Title: GPTs and the probabilistic foundations of quantum theory - Lecture

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Collection: GPTs and the probabilistic foundations of quantum theory - mini-course

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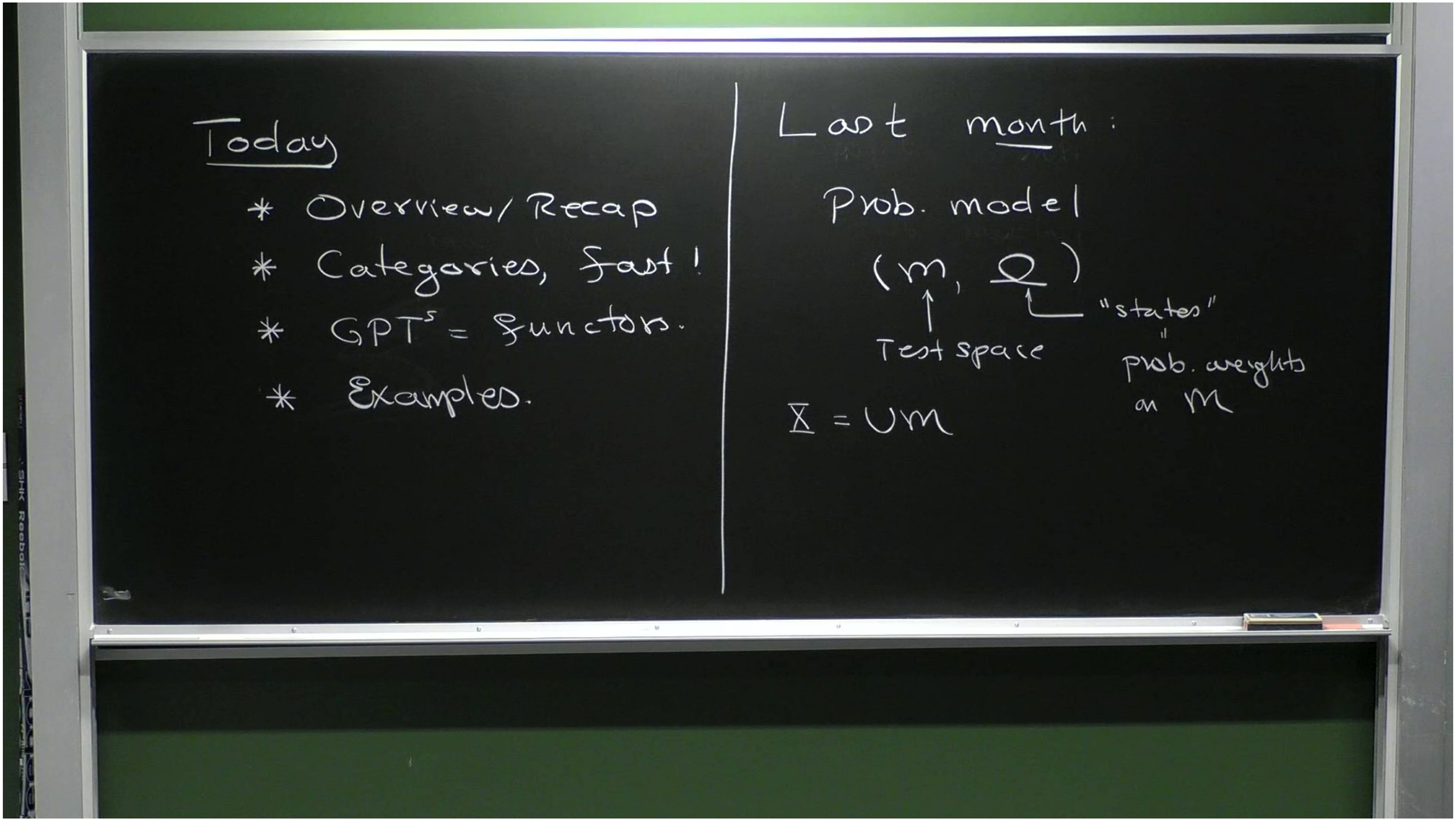
Today
* Overview/Recap
* Categories, fast!
* GPT's = functors.
* Examples.
Last month:
Prob. model
(M, Ω)
Test space
"states"
prob. weights on M
X = ∪M

morphisms:

$$(\mathcal{M}, \Omega) \xrightarrow{\varphi} (\mathcal{M}', \Omega')$$

$$X = \cup \mathcal{M} \xrightarrow{\varphi} X' = \cup \mathcal{M}'$$

* If $a \in \mathcal{E}v(\mathcal{M}) \Rightarrow \varphi(a) \in \mathcal{E}v(\mathcal{M}')$
* $\forall \beta \in \Omega'$, $\beta \circ \varphi = t\,\alpha$ for some $\alpha \in \Omega$, $0 < t \leq 1$

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* $\forall E, F \in \mathcal{M}$, $\varphi(E), \varphi(F)$ equiprobable

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morphisms compose!

identity a morphism!

Have a category

$\underline{\text{Prob}}$

A category consists of:

* A class of objects $A, B, C, \ldots$
* for any objects A, B, a set of "morphisms", or "arrows"

$$A \xrightarrow{f} B$$

$\mathcal{C}(A,B)$ - set of these.

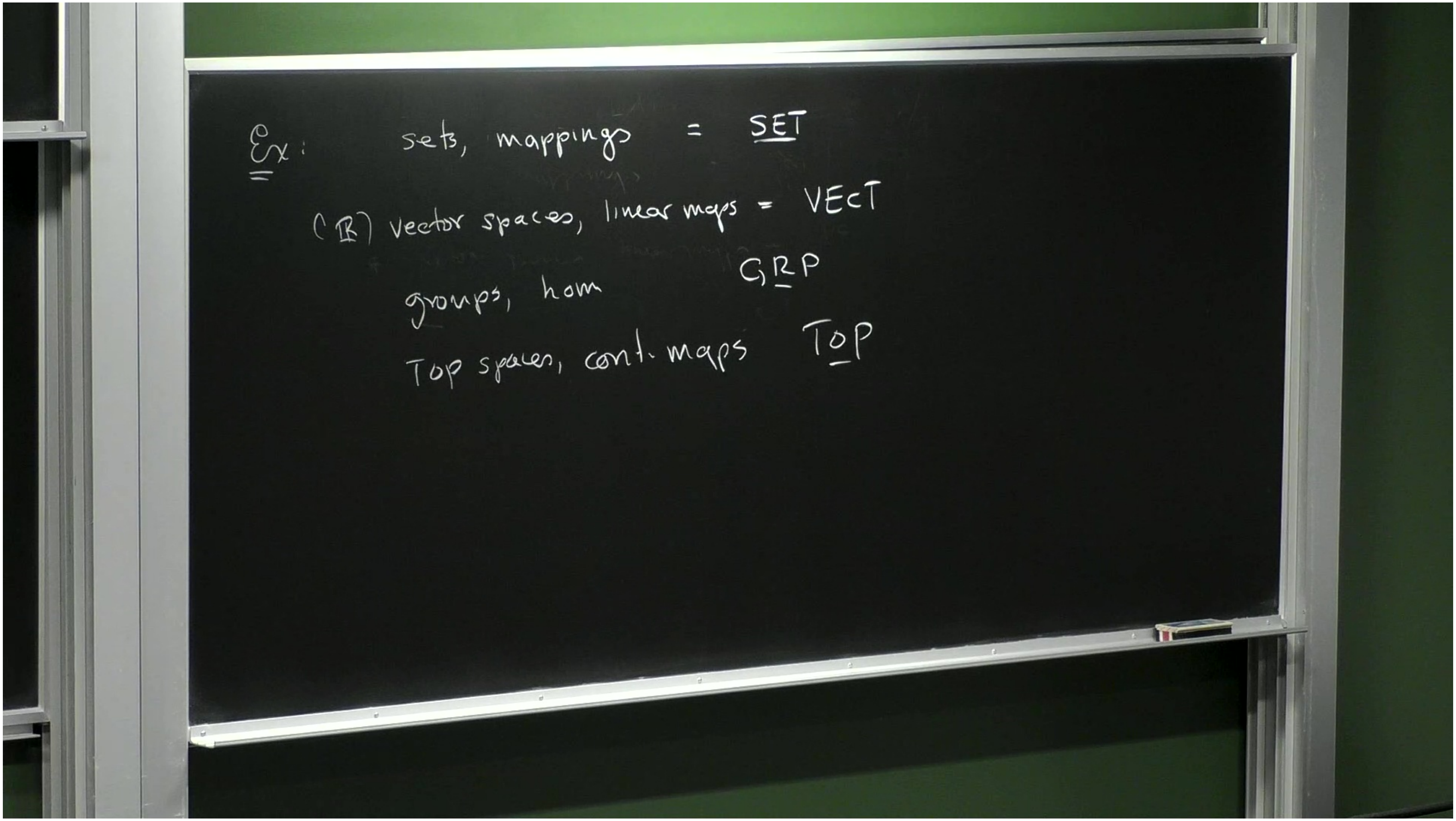
* composition rule:

$$\mathcal{C}(A,B) \times \mathcal{C}(B,C) \xrightarrow{\circ} \mathcal{C}(A,C)$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

- associative!
- $\forall A \;\; \exists\, id_A \in \mathcal{C}(A,A)$

$$id_B \circ f = f = f \circ id_A$$


Ex: sets, mappings = SET
(ℝ) vector spaces, linear maps = VECT
groups, hom GRP
Top spaces, cont. maps TOP

A **functor** from cat $\mathcal{C}$ to cat $\mathcal{D}$

is a rule assigning

* to each object $A \in \mathcal{C}$, an object $\left.\begin{matrix} FA \\ F(A) \end{matrix}\right\} \in \mathcal{D}$
* $\forall A, B \in \mathcal{C}$, $\forall A \xrightarrow{f} B$, a morphism $FA \xrightarrow{Ff} FB$
* $F(f \circ g) = Ff \circ Fg$
* $F(\mathrm{id}_A) = \mathrm{id}_{FA}$

$\mathcal{C}$ cat:

$\mathcal{C}^{op}$

$\mathcal{C}^{op}(A,B) = \mathcal{C}(B,A)$

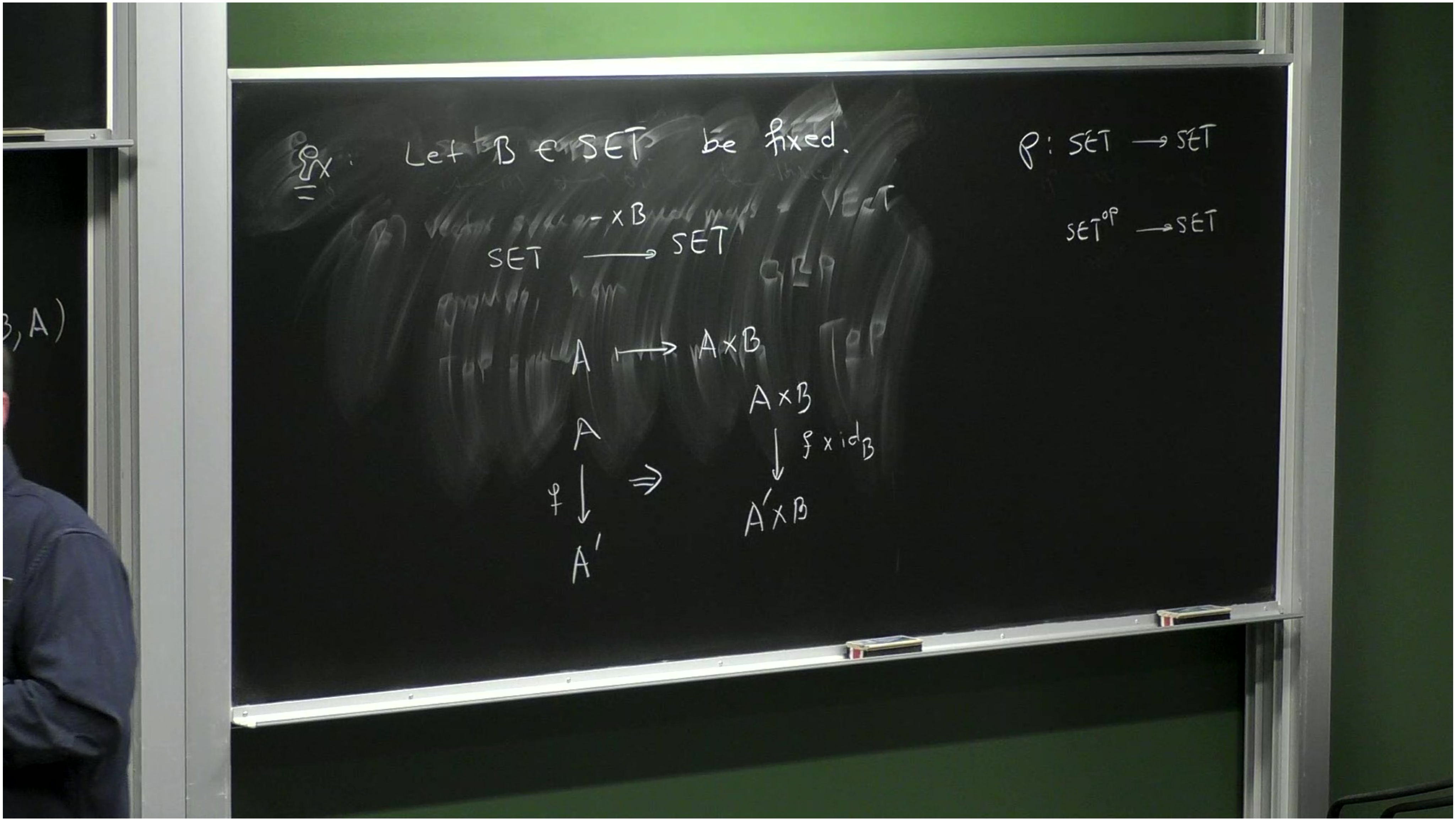
Ex (?)

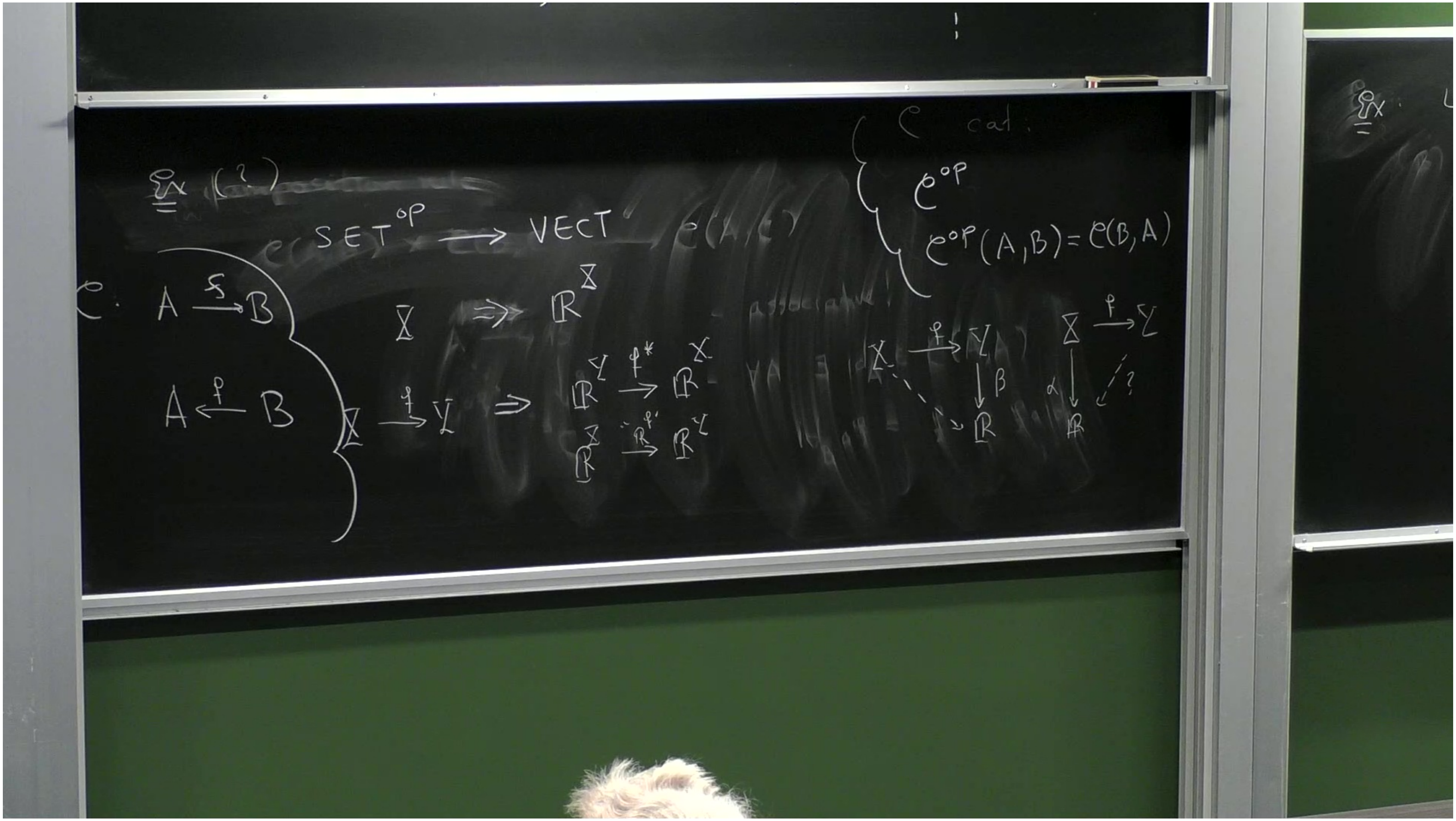
$$SET^{op} \longrightarrow VECT$$

$$X \Rightarrow \mathbb{R}^X$$

$$X \xrightarrow{f} Y \Rightarrow \mathbb{R}^Y \xrightarrow{f^*} \mathbb{R}^X$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed] & Y \arrow[d, "\beta"] \\ & \mathbb{R} \end{tikzcd}$$


Ex: Let B ∈ SET be fixed.
ρ: SET → SET
SET^op → SET
-×B
SET → SET
A ↦ A×B
A ⇒ A×B
f ↓ f×id_B
A' A'×B



Ex (?)
SET^op ⟶ VECT
C: A →f B
A ←f B
X ⇒ R^X
X →f Y ⇒ R^Y →f* R^X
R^X →R^f R^Y
C cat:
C^op
C^op(A,B) = C(B,A)

If $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ are functors,

a natural transformation from F to G

is a collection of morphisms $\{ \varphi_A \in \mathcal{D}(FA, GA) \}$

so that

$$\mathcal{C} \qquad\qquad \mathcal{D}$$

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "\varphi_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\varphi_B"'] & GB \end{tikzcd}$$

commutes

$\varphi : F \Rightarrow G$

Ex: Here are two functors

$$SET \longrightarrow SET$$

$a : \{0,1\} \longrightarrow A$

* $FA = A \times A$

* $GA = A^2 = A^{\{0,1\}}$

$$A^2 \xrightarrow{\varphi_A} A \times A$$

$$\varphi_A(a) = (a(0), a(1))$$

If $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ are functors,

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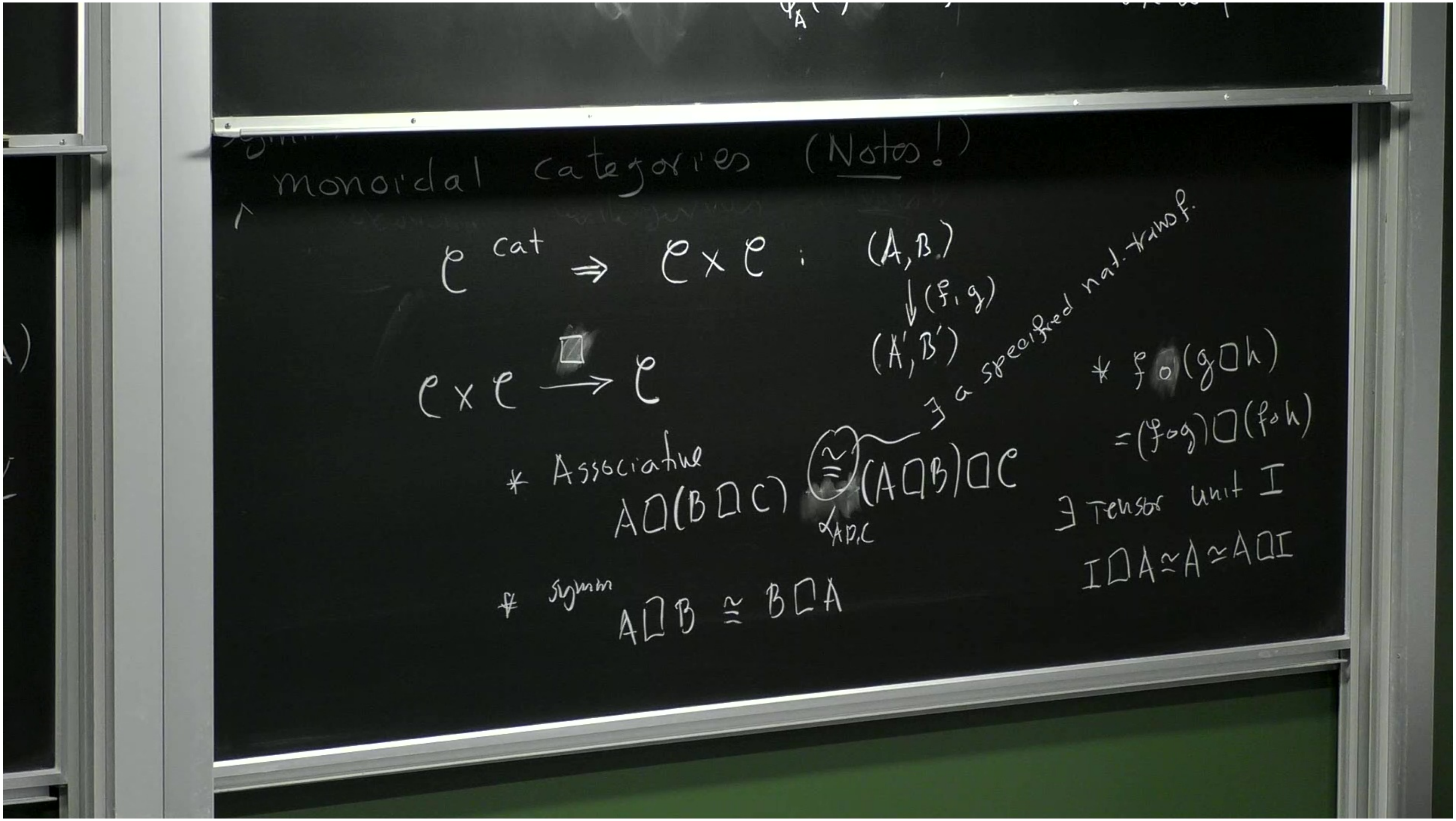
commutes.

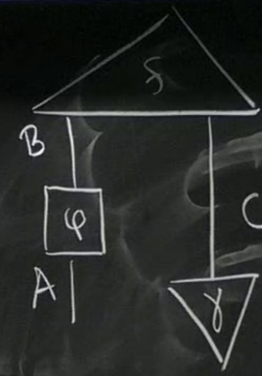

monoidal categories (Notes!)
$\mathcal{C}$ cat $\Rightarrow$ $\mathcal{C}\times\mathcal{C}$: $(A,B) \xrightarrow{(f,g)} (A',B')$
$\mathcal{C}\times\mathcal{C} \xrightarrow{\square} \mathcal{C}$
* Associative
$A\square(B\square C) \underset{\alpha_{A,B,C}}{\cong} (A\square B)\square C$
∃ a specified nat. transf.
* symm $A\square B \cong B\square A$
* $f\circ(g\square h) = (f\circ g)\square(f\circ h)$
∃ Tensor unit I
$I\square A \simeq A \simeq A\square I$

$* \ \forall E, F \in M, \ \varphi(E), \varphi(F)$ equiprobable

Graphical Calculus

objects = wires

morphisms = boxes



$$A \xrightarrow{\varphi} B$$

$$A \cong A \square I \xrightarrow{id_A \square \gamma} A \square C \xrightarrow{\varphi \square id_C} B \square C \xrightarrow{\xi} I$$

$$\xi \circ (\varphi \square id_C) \circ (id_A \square \gamma)$$

Def: A Prob. theory is a functor

If $(\mathcal{C}, \square)$ is S-monoidal: $\quad m : \mathcal{C} \longrightarrow \text{Prob}.$

systems/processes

let $(mA)(mB) = m(A \square B)$

want:

$mf = mf'$

$\Rightarrow m(f \square g) = m(f' \square g)$

Injective on objects

M is NS if

$mA \; m \; B := m(A \square B)$

NS composite

$\forall A, B$

$A \mapsto m(A) = (m(A), S(A))$

$\downarrow \varphi \Rightarrow m\varphi : m(A) \to m(B)$

B

* $\forall E, F \in \mathcal{M}$, $\varphi(E), \varphi(F)$ equiprobable

$$\Rightarrow \quad \pi_{A,B} : \mathcal{M}A \underset{NS}{\times} \mathcal{M}B \longrightarrow \mathcal{M}(A \,\square\, B)$$

i.e., $\exists$ nat. transf

$$\pi : \mathcal{M}- \underset{NS}{\times} \mathcal{M}- \longrightarrow \mathcal{M}(- \,\square\, -)$$