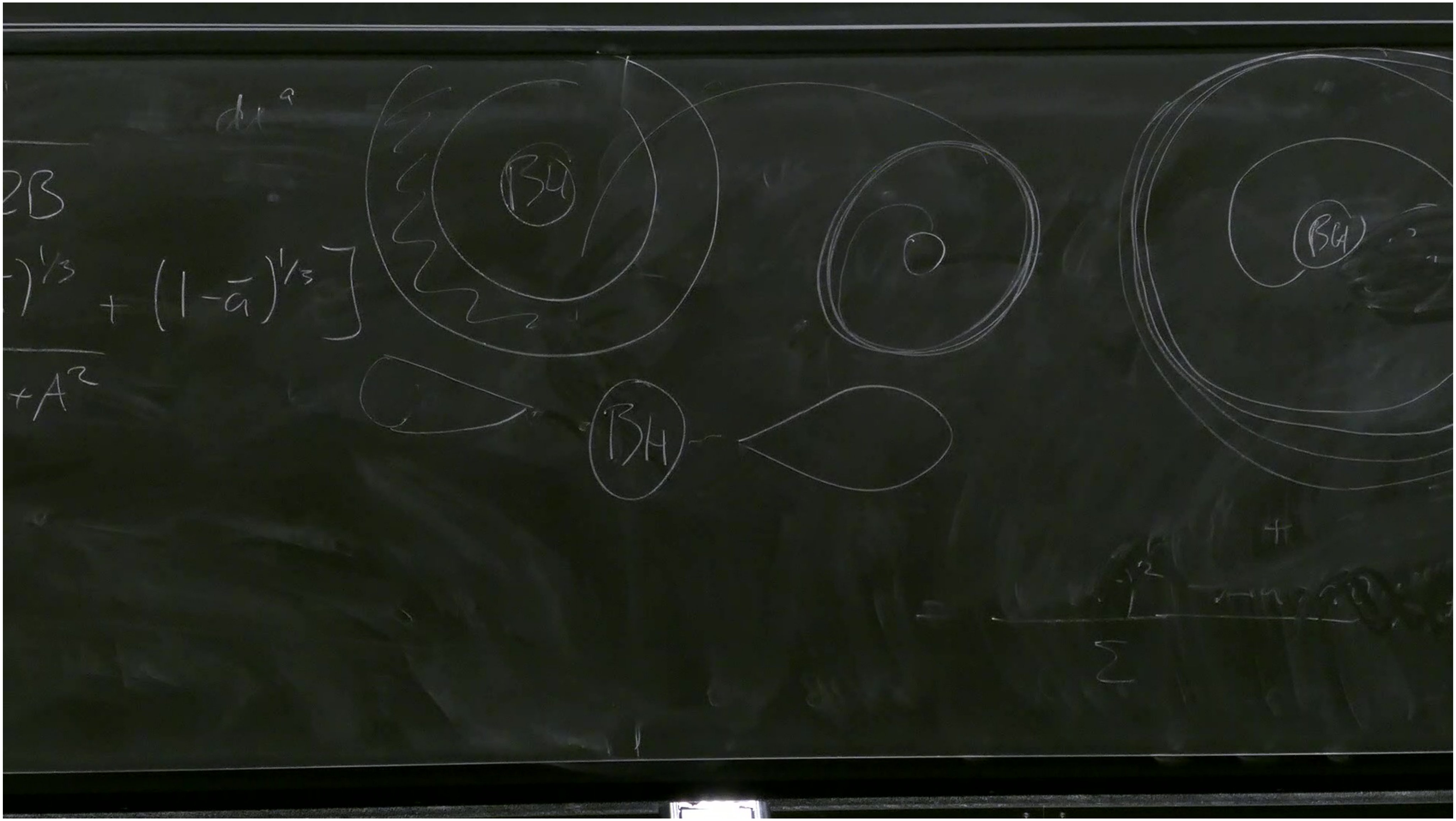
$$A = 1 + (1-\bar{a}^2)^{1/3}\left[(1+\bar{a})^{1/3} + (1-\bar{a})^{1/3}\right]$$

$$B = \sqrt{3\bar{a}^2 + A^2}$$

$$r_{c,min} < r < r_{ISCO}$$

unstable


BH
f
t
Σ


2B
$+ (1-\bar{a})^{1/3}$
$+A^2$
BH
BH
BH
Σ