Title: GPTs and the probabilistic foundations of quantum theory - Lecture

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Collection: GPTs and the probabilistic foundations of quantum theory - mini-course

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Today

* Recap
* Boxworld
* "Convex operational"

! * Prob. Theories à la mode $\binom{\text{de}}{\text{du}}$ Pavia

Story so far:

$\mathcal{C}$ a SMC

A prob. theory:
a functor (inj on obj!)

$$F : \mathcal{C} \longrightarrow \mathbf{Prob}$$

let $F(A)F(B) = F(A \otimes B)$

Need:

$$Ff = Ff' \Rightarrow \forall g \; F(f \otimes g) = F(f' \otimes g)$$

"for g, g'"

more: want

$F(A)\,F(B)$

to be a NS comp:

need, $\forall A, B$. a Prob-morph

$$\pi_{A,B} : F(A) \times_{NS} F(B) \longrightarrow F(A \otimes B)$$

Strong comp: have all product states.

Finally: want π_{AB}-s comp^s of a nat. transf

$$F(-) \times_{NS} F(-) \xrightarrow{\pi} F(- \otimes -)$$

For any models A, B in $\underline{\underline{\mathrm{Prob}}}$

$$A \times_{NS} B = (m(A) \times m(B),\ \Omega_{NS}(m(A) \times m(B))$$

$$\mathbb{V}(A \times_{NS} B)$$

$$= \mathbb{V}(A) \otimes_{max} \mathbb{V}(B)$$

"gbit" : any model of the form

- 2 tests
- 2 outcomes per test
- All prob.-weights $\square$

E_0

$$\mathcal{B} = \{\{x,y\},\{u,v\}\}$$

$$= \{\{(0|0),(1|0)\}, \{(0|1),(1|1)\}\}$$

$$\mathcal{B} \times_{NS} \mathcal{B}$$

$$= \left\{ \left\{ ((0|0),(0|0)), ((0|0),(1|0)), \dots \right. \right.$$

$$(0,0\,|\,0,0) \quad \dots$$

$$(i,j\,/\,k,l) \qquad i,j,k,l \in \{0,1\}$$

4 tests, 4 outcomes...

For any models A, B in $\underline{\mathrm{Prob}}$

$$A \times_{NS} B = (m(A) \times m(B),\; \Omega_{NS}(m(A) \times m(B))$$

$$\mathbb{V}(A \times_{NS} B) = \mathbb{V}(A) \otimes_{\max} \mathbb{V}(B)$$

$$\mathcal{B}_1 \times_{NB} \mathcal{B}_2 \times_{NB} \cdots \times_{NB} \mathcal{B}_n \cdots =: \mathcal{B}^n$$

$\mathcal{C}$: objects = $\mathcal{B}^n$'s, possibly labeled ...
(w\ all states)

morphisms = all morphisms allowed
(in Prob)

subcat. of Prob

$$I = \mathcal{B}^0 = \{\{1\}\}$$

$$F : \mathcal{C} \longrightarrow \mathrm{Prob}$$

inclusion!

$$\mathcal{B}_1 \times_{NB} \mathcal{B}_2 \times_{NB} \cdots \times_{NB} \mathcal{B}_n \cdots =: \mathcal{B}^n$$

Bxwld

$\mathcal{C}$: objects = $\mathcal{B}^n$'s, possibly labeled ...
(w/ all states)

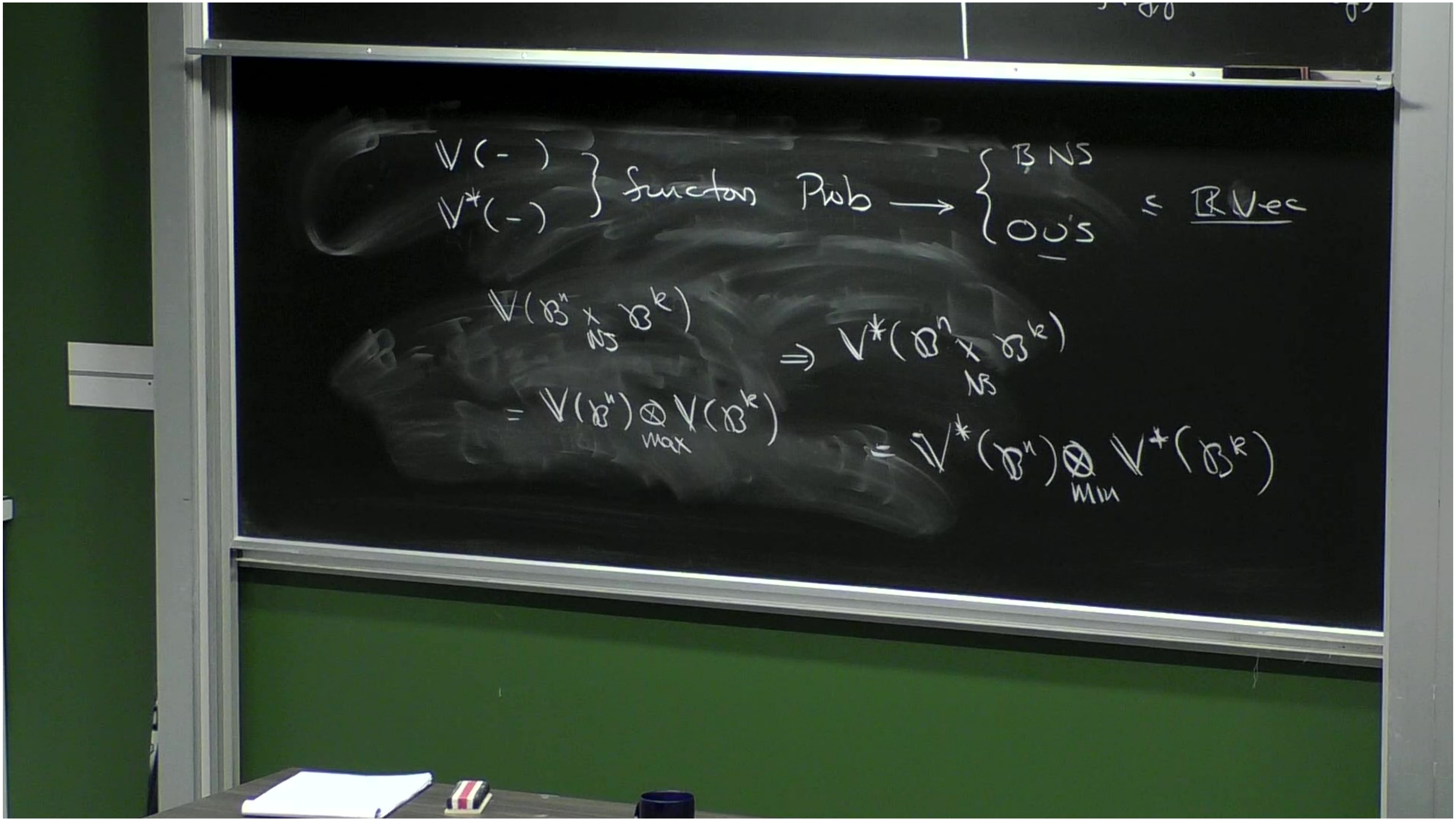
morphisms = $\underset{\approx}{\text{all}}$ morphisms allowed
(in Prob)

subcat. of Prob

$I = \mathcal{B}^0 = \{\{1\}\}$

$F : \mathcal{C} \longrightarrow \text{Prob}$

inclusion!


$\mathbb{V}(-)$, $\mathbb{V}^*(-)$ } functors $\mathbf{Prob} \longrightarrow$ { $\mathbf{BNS}$, $\mathbf{OU's}$ } $\subseteq \mathbb{R}\mathbf{Vec}$
$\mathbb{V}(\mathcal{B}^n \times_{NS} \mathcal{B}^k) = \mathbb{V}(\mathcal{B}^n) \otimes_{max} \mathbb{V}(\mathcal{B}^k)$
$\Rightarrow \mathbb{V}^*(\mathcal{B}^n \times_{NS} \mathcal{B}^k) = \mathbb{V}^*(\mathcal{B}^n) \otimes_{min} \mathbb{V}^*(\mathcal{B}^k)$

"Convex Op. Theories"

K cpt convex set.

$\mathrm{Aff}_b(K)$ = space of bdd affine functionals $f : K \to \mathbb{R}$.

$f \in \mathrm{Aff}_b(K)$ an effect

$\Leftrightarrow 0 \leq f(\alpha) \leq 1 \quad \forall \alpha \in K$

$$m(K) = \Big\{ E \subseteq \mathrm{Aff}_b(K) \;\Big|\; \forall f \in E \text{ an effect}, \quad \sum_{f \in E} f = \underset{\substack{\uparrow \\ \text{effect}}}{1} \Big\}$$

$\mathcal{B}^n$

$\forall \alpha \in K$, let

$$\hat{\alpha}(f) = f(\alpha) \quad \forall \text{ effects } f.$$

$$\hat{K} = \{\hat{\alpha} \mid \alpha \in K\}$$

$$F(K) = \left(m(K), \hat{K}\right)$$

$\underline{\mathbf{CConv}}$ = cat. of compact convex sets,

morphisms $K \xrightarrow{\varphi} K'$

bounded affine mappings

$$\underline{\mathbf{CConv}}^{op} \xrightarrow{F} \mathbf{Prob}$$

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[dr] & K' \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

$$F(\varphi)\begin{cases} \varphi^*(f) \\ = f \circ \varphi \end{cases}$$

$\Leftrightarrow \; 0 \leq f(\alpha) \leq 1 \quad \forall \alpha \in K$

$f \in E$ $\uparrow$ effect $\}$

Namioka / Phelps ('68)

$\otimes_{min}$, $\otimes_{max}$ on $\mathcal{C}$conv.

$F(K \otimes_{min} K')$ $\quad$ $F(K \otimes_{max} K')$

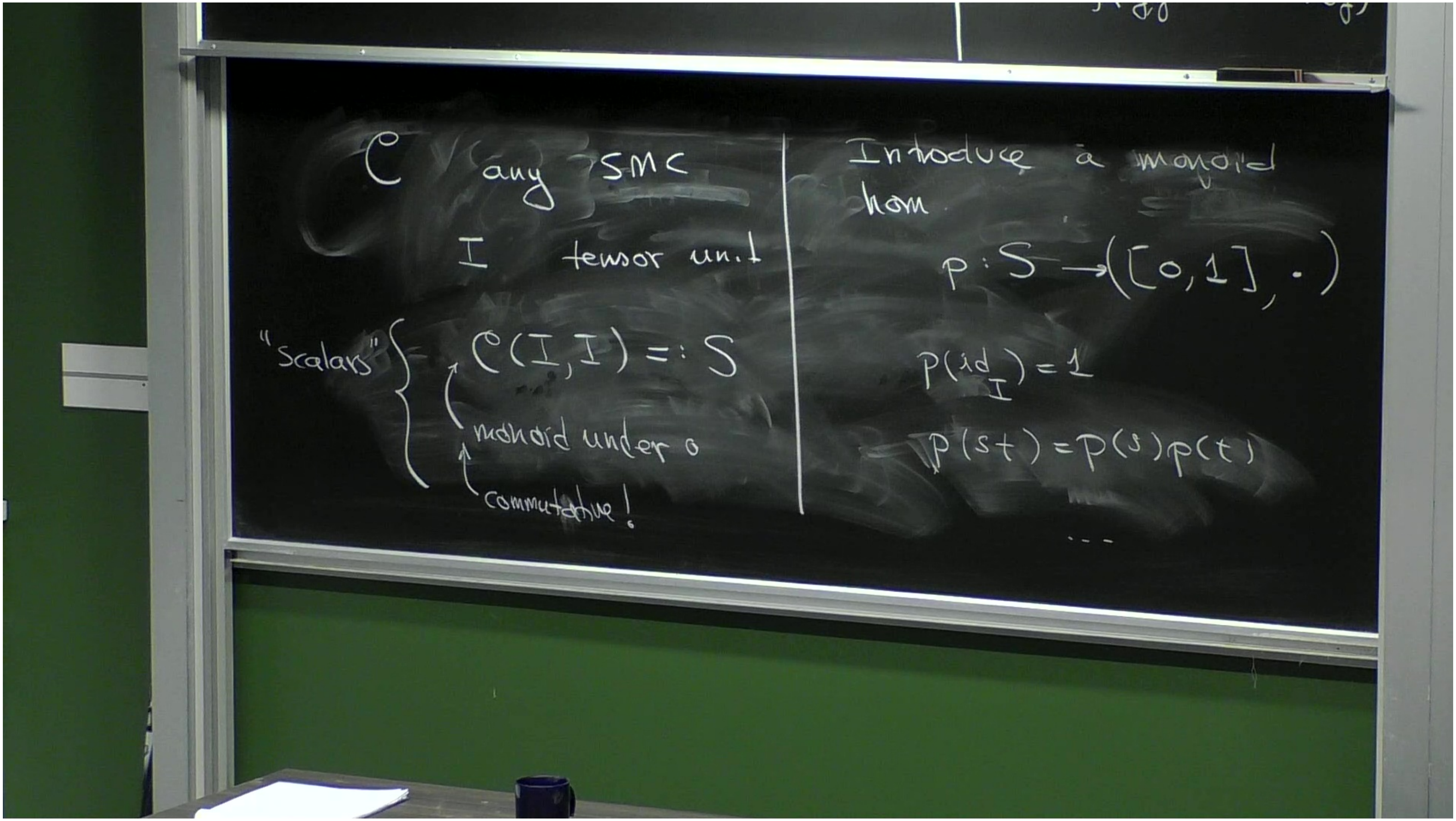
NS comp.

$\mathbb{V}(F(K \otimes_{min} K'))$

$= \mathbb{V}(F(K)) \otimes_{min} \mathbb{V}(F(K'))$

$\mathbb{V}(F(K \otimes_{max} K'))$

$= \mathbb{V}(F(K)) \otimes_{max} \mathbb{V}(F(K'))$


$\mathcal{C}$ any SMC
I tensor unit
"Scalars" $\mathcal{C}(I,I) =: S$
monoid under $\circ$
commutative!
Introduce a monoid hom
$p : S \to ([0,1], \cdot)$
$p(id_I) = 1$
$p(st) = p(s)p(t)$
...

$\Leftrightarrow \quad 0 \leq f(\alpha) \leq 1 \quad \forall \alpha \in K$

$f \in E$ $\uparrow$ effect $\}$

OK, think of

$\alpha : I \to A$

"preparation"

$a : A \to I$

"effect"

$a \circ \alpha \in S$

$p(a \circ \alpha) =$ "$pr(a|\alpha)$"

$\forall \alpha$, define $\hat{\alpha} \in [0,1]^{\mathcal{E}(A,I)}$ — compact

$\hat{\alpha}(a) = p(a \circ \alpha) = pr(a|\alpha)$

$$K(A) := \overline{\mathrm{con}}\left(\{\hat{\alpha} \mid \alpha \in \mathcal{C}(I, A)\}\right)$$

convex

$\xrightarrow{\varphi} K$

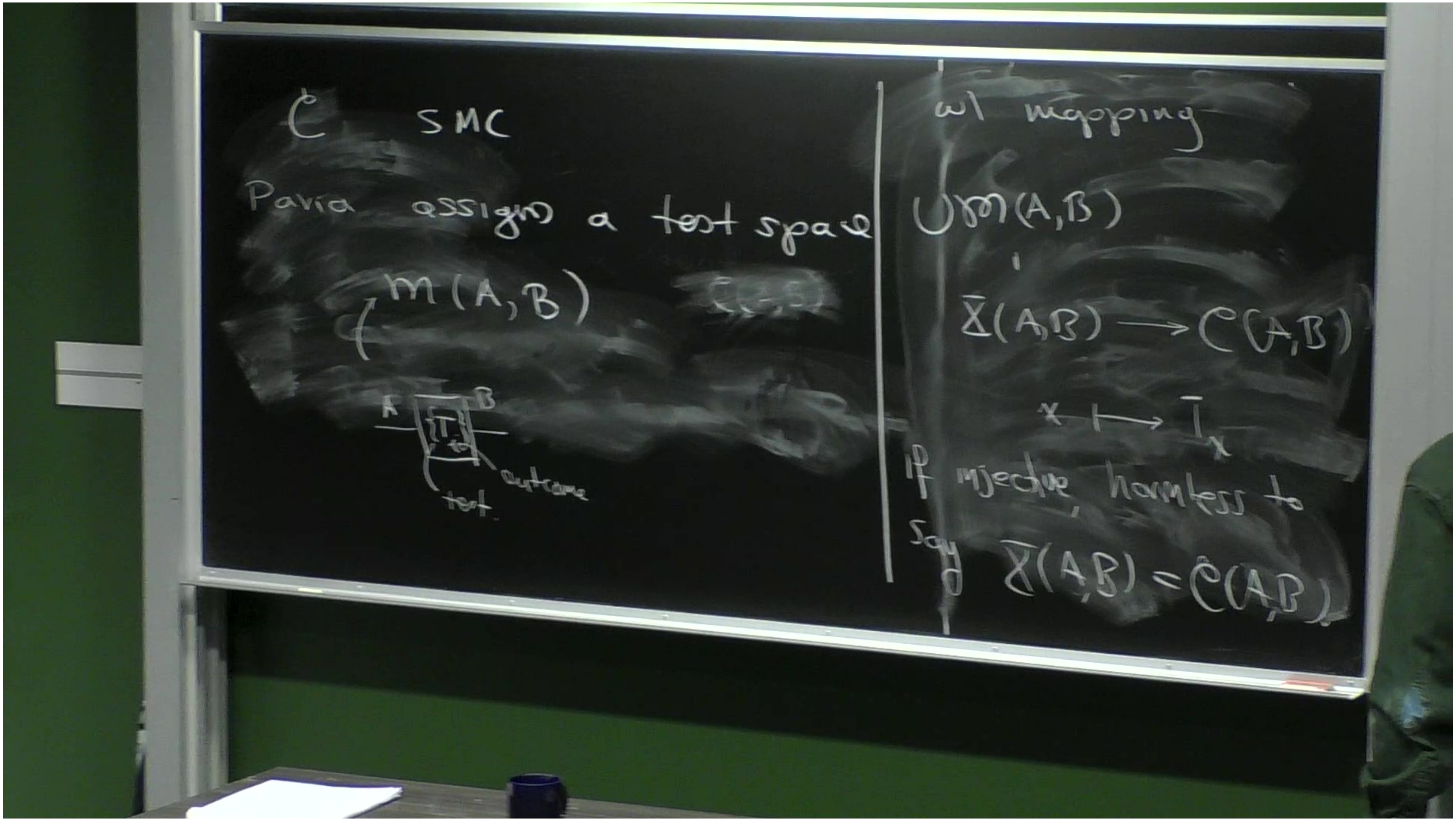
$\to K'$

$\downarrow f$

(φ)

$\mathbb{R}$

$f \circ \varphi$


$\mathcal{C}$ SMC
Pavia assign a test space
$m(A,B)$
A
B
$\{T_i\}$
outcome
test
w/ mapping
$\bigcup m(A,B)$
$\bar{X}(A,B) \longrightarrow \mathcal{C}(A,B)$
$x \longmapsto \bar{T}_x$
If injective, harmless to say $\bar{X}(A,B) = \mathcal{C}(A,B)$

$\Leftrightarrow \; 0 \leq f(\alpha) \leq 1 \;\; \forall \alpha \in K$

$f \in E$ $\uparrow$ effect $\}$

Also posit

$p : S \to [0,1]$

$\rightsquigarrow$ can assign prob^s.

Q: Is causality $\xrightarrow{\quad\quad\quad}$

$\overset{\text{Pavia}}{\Longleftrightarrow}$ to $M(A,B)M(B,C)$

$\subseteq M(A,C)$

... get

$\underbrace{\mathcal{C}^{op} \times \mathcal{C}} \longrightarrow \mathrm{Prob}.$

Need: "causality" so that $\forall$

if $\{a_x\} \in M(A,I)$

$\bar{\alpha} \in \mathcal{C}(I,A)$, indep $\{a_x\}$

$\sum_x p(a_x \circ \alpha)$

(causal)

Q: Is "Pavia" a NS theory?

(π-s?)

Q: What would a "non-causal" theory look like in terms of test spaces?

w/ mapping

$\bigcup \mathfrak{M}(A,B)$

$\subseteq$

$\overline{X}(A,B) \longrightarrow \mathcal{C}(A,B)$

$x \longmapsto \overline{T}_x$

If injective, harmless to say $\overline{X}(A,B) \subseteq \mathcal{C}(A,B)$.