Title: GPTs and the probabilistic foundations of quantum theory - Lecture

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Collection: GPTs and the probabilistic foundations of quantum theory - mini-course

Date: March 26, 2024 - 9:00 AM

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Lectures 6, 7
April 16, 18

Today

* Recap
* Morphisms & Composites
* Linearizing
? * Preview

Last Time:

Non-signaling (N-S) states

Two very simple "composite models"

$$A \underset{NS}{\times} B = \left(M(A) \times M(B) \, , \, \Omega_{NS} \right)$$

$$M(A) \times M(B) = \left\{ E \times F \mid E \in M(A), \ F \in M(B) \right\}$$

Ω_{NS}: NS prob. weights, ω

$\omega_1 \in \Omega(A)$

$\omega_2 \in \Omega(B)$

also

$$\overleftrightarrow{AB} = \left(\overleftrightarrow{m(A)\,m(B)},\ \Omega_{NS}\right)$$

$$\Downarrow$$

bilateral product $\quad pr(\ \text{"}\ )$

$= Pr_{NS}$

) states

"composite models"

$$A) \times m(B) \quad , \quad \Omega_{NS} \quad)$$

$$\Vert$$

$$\left\{ E \times F \;\middle|\; \begin{array}{l} E \in m(A) \\ F \in m(B) \end{array} \right\}$$

$\uparrow$ NS prob. weights, ω

$\omega_{1|y} \in \Omega(A)$

$\omega_{2|x} \in \Omega(B)$

) states

"composite models"

$) \times m(B)$, Ω_{NS})

"

$E \times F \mid E \in m(A)$, $F \in m(B)$ }

$\uparrow$ NS prob. weights, ω

$\omega_{1|y} \in \Omega(A)$

$\omega_{2|x} \in \Omega(B)$

also $\bigcup_{x \in E} \{x\} \times F_x$

$\overleftrightarrow{AB} = (\overleftrightarrow{m(A) m(B)}, \Omega_{NS})$

$\downarrow$

bilateral product $\quad pr($ " $)$

$= pr_{NS}$

$\underline{\underline{N}}$:

$$\mathbb{X}(A \times_{NS} B) = \mathbb{X}(\overleftrightarrow{AB})$$

$$= \mathbb{X}(A) \times \mathbb{X}(B)$$

Too small!!

NB:

$$\mathbb{X}(A \times_{NS} B) = \mathbb{X}(\overleftrightarrow{AB})$$

$$= \mathbb{X}(A) \times \mathbb{X}(B)$$

Too small!!

Ex (QM):

$$A = (\mathcal{F}(\mathcal{H}), \Omega(\mathcal{H}))$$

$$B = (\mathcal{F}(R), \Omega(R))$$

$$\Downarrow$$

$$\text{"}AB\text{"} = (\mathcal{F}(\mathcal{H} \otimes R), \Omega(\mathcal{H} \otimes R))$$

$\mathbb{X}(AB)$ = unit sphere of $\mathcal{H} \otimes R$

— "entangled outcomes"

Ex (Classical)

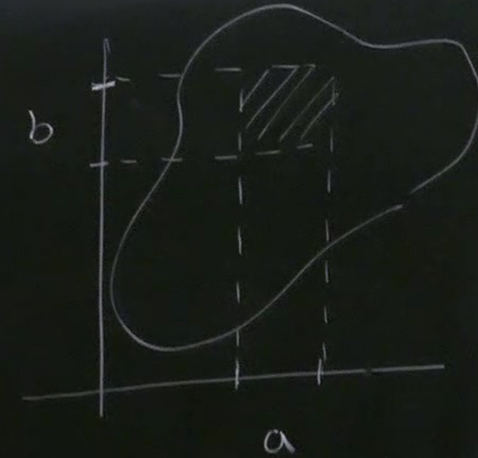
$A = (m(S, \Sigma), \ Pr)$

$B = (m(S', \Sigma'), \ Pr)$

$\text{"}AB\text{"} = (m(S \times S', \Sigma \otimes \Sigma'), \ Pr)$

σ-alg gen. by

$\Sigma \times \Sigma' = \{ a \times b \mid a \in \Sigma, b \in \Sigma' \}$



N: In each case, <u>do</u> have a mapping

$$\pi : \Sigma(A) \times \Sigma(B) \longrightarrow \Sigma(AB)$$

* QM: $\pi(x,y) = x \otimes y \in \mathcal{H} \otimes \mathcal{K}$.

* Class: $\pi(a,b) = a \times b$

N: In each case, do have a mapping

$$\pi : \Sigma(A) \times \Sigma(B) \longrightarrow \Sigma(AB)$$

* QM: $\pi(x,y) = x \otimes y \in \mathcal{H} \otimes \mathcal{K}$
* Class: $\pi(a,b) = a \times b$

$\underline{\underline{N}}$: $\pi(E \times F) \in \mathcal{M}(AB) \quad \forall E \in \mathcal{M}(A),\ F \in \mathcal{M}(B)$

$\underline{\underline{NB}}$:

$$\pi : \underline{Q}(AB) \longrightarrow \Pr(\mathcal{M}(A) \times \mathcal{M}(B))$$

$$\pi^*(\omega) = \omega \circ \pi$$

$$\pi^*(\omega)(x, y) = \omega(\pi(x, y))$$

$$= \begin{cases} \omega(x \otimes y) \\ \omega(a \times b) \end{cases}$$

$\pi(E \times F) \in \mathcal{M}(AB) \quad \forall E \in \mathcal{M}(A),\ F \in \mathcal{M}(B)$

$\underline{\underline{NB}}$:

$$\pi : \Omega(AB) \longrightarrow Pr(\mathcal{M}(A) \times \mathcal{M}(B))$$

$$\pi^*(\omega) = \omega \circ \pi$$

$$\pi^*(\omega)(x, y) = \omega(\pi(x, y)) = \left\{ \begin{array}{l} \omega(x \otimes y) \\ \omega(a \times b) \end{array} \right.$$

In fact, $\underline{\underline{NS}}$ ←

In $\underline{\underline{fact}}$, $\in \Omega_{NS}$

Test-preserving

Mappings / morphisms of models.

$A \xrightarrow{\varphi} B$:

$\varphi : \mathbb{X}(A) \longrightarrow \mathbb{X}(B)$

such that:

* $x \perp y \Rightarrow \varphi(x) \perp \varphi(y)$ "locally injective"
* $\varphi(E) \in m(B) \quad \forall E \in m(A)$
* $\forall \beta \in \Omega(B), \quad \varphi^*(\beta) = \beta \circ \varphi \in \Omega(A)$

odels.

"locally injective"

$n(A)$

$\in \Omega(A)$

In our examples: we have

$$\pi : A \times_{NS} B \longrightarrow AB$$

was a morphism.

Def: A NS composite of A, B is a pair (AB, π)

es: we have

$\longrightarrow AB$

NS composite of

is a pair

(AB, π)

N:

$A \times_{NS} B$ is a composite.

$\overleftrightarrow{AB}$ " "

Given A, have

states $\longrightarrow$ $\mathbb{V}(A)$ = span of $\Omega(A)$ in $\mathbb{R}^{X}$

Effects $\longrightarrow$ $\mathbb{V}^*(A) := \mathbb{V}(A)^*$

$\hookrightarrow$ $a : \mathbb{V} \to \mathbb{R}$ linear

$0 \leq a \leq u$ positive, $a(\alpha) \leq 1 \ \forall \alpha \in \Omega$

unit effect u: $u(\alpha) \equiv 1$

$\varphi : A \to B$ "(test preserving) morph:"

$$\varphi^* : \Omega(B) \to \Omega(A)$$

$$\varphi^{**} : [0, u_A] \to [0, u_B]$$

$$\begin{tikzcd} \Omega(B) \arrow[r, "\varphi^*"] \arrow[dr, dashed] & \Omega(A) \arrow[d, "a"] \\ & \mathbb{R} \end{tikzcd}$$

indeed,

$$\varphi^* : \mathbb{V}(B) \to \mathbb{V}(A)$$

$$\varphi = \varphi^{**} : \mathbb{V}^*(A) \to \mathbb{V}^*(B) : \quad \varphi(u_A) = u_B$$

$A \underset{NS}{\times} B \xrightarrow{\pi} AB$ models.

$\Downarrow$

$\mathbb{V}^*(A \underset{NS}{\times} B) \xrightarrow{\pi} \mathbb{V}^*(AB)$

$\mathbb{V}(AB) \xrightarrow{\pi^*} \mathbb{V}(A \underset{NS}{\times} B)$

Namioka, Phelps '68

upshot: $\mathbb{V}, \mathbb{W}$ two ordered vector spaces (f. d.)

Two cones:

$$(\mathbb{V} \otimes_{min} \mathbb{W})_+ = \text{cone spanned by } \alpha \otimes \beta \quad \alpha \in \mathbb{V}$$

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wo ordered vector spaces (f. d.)

cone spanned by $\alpha \otimes \beta$ $\alpha \in \mathbb{V}_+$, $\beta \in \mathbb{W}_+$

$= \left\{ \sum_i t_i \alpha_i \otimes \beta_i \mid \alpha_i, \beta_i \geq 0, \ t_i \geq 0 \text{ in } \mathbb{R} \right\}$

elps '68

two ordered vector spaces (f. d.)

$$V)_+ = \text{cone spanned by } \alpha \otimes \beta \quad \alpha \in \mathbb{V}_+ ,\ \beta \in \mathbb{W}_+$$

$$= \left\{ \sum_i t_i \alpha_i \otimes \beta_i \;\middle|\; \alpha_i, \beta_i \geq 0,\ t_i \geq 0 \text{ in } \mathbb{R} \right\}$$

$$)_+ = \left\{ \omega \in \mathbb{V} \otimes \mathbb{W} \;\middle|\; \forall a \in \mathbb{V}_+^*,\ b \in \mathbb{W}_+^* \quad (a \otimes b)(\omega) \geq 0 \right\}$$

$$\xrightarrow{\varphi^*} \mathcal{S}(A) \xrightarrow{a} \mathbb{R}$$

Thm (

Thm (not hard):

$$(\mathbb{V} \otimes_{\min} \mathbb{W})^* = \mathbb{V}^* \otimes_{\max} \mathbb{W}^*$$

$$(\mathbb{V} \otimes_{\max} \mathbb{W})^* = \mathbb{V}^* \otimes_{\min} \mathbb{W}^*$$

Thm (K, F, R ; AW):

$$\mathbb{V}(A \times_{NS} B) \cong \mathbb{V}(A) \otimes_{max} \mathbb{V}(B)$$

$$\| $$

$$\mathbb{V}(\overleftrightarrow{AB})$$

Cor: $\mathbb{V}^*(A \times_{NS} B) \cong \mathbb{V}^*(A) \otimes_{min} \mathbb{V}^*(B)$

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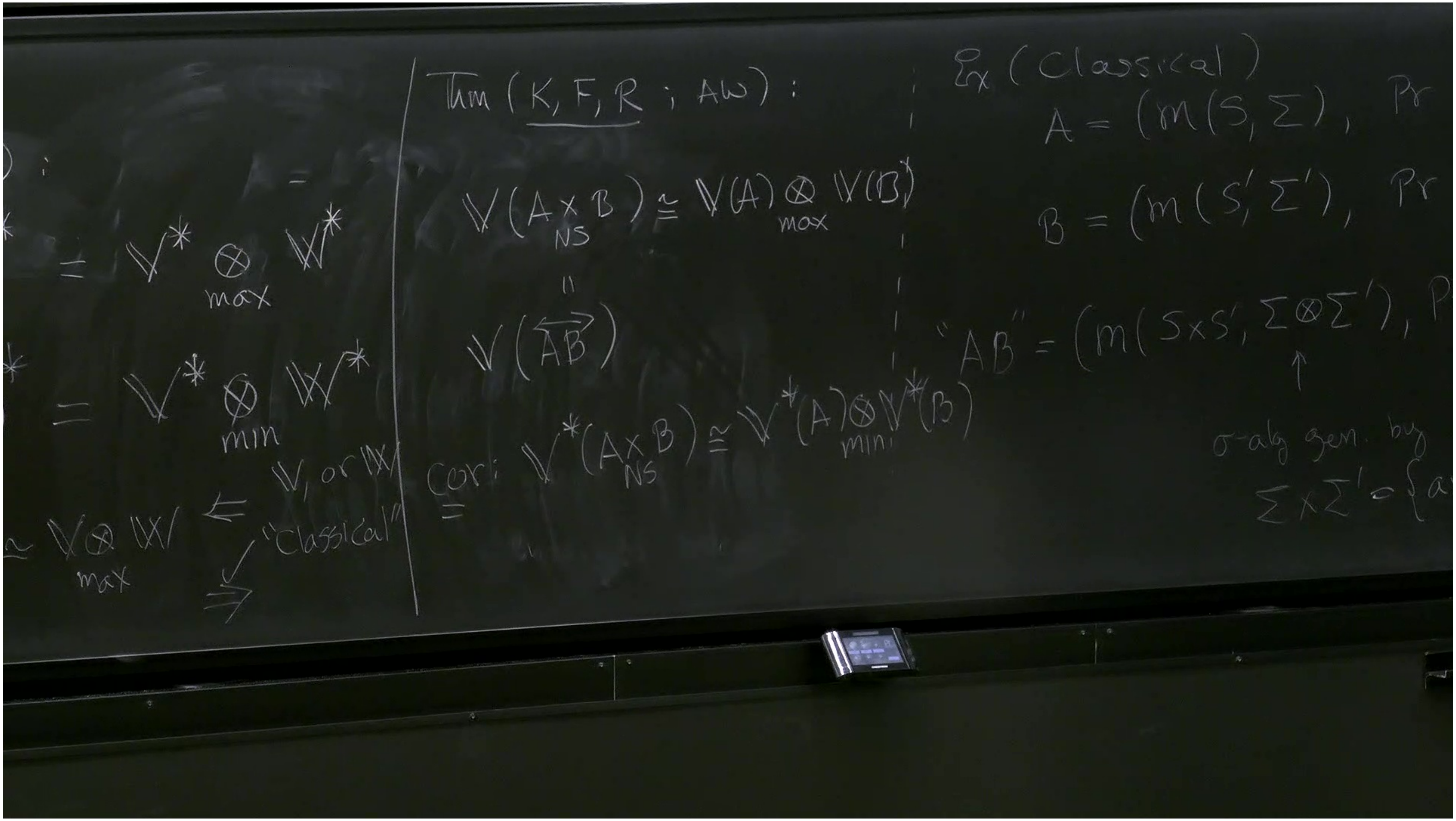
Thm (K, F, R ; AW) :
$\mathbb{V}(A \times_{NS} B) \cong \mathbb{V}(A) \otimes_{max} \mathbb{V}(B)$
$\mathbb{V}(\overleftrightarrow{AB})$
Cor: $\mathbb{V}^*(A \times_{NS} B) \cong \mathbb{V}^*(A) \otimes_{min} \mathbb{V}^*(B)$
$\cong \mathbb{V}^* \otimes_{max} \mathbb{W}^*$
$= \mathbb{V}^* \otimes_{min} \mathbb{W}^*$
$\mathbb{V}$, or $\mathbb{W}$
$\cong \mathbb{V} \otimes_{max} \mathbb{W} \Leftarrow$ "Classical"
Ex (Classical)
$A = (m(S, \Sigma),$ Pr
$B = (m(S', \Sigma'),$ Pr
"AB" $= (m(S \times S', \Sigma \otimes \Sigma'),$ P
σ-alg gen. by
$\Sigma \times \Sigma' = \{a$

strong composite:

ing) morph: $(\mathbb{B})$

$$\forall \alpha \in \mathcal{L}(A),\ \beta \in \mathcal{L}(\mathbb{B})$$

$$\exists \text{ canonical } ``\alpha \otimes \beta" \in \mathcal{L}(A\mathbb{B})$$