Title: Quantum Field Theory for Cosmology - Lecture 20240404

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (PHYS785/AMATH872)

Date: April 04, 2024 - 4:00 PM

URL: https://pirsa.org/24040003

QFT for Cosmology, Achim Kempf, Lecture 24

The Hawking effect

In 1915, Schwarzschild found this black hole solution to the Einstein equation:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + d\varphi^2 \sin^2\theta\right)$$

Mass of black hole

Singularity: $r = 0$

Horizon: $r = 2M$

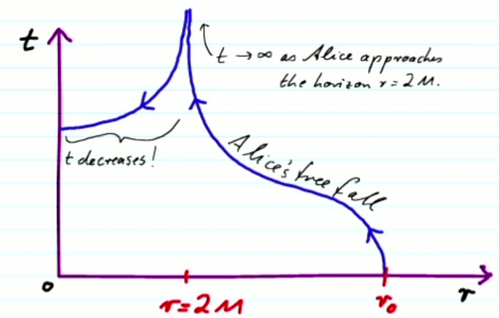
Here, $x = (t, r, \varphi, \theta)$ are called the Schwarzschild coordinates.

Schwarzschild coordinates long misled intuition:

- The singularity at $r = 2M$ is not real: it disappears in other coordinate systems. The curvature is smooth across $r = 2M$.

- Due to the sign changes across $r = 2M$, for $r < 2M$ dt is spacelike and dr is timelike for $r < 2M$.

- Consider, for example, a traveler, Alice, who is freely falling from $r = r_0$ to $r = 0$:



$$r(\alpha) = \frac{r_0}{2}\left(1 + \cos(\alpha)\right)$$

$$t(\alpha) = \left(\frac{r_0}{2} + 2M\right)w\alpha + \frac{r_0}{2}w\sin(\alpha) + 2M\log\left|\frac{w + \tan(\alpha/2)}{w - \tan(\alpha/2)}\right|$$

$$\tau(\alpha) = \frac{r_0}{2}\left(\frac{r_0}{2M}\right)^{1/2}\left(\alpha + \sin(\alpha)\right)$$

Here: $0 < \alpha < \pi$ and $w := \left(\frac{r_0}{2M} - 1\right)^{1/2}$

- For quantization, need better choices of coordinate systems!

Simplification: For now, we drop the φ and θ coordinates.

First design of a new cds (T, R) – Alice's choice (for $r_0 = 2M$):

- Require $g_{\mu\nu}(T, R)$ to be regular across $r = 2M$.

- Require $g_{\mu\nu}(0, 0) = \eta_{\mu\nu}$ at $r = 2M$. If there's really no singularity at $r = 2M$ this must be possible.

- Extend this cds so that ... because

$\Rightarrow$ **Alice's choice are the Kruskal-Szekeres coordinates (T, R):**

$$T(t, r) := 4M\left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r - 2M}{4M}}\left(\sinh\left(\frac{t}{4M}\right)\theta(r - 2M) + \cosh\left(\frac{t}{4M}\right)\theta(2M - r)\right)$$

$$R(t, r) := 4M\left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r - 2M}{4M}}\left(\cosh\left(\frac{t}{4M}\right)\theta(r - 2M) + \sinh\left(\frac{t}{4M}\right)\theta(2M - r)\right)$$

This map is, in principle, invertible, to obtain $t(T, R)$, $r(T, R)$.

File Edit View Insert Actions Tools Help

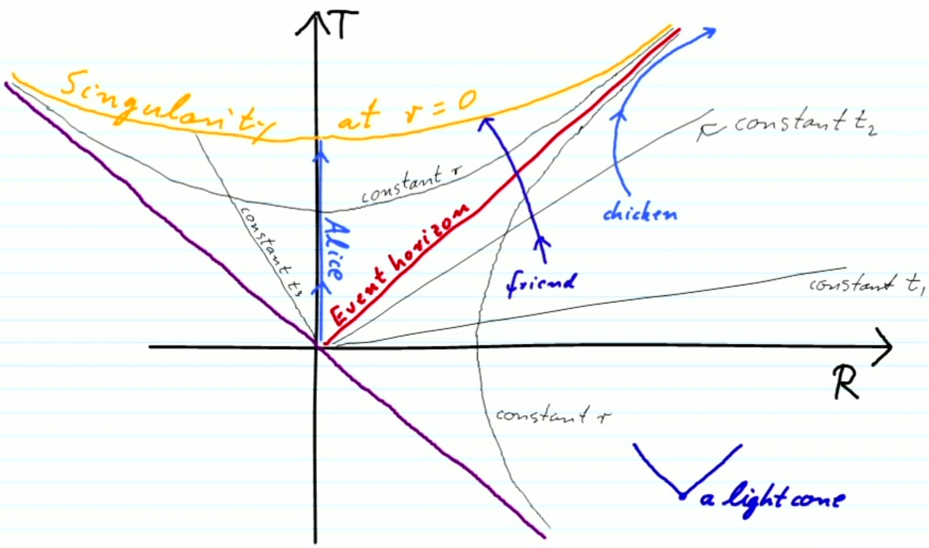
- Require $g_{\mu\nu}(0,0) = \eta_{\mu\nu}$ at $r = 2M$. If there's singularity at $r=2M$ this must be possible.

- Extend this cds so that $g_{\mu\nu}(T,R) = f(T,R)\,\eta_{\mu\nu}$ because then we know:
  - the action
  - the Klein Gordon equation
  - the solution space of the K.G. equation.
  - the mode fctn of the vacuum in this cds.

This map is, in principle, invertible, to obtain $t(T,R)$, $r(T,R)$.

The Schwarzschild metric now takes this form:

$$ds^2 = \underbrace{\frac{2M}{r(T,R)}\, e^{1-\frac{r(T,R)}{2M}}}_{\text{Conformal prefactor} = 1 \text{ as } r = 2M} \underbrace{\left(dT^2 - dR^2\right)}_{=\eta_{\mu\nu}} \quad \text{Obeys all conditions!}$$



- Alice was at rest at the event horizon.
- The singularity is at $T(R) = \left(R^2 + \frac{16M^2}{e}\right)^{1/2}$ and is spacelike.

Alice's light cone coordinates:

$$u := T - R, \quad v := T + R$$

Metric: $$ds^2 = \underbrace{\frac{2M}{r(u,v)}\, e^{1-\frac{r(u,v)}{2M}}}_{\substack{\text{conformal factor} \\ \text{(which is 1 at horizon)}}} \underbrace{du\,dv}_{\substack{\text{light cone} \\ \text{Minkowski}}}$$

$\Rightarrow$ The action $S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\sqrt{g}\,d^2x$ becomes:

$$= \frac{1}{2}\int_{T>-R} \left(\partial_T \phi(T,R)\right)^2 - \left(\partial_R \phi(T,R)\right)^2 dT\,dR$$

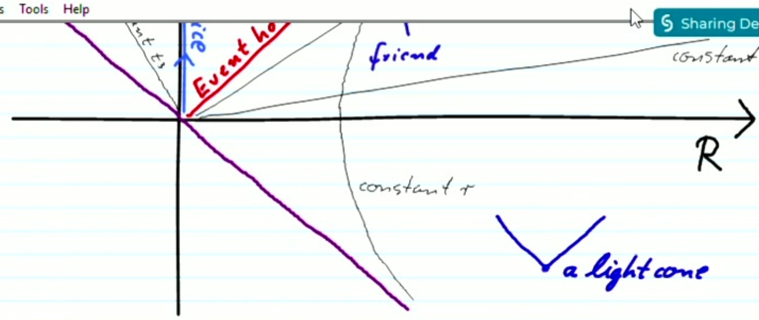
$$= 2\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\partial_u \phi(u,v)\right)\left(\partial_v \phi(u,v)\right) dv\,du$$

$\leftarrow$ b/c region $T>-R$ means $T+R>0$, i.e. $v>0$.

$\Rightarrow$ Eqn of motion: $\partial_u \partial_v \phi(u,v) = 0$

3 / 25

File Edit View Insert Actions Tools Help



- Alice was at rest at the event horizon.
- The singularity is at $T(R) = \left(R^2 + \frac{16M^2}{e}\right)^{1/2}$ and is spacelike.

conformal factor (which is 1 at horizon) Minkowski

⇒ The action $S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\sqrt{g}\,d^2x$ becomes:

$$= \frac{1}{2}\int_{T>-R} \left(\partial_T\phi(T,R)\right)^2 - \left(\partial_R\phi(T,R)\right)^2 \, dT\,dR$$

$$= 2\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\partial_u\phi(u,v)\right)\left(\partial_v\phi(u,v)\right) dv\,du$$

← b/c region $T>-R$ means $T+R>0$, i.e. $v>0$.

⇒ Eqn of motion: $\partial_u\partial_v\phi(u,v) = 0$

⇒ Solution for the QFT found as before:

$$\hat{\phi}(u,v) = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}}\frac{1}{(2\omega)^{1/2}}\left(e^{-i\omega u}\hat{a}_\omega + e^{i\omega u}\hat{a}^+_\omega + \text{left movers}\right)$$

obeys the 3 conditions: EoM, CCRs and hermiticity.

Alice's notion of vacuum

- For Alice, as she crosses the horizon, $g_{\mu\nu} = \eta_{\mu\nu}$.
- If her detectors are not clicking, the state of the field is $|0_{Alice}\rangle$, obeying $a_\omega|0_{Alice}\rangle = 0 \;\forall\omega$.

One problem though: In this cds, far away, i.e., as $r\to\infty$, the metric doesn't become the Minkowski $\eta_{\mu\nu}$.

Bob's choice of coordinate system

Bob is far from the black hole.

He wants a cds in which:

- $g_{\mu\nu}(x) \to \eta_{\mu\nu}$ as $r\to\infty$.
- $g_{\mu\nu}(x) = f(x)\,\eta_{\mu\nu}$ everywhere.

This is so that in his cds too

- photons travel at 45°
- we know action, K.G. equation and mode functions.

⇝ Bob's choice is the Tortoise coordinate system.

5/25

5

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

Alice's notion of vacuum

- For Alice, as she crosses the horizon, $g_{\mu\nu} = \eta_{\mu\nu}$.
- If her detectors are not clicking, the state of the field is $|0_{Alice}\rangle$, obeying $a_\omega |0_{Alice}\rangle = 0 \;\forall \omega$.

One problem though: In this cds, far away, i.e., as $r \to \infty$, the metric doesn't become the Minkowski $\eta_{\mu\nu}$.

- $g_{\mu\nu}(x) = f(x)\,\eta_{\mu\nu}$ everywhere.

This is so that in his cds too
- photons travel at $45°$
- we know action, K.G. equation and mode functions.

$\rightsquigarrow$ Bob's choice is the Tortoise coordinate system.

Tortoise cds (t^*, r^*):

- In terms of the Schwarzschild cds:

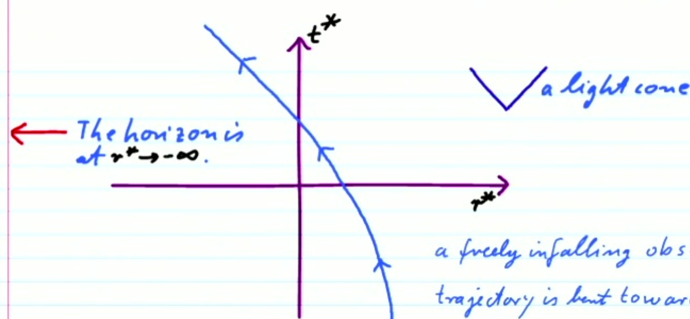
$$t^* := t$$

$$r^* := r - 2M + 2M \log\Big(\underbrace{\tfrac{r}{2M} - 1}_{\text{must require } r > 2M\,!}\Big)$$

$\Rightarrow$ Important: This is in principle invertible, to obtain $r = r(r^*)$ but only for $r > 2M$!

$\Rightarrow$ The tortoise cds only cover the BH's outside!

Metric: $$ds^2 = \underbrace{\left(1 - \frac{2M}{r(r^*)}\right)}_{\text{Conformal factor} \to 1 \text{ as } r\to\infty \text{, as planned but } \to 0 \text{ at horizon.}} (dt^{*2} - dr^{*2})$$



a freely infalling observer's trajectory is bent towards $45°$ as he speeds up.

Bob's light cone coordinates: $\bar{u} := t^* - r^*$, $\bar{v} := t^* + r^*$

The metric is then: $$ds^2 = \underbrace{\left(1 - \frac{2M}{r(\bar{u},\bar{v})}\right)}_{\to 1 \text{ as } r\to\infty \text{ and } \to 0 \text{ as } r \to 2M} d\bar{u}\, d\bar{v}$$

$$\boxed{\text{Important later: } u = -4M e^{-\frac{\bar{u}}{4M}},\; v = 4M e^{\frac{\bar{v}}{4M}}}$$

$$r^* := r - 2M + 2M\log\left(\frac{r}{2M}-1\right)$$

⇒ Important: This is in principle invertible, to obtain

$$r = r(r^*)$$

but only for $r > 2M$!

⇒ The tortoise cds only cover the BH's outside!

Metric: $$ds^2 = \left(1-\frac{2M}{r(r^*)}\right)\left(dt^{*2} - dr^{*2}\right)$$

Conformal factor $\to 1$ as $r\to\infty$, as planned but $\to 0$ at horizon.

trajectory is bent towards 45° as he speeds up.

Bob's light cone coordinates: $\bar{u} := t^* - r^*$, $\bar{v} := t^* + r^*$

The metric is then: $$ds^2 = \underbrace{\left(1-\frac{2M}{r(\bar u,\bar v)}\right)}_{\to 1 \text{ as } r\to\infty \text{ and } \to 0 \text{ as } r\to 2M} d\bar u\, d\bar v$$

$$\boxed{\text{Important later: } u = -4Me^{-\frac{\bar u}{4M}},\quad v = 4Me^{\frac{\bar v}{4M}}}$$

⇒ The action:

$$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\sqrt{g}\,d^2x \quad \text{becomes:}$$

$$= \frac{1}{2}\int_{\mathbb{R}^2}\left(\partial_{t^*}\phi(t^*,r^*)\right)^2 - \left(\partial_{r^*}\phi(t^*,r^*)\right)^2 dt^*\,dr^*$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\partial_{\bar u}\phi(\bar u,\bar v)\right)\left(\partial_{\bar v}\phi(\bar u,\bar v)\right) d\bar v\, d\bar u$$

⇒ Eqn of motion: $\partial_{\bar u}\partial_{\bar v}\phi(\bar u,\bar v) = 0$

⇒ Solution for the QFT found as before:

$$\hat\phi(\bar u,\bar v) = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}}\frac{1}{(2\omega)^{1/2}}\left(e^{-i\omega\bar u}\,\hat b_\omega + e^{i\omega\bar u}\,\hat b^\dagger_\omega + \text{left movers}\right)$$

obeys the 3 conditions: EoM, CCRs and hermiticity.

Bob's notion of vacuum

- For Bob, out at $r\to\infty$, the metric is $g_{\mu\nu} = \eta_{\mu\nu}$.
- If Bob's detectors are not clicking, the state of the field is $|0_{Bob}\rangle$, obeying $\hat b_\omega|0_{Bob}\rangle = 0\ \forall\omega$.

9 / 25

9

File Edit View Insert Actions Tools Help

$$= \frac{?}{2}\int_{\mathbb{R}^2} (\partial_{\bar u}\phi(\cdots)) \ \ (\partial_{\bar v}\phi(\cdots))$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\partial_{\bar u}\phi(\bar u,\bar v))(\partial_{\bar v}\phi(\bar u,\bar v))\, d\bar v\, d\bar u$$

$\Rightarrow$ Eqn of motion: $\partial_{\bar u}\partial_{\bar v}\phi(\bar u,\bar v) = 0$

obeys the 3 conditions: EoM, CCRs and hermiticity.

Bob's notion of vacuum

- For Bob, out at $r\to\infty$, the metric is $g_{\mu\nu} = \eta_{\mu\nu}$.
- If Bob's detectors are not clicking, the state of the field is $|0_{Bob}\rangle$, obeying $b_\omega |0_{Bob}\rangle = 0\ \forall\omega$.

Real black holes:

- they have complicating properties, such as
  - a ring down
  - peculiar velocity
  - angular momentum
  - charges
  - and even mass changes (through absorption or emission).

Simple model:

- Let us neglect all these.
- Also assume that the rest of the universe is empty.

$\Rightarrow$ In good approximation the spacetime should be described by the Schwarzschild metric.

Which is then the state $|\psi\rangle$ of the quantum field?

Q: Is $|\psi\rangle = |0_{Alice}\rangle$ or perhaps $|\psi\rangle = |0_{Bob}\rangle$?

A: Probably both: $|\psi\rangle = |0_{Alice,right}\rangle \otimes |0_{Bob,left}\rangle$

Here: $a_{\omega,right}|0_{Alice,right}\rangle = 0\ \forall\omega$

$b_{\omega,left}|0_{Bob,left}\rangle = 0\ \forall\omega$

Why?

We cannot reliably calculate through the collapse process, because it involves tracking waves being infinitely blue-shifted at the horizon ($\to$ see the Transplanckian problem for BHs).

11 11 / 25

File Edit View Insert Actions Tools Help

Sharing Desktop ■ Stop Sharing

- angular momentum
- charges
- and even mass changes (through absorption or emission).

Simple model:

- Let us neglect all these.
- Also assume that the rest of the universe is empty.

⇒ In good approximation the spacetime should be described by the Schwarzschild metric.

Heuristic arguments yield:

- If, as we assume, the rest of the universe has no stars etc, then there should be no flux of quanta into the black hole.

  ⇒ The left-moving (i.e. ingoing) modes should be in the state

  $$|0_{Bob, left}\rangle$$

- Can the right moving (i.e. outgoing) modes be in the state $|0_{Bob, right}\rangle$?

  No!

A: Probably both: $|\psi\rangle = |0_{Alice, right}\rangle \otimes |0_{Bob, left}\rangle$

Here: $a_{\omega, right}|0_{Alice, right}\rangle = 0 \ \forall \omega$

$b_{\omega, left}|0_{Bob, left}\rangle = 0 \ \forall \omega$

Why?

We cannot reliably calculate through the collapse process, because it involves tracking waves being infinitely blue-shifted at the horizon (→ see the Transplanckian problem for BHs).

Recall:

$$\boxed{\underset{\uparrow \text{Alice's}}{u} = -4M e^{-\frac{\overset{\text{Bob's}}{\bar{u}}}{4M}}, \quad v = 4M e^{\frac{\bar{v}}{4M}}}$$

Compare with (from the previous lecture):

$$\boxed{\underset{\uparrow \text{inertial}}{u} = -\frac{1}{a} e^{-a \underset{\uparrow \text{accelerated}}{\bar{u}}}}$$

⇒ Alice's and Bob's cds relate in the same way as the inertial and accelerated before, with $a = \frac{1}{4M}$

⇒ $|0_{Bob, right}\rangle$ has divergent vacuum energy towards the horizon!

File Edit View Insert Actions Tools Help

$\Rightarrow$ The left-moving (i.e. ingoing) modes should be in the state

$$|0_{Bob,left}\rangle$$

- Can the right moving (i.e. outgoing) modes be in the state $|0_{Bob,right}\rangle$?

No!

$\Rightarrow$ If the QFT is in the state $|0_{Bob,right}\rangle$, then:

- Via the Einstein equation, this would contradict our assumption of spacetime being Schwarzschild, which solves:

$$G_{\mu\nu}(g_{Schwarzschild}) = T_{\mu\nu} \quad \text{with } T_{\mu\nu} = 0.$$

- During the collapse, the quantum state will be energetically prevented to evolve into the state

$$|0_{Bob,right}\rangle$$

(in the Schrödinger picture).

$$\boxed{\underset{\uparrow\,\text{inertial}}{u} = -\frac{1}{a}\, e^{-a\underset{\uparrow\,\text{accelerated}}{\tilde{u}}}}$$

$\Rightarrow$ Alice's and Bob's cds relate in the same way as the inertial and accelerated before, with $a = \frac{1}{4M}$

$\Rightarrow$ $|0_{Bob,right}\rangle$ has divergent vacuum energy towards the horizon!

- Alice would see a diverging amount of quantum field fluctuations and particles as she crosses the horizon.
  - $\Rightarrow$ She would be able to tell the location of the horizon by local measurements in a free-falling lab.
  - $\Rightarrow$ This would contradict the equivalence principle.

Q: What state do the right-moving (outgoing) modes have to be in, so that

- Their contribution to $T_{\mu\nu}$ is smooth across the horizon.
- Alice does not see a burst of particles from the horizon.

A: $|0_{Alice,right}\rangle$ has these properties, (via previous lecture's results).

15 / 25

15

File Edit View Insert Actions Tools Help

$G_{\mu\nu}(g_{\text{Schwarzschild}}) = T_{\mu\nu}$ with $T_{\mu\nu} = 0$.

- During the collapse, the quantum state will be energetically prevented to evolve into the state

$$|0_{Bob,right}\rangle$$

(in the Schrödinger picture).

$\Rightarrow$ Plausible is that the state of the QFT is:

$$\boxed{|\psi\rangle = |0_{Alice,right}\rangle \otimes |0_{Bob,left}\rangle}$$

Q: What, therefore, should we see at rest from far?

A: Our natural cds is Bob's then.

$\Rightarrow$ We see no ingoing (left-moving) radiation.

But we can repeat the calculations of the previous lecture for the outgoing modes, using $a = 1/4M$

$\Rightarrow$ We see an outflux of quanta of temperature:

$$\boxed{T_H = \frac{1}{8\pi M}}$$

Recall: $T_u = \frac{a}{2\pi}$

a free falling ...

$\Rightarrow$ This would contradict the equivalence principle.

Q: What state do the right-moving (outgoing) modes have to be in, so that

- Their contribution to $T_{\mu\nu}$ is smooth across the horizon.
- Alice does not see a burst of particles from the horizon.

A: $|0_{Alice,right}\rangle$ has these properties, (via previous lecture's results).

Summary of Unruh-Hawking connection:

Minkowski space	Schwarzschild spacetime
Accelerated observer's vacuum: "Rindler vacuum"	Bob's vacuum: "Boulware vacuum"
Inertial observer's vacuum: "Minkowski vacuum"	Alice's vacuum: "Kruskal vacuum"

Remarks:

- The state $|0_{Bob,right}\rangle$ (outgoing) was disqualified due to its contribution to $T_{\mu\nu}$ which would diverge towards the horizon.

  Is $|0_{Bob,left}\rangle$ having the same problem?
  No, it would have that problem at the past horizon but a real black hole doesn't have one (unlike an accelerated observer.)

A : Our natural cds is Bob's then.

⇒ We see no ingoing (left-moving) radiation.

But we can repeat the calculations of the previous lecture for the outgoing modes, using $a = 1/4M$

⇒ We see an outflux of quanta of temperature:

$$\boxed{T_H = \frac{1}{8\pi M}}$$

Recall: $T_U = \frac{a}{2\pi}$

Inertial observer's vacuum: "Minkowski vacuum" Alice's vacuum: "Kruskal vacuum"

Remarks: ▫ The state $|0_{Bob,right}\rangle$ (outgoing) was disqualified due to its contribution to $T_{\mu\nu}$ which would diverge towards the horizon.

Is $|0_{Bob,left}\rangle$ having the same problem?
No, it would have that problem at the past horizon but a real black hole doesn't have one (unlike an accelerated observer.)

▫ We dropped the angles φ, θ. Do they matter?

Yes, it leads to a weakening of Hawking radiation:

The mode decomposition now involves the analog of Fourier for angles: spherical harmonics.

⇝ The Klein Gordon equation becomes:

$$\left(\Box + \underbrace{\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{\ell(\ell+1)}{r^2}\right)}_{=: V_\ell(r)}\right)\phi_{\ell m}(t,r) = 0$$

⇒ This effective potential needs to be overcome by Hawking radiation ⇒ Grey body factor.

Problem:

After a black hole Hawking evaporates, it leaves behind only heat radiation.

=> a pure initial state of a collapsing star can evolve into a mixed state?

Violation of unitarity? Can nature erase information?

Not necessarily:

Consider a laser pulse heating a piece of charcoal from T=0 to finite T.

Then charcoal cools back to T=0 by radiating away the heat.

Evolution is unitary => emitted radiation is in pure state.

=> early emitted radiation was entangled with charcoal, so high entanglement entropy

but late radiation is entangled with itself so low entanglement entropy

Same possible for Black Holes: radiation entropy first rises then falls to zero.

Holography studies found evidence of this "page curve" of the radiation's entropy.

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

$$\left(\Box + \underbrace{\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{\ell(\ell+1)}{r^2}\right)}_{=\,V_\ell(r)}\right)\phi_{\ell,m}(t,r)=0$$

⇒ This effective potential needs to be overcome by Hawking radiation ⇒ Grey body factor.

Then charcoal cools back to T=0 by radiating away the heat.

Evolution is unitary => emitted radiation is in pure state.

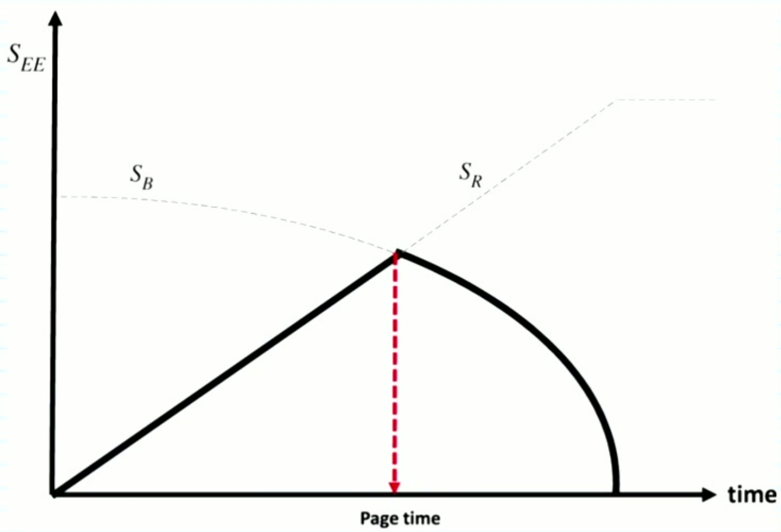
=> early emitted radiation was entangled with charcoal, so high entanglement entropy

but late radiation is entangled with itself so low entanglement entropy

Same possible for Black Holes: radiation entropy first rises then falls to zero.

Holography studies found evidence of this "page curve" of the radiation's entropy.

The page curve of the entanglement entropy of the Hawking radiation over time



Real black holes:

Observations by

Event Horizon Telescope Array

Black Hole M87*

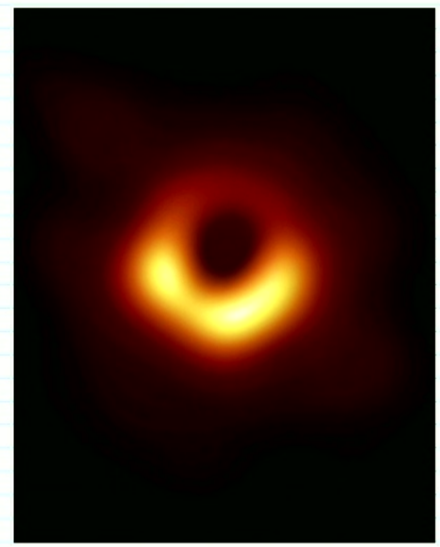
55 million lightyears away

We see the photons escaping from the photon sphere (r=3m), observed in mm band.

Why not Hawking radiation?

6.5 billion solar masses

=> Peak Hawking radiation has tens of billions of km wavelength



21 / 25

21