Title: Strong Gravity Lecture

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Collection: Strong Gravity 2023/24

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Review and Conventions

Geometric units: $G = c = 1$

Index conventions: $a, b, c, \ldots$ 4 index (time + space)

$i, j, k, \ldots$ 3 index (spatial)

$$u^a = (u^t, u^i)$$

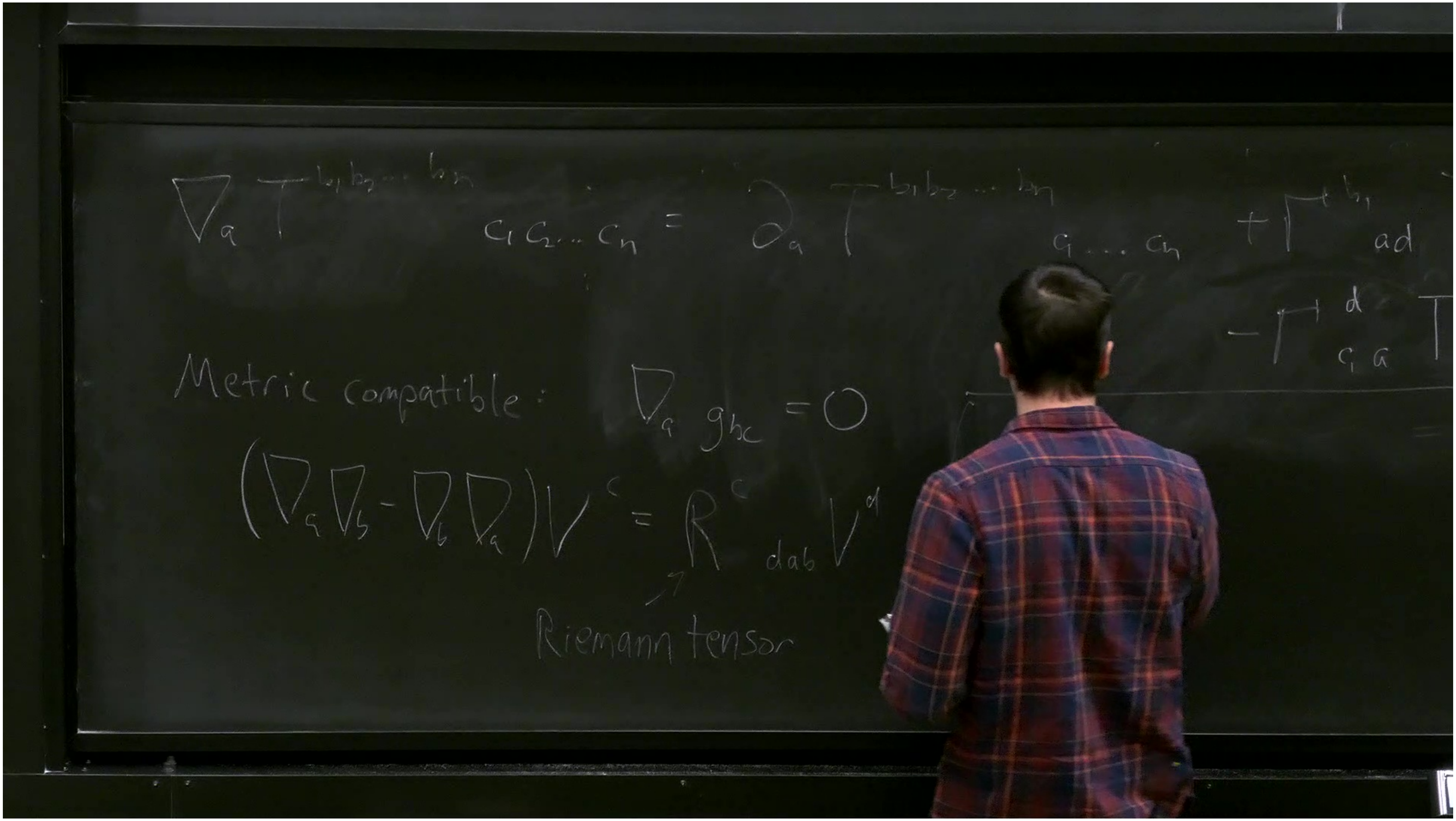
ace)

East coast metric $(-+++)$

Metric: $ds^2 = g_{ab}\, dx^a dx^b$

Christoffel symbols:

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right)$$


$\nabla_a T^{b_1 b_2 \dots b_n}{}_{c_1 c_2 \dots c_n} = \partial_a T^{b_1 b_2 \dots b_n}{}_{c_1 \dots c_n} + \Gamma^{b_1}{}_{ad}$
$- \Gamma^{d}{}_{c_1 a}$
Metric compatible: $\nabla_a g_{bc} = 0$
$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R^c{}_{dab} V^d$
Riemann tensor

$$= \partial_a T^{b_1 b_2 \ldots b_n}{}_{c_1 \ldots c_n} + \Gamma^{b_1}{}_{ad} T^{d b_2 \ldots b_n}{}_{c_1 \ldots c_n} + \ldots$$

$$- \Gamma^{d}{}_{c_1 a} T^{b_1 \ldots b_n}{}_{d c_2 \ldots c_n} - \ldots$$

$$g_{bc} = 0$$

$$R_{dab}{}^{c} V^{d}$$

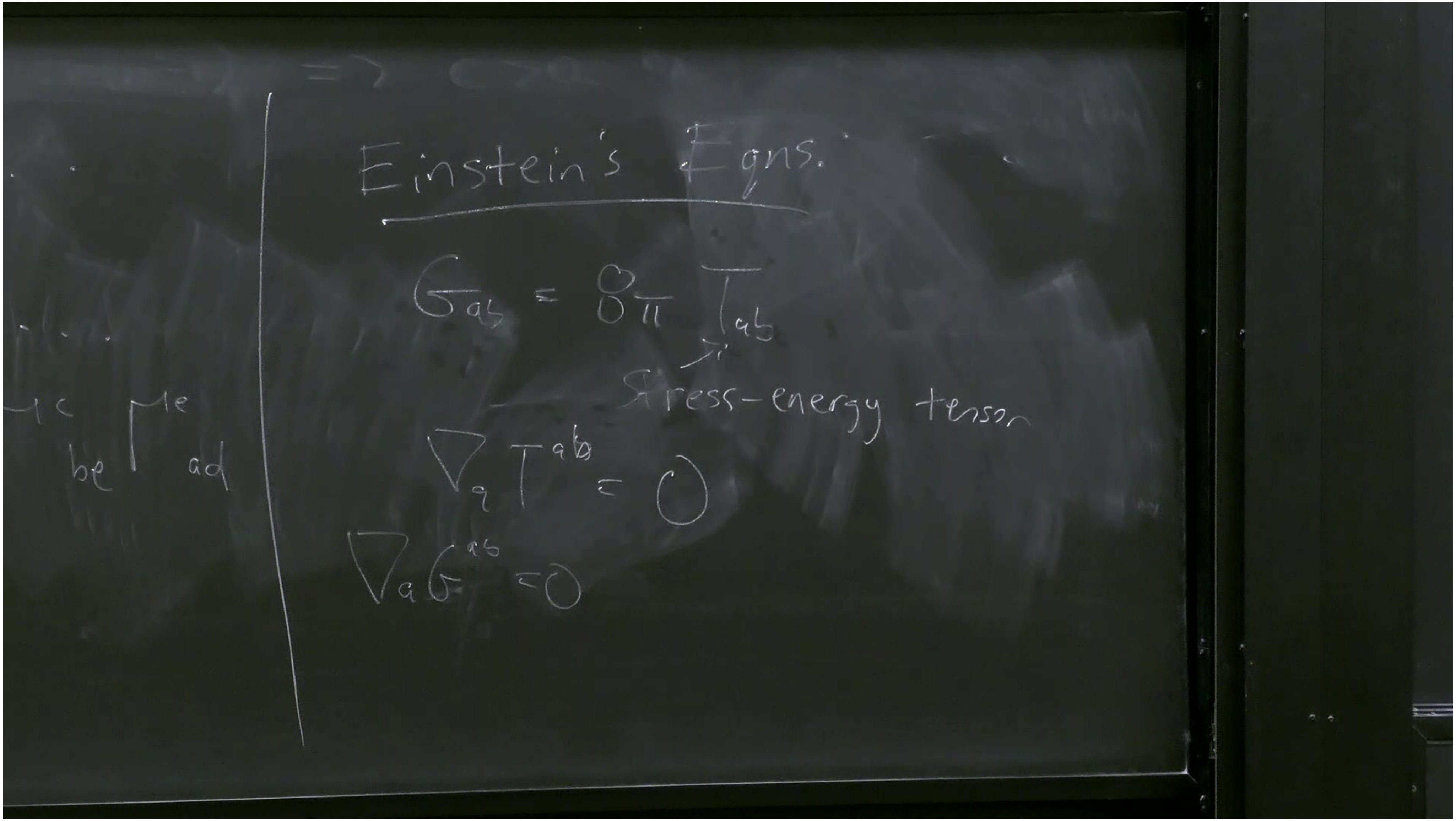
tensor

$$R^{c}{}_{dab} = \partial_a \Gamma^{c}{}_{bd} - \partial_b \Gamma^{c}{}_{ad} + \Gamma^{c}{}_{ae} \Gamma^{e}{}_{bd} - \Gamma^{c}{}_{be} \Gamma^{e}{}_{ad}$$

Ricci tensor: $R_{ab} = R^{c}{}_{acb}$

Ricci Scalar: $R = g^{ab} R_{ab}$

Einstein tensor: $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$


Einstein's Eqns.
$G_{ab} = 8\pi T_{ab}$
stress-energy tensor
$\nabla_a T^{ab} = 0$
$\nabla_a G^{ab} = 0$

Riemann tensor

Ricci Scalar: $R = g^{ab} R_{ab}$

Einstein tensor: $G_{ab} = R_{ab} - \frac{1}{2}$

Perfect Fluid S.E. tensor

$$T_{ab} = (\rho + P) u_a u_b + P g_{ab}$$

Null Energy Condition: $T_{ab} k^a k^b \geq 0$ for k^a future pointing null

$$(\rho + P)(u_a k^a)^2 \geq 0$$

Kerr Black holes

- Only possible stationary vacuum Black hole solution
  (Israel, Carter, Hawking, Robinson 1967-1971)
- Perturbations of Kerr decay rapidly (later quasinormal modes)
- Final state of generic collapse Kerr BH (M + J) + gravitional waves

Metric in Boyer-Lindquist Coördinates

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dt\,d\phi + \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 + a^2 - 2Mr$$

$$a = J/M \quad (\text{total Ang. mom.} / \text{mass})$$

$$\bar{a} = J/M^2$$

$\bar{a} = J/M$

$a = 0, \Rightarrow$ Schwarzshild

Fix $\bar{a}$, $M \to 0$

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\right) dr^2 + \left(r^2 + a^2\cos^2\theta\right) d\theta^2 + \left(r^2 + a^2\right)\sin^2\theta \, d\phi^2$$

$$x = (r^2 + a^2)^{1/2} \sin\theta \cos\phi$$

$$y = (r^2 + a^2)^{1/2} \sin\theta \sin\phi$$

$$z = r\cos\theta$$

K is Killing: $\mathcal{L}_K \, g_{ab} = 0$

$$= K^c \partial_c g_{ab} + (\partial_a K^c) g_{cb} + (\partial_b K^c) g_{ac}$$

$$\partial_a \to \nabla_a$$

$$\mathcal{L}_K g_{ab} = 0 + \nabla_a K_b + \nabla_b K_a = 2 \nabla_{(a} K_{b)}$$

Killing vectors of Kerr

$\vec{\phi}^a := (\partial_\phi)^a \iff$ axisymmetric

$\vec{t}^a := (\partial_t)^a \iff$ stationary

Killing tensor: $\nabla_{(a}K_{bc)} = 0$

Kerr: $K_{ab} = r^2 g_{ab} + 2\Sigma\, l_{(a}n_{b)}$

Horizons

Coördinates : $\Sigma = 0$, $\Delta = 0$

$R_{abcd} R^{abcd}$ blows up at $\Sigma = 0$

$r^2 + a^2 \cos^2\theta = 0$ $r = 0$, $\theta = \frac{\pi}{2}$

$r = \text{constant}$

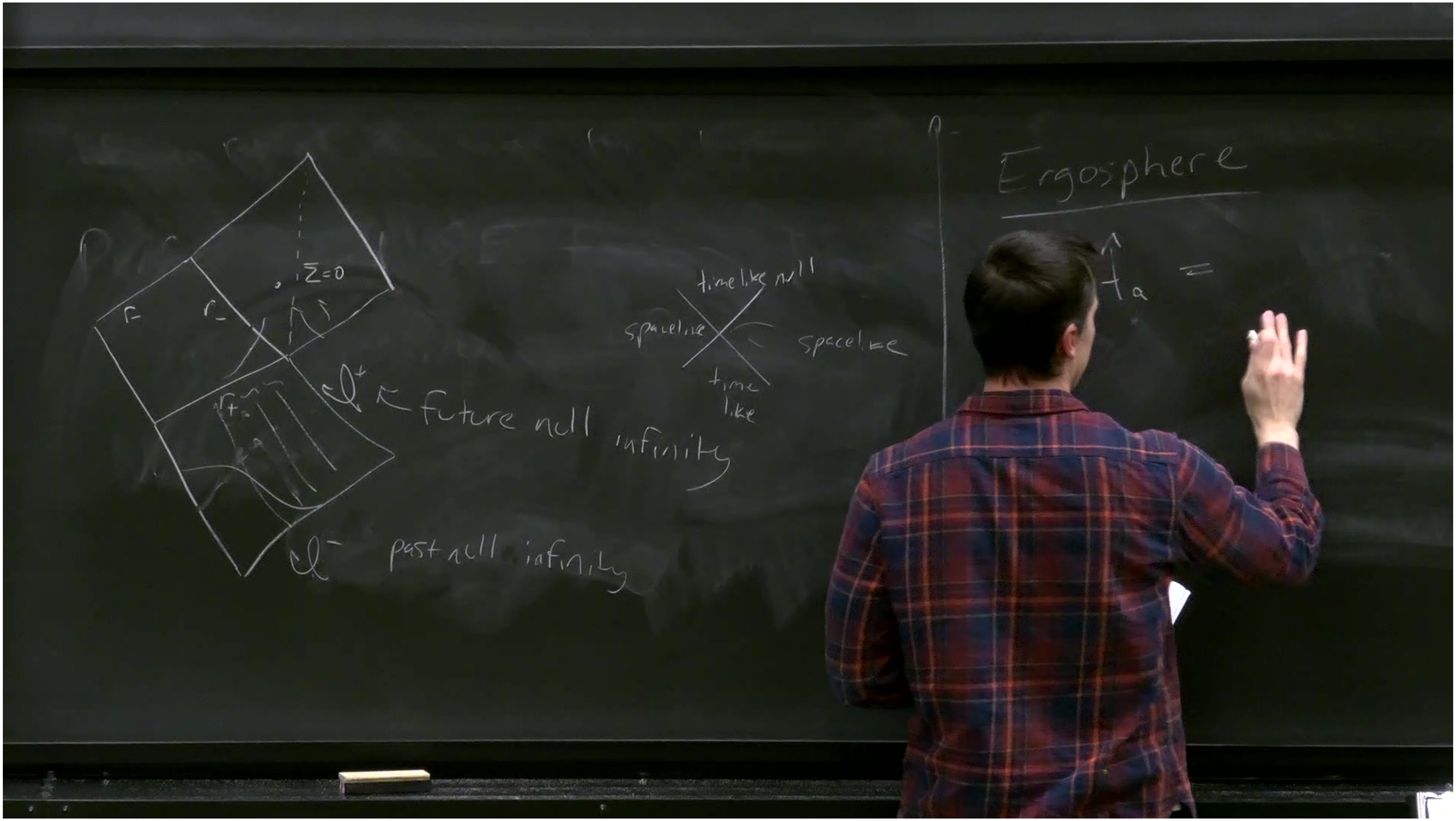
$$g^{ab}\,\partial_a r\,\partial_b r = g^{rr} = \frac{\Delta}{\Sigma}$$

$= 0 \Rightarrow$ null surface

$$\Delta = (r - r_+)(r - r_-)$$

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$ (Outer / inner horizons)

$|\bar{a}| > 1 \Rightarrow$ no horizon $\Rightarrow$ naked singularity


Ergosphere
$\hat{t}_a =$
$\Sigma = 0$
r_-
r_-
r_+
timelike null
spacelike
spacelike
time like
$\mathscr{I}^+$ ← future null infinity
$\mathscr{I}^-$ past null infinity

spacelike

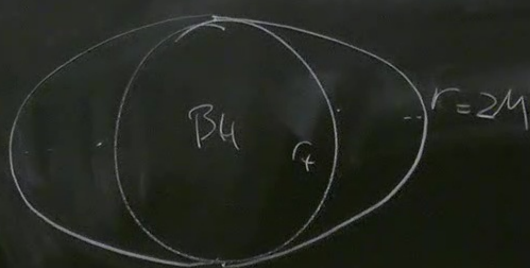
Ergosphere

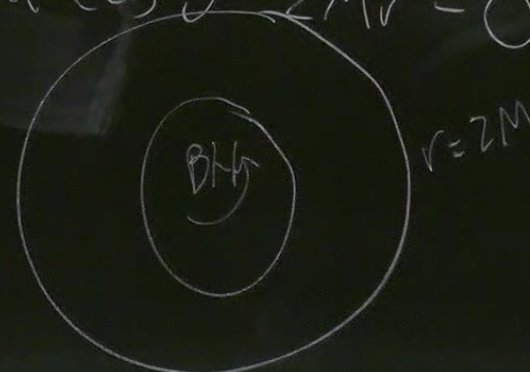
at $r = r_+$

$$\hat{t}^a \hat{t}_a = g_{tt} = \frac{a^2 \sin^2\theta}{\Sigma} \geq 0$$

null $\hat{t}^a \hat{t}_a = 0 \Rightarrow \Delta = a^2 \sin^2\theta$

$$r^2 + a^2 \cos^2\theta - 2Mr = 0$$



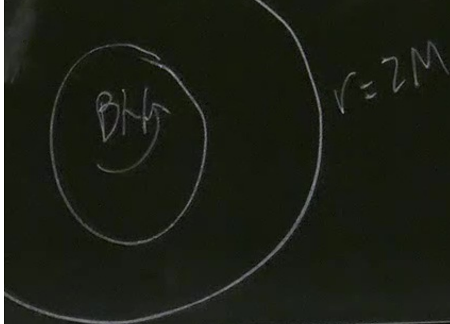


$= r_+$

$\theta \geq 0$

$= a^2 \sin^2\theta$

$\cos^2\theta - 2Mr = 0$



U^a timelike 4-velocity

$$-1 = U^a U_a = \underbrace{g_{tt}(U^t)^2 + g_{rr}(U^r)^2 + g_{\theta\theta}(U^\theta)^2 + g_{\phi\phi}(U^\phi)^2}_{\geq 0} \underbrace{+ 2 g_{t\phi} U^t U^\phi}_{< 0}$$

$$g_{t\phi} < 0, \quad U^t = \frac{dt}{d\tau} > 0, \quad U^\phi = \frac{d\phi}{d\tau} > 0$$