Title: Quantum Field Theory for Cosmology - Lecture 20240402

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (PHYS785/AMATH872)

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QFT for Cosmology, Achim Kempf, Lecture 23

Plan: Unruh effect $\longrightarrow$ Hawking effect

Unruh effect in 1+1 dimensions

The metric: In inertial, cartesian cds x^μ: $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix}$

Consider an observer's trajectory $x^\mu(\tau)$
and use the observer's proper time τ as the parameter.

□ **Velocity** $\dot{x}^\mu(\tau) := \frac{dx^\mu(\tau)}{d\tau}$

Proposition: $\eta_{\mu\nu}\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) = 1 \quad \forall\tau$

Proof: At any point in time, τ, in rest frame: $x^\mu(\tau) = (\tau, 0)$

$\Rightarrow \dot{x}^\mu(\tau) = (1, 0)$

$\Rightarrow \eta_{\mu\nu}\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) = 1$ which is a scalar

$\Rightarrow \eta_{\mu\nu}\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) = 1$ in all cd systems for all τ.

□ **Acceleration:** $\ddot{x}^\mu(\tau) := \frac{d\dot{x}^\mu(\tau)}{d\tau}$

Proposition: $\ddot{x}_\mu(\tau)\dot{x}^\mu(\tau) = 0 \quad \forall\tau$

Proof: $0 = \frac{d}{d\tau}\left(\dot{x}_\mu(\tau)\dot{x}^\mu(\tau)\right) = 2\ddot{x}_\mu(\tau)\dot{x}^\mu(\tau)$

"proper acceleration"
$a(\tau) := \frac{d^2x^1(\tau)}{d\tau^2}$

$\Rightarrow$ In rest frame: $\dot{x}^\mu(\tau) = (1, 0)$ and $\ddot{x}^\mu(\tau) = (0, a(\tau))$

$\Rightarrow$ In every frame: $\ddot{x}_\mu(\tau)\ddot{x}^\mu(\tau) = -a^2(\tau)$

Special case of uniform acceleration: $a(\tau) = a \quad \forall\tau$

Proposition: A trajectory of uniform acceleration a is given by:

$$x^\mu(\tau) = (t(\tau), x(\tau)) = \left(\tfrac{1}{a}\sinh(a\tau), \tfrac{1}{a}\cosh(a\tau)\right)$$

This trajectory also obeys:

$$x_\mu(\tau)x^\mu(\tau) = x^0(\tau)^2 - x^1(\tau)^2 = -\frac{1}{a^2}$$

i.e., it is a hyperbola of deceleration followed

⇒ τ really is the proper time

And, crucially:

$$\ddot{X}_\mu(\tau) = (a\sinh(a\tau), a\cosh(a\tau)) \text{ obeying } \ddot{x}_\mu \ddot{x}^\mu = -a^2 \checkmark$$

Notice: Our uniformly accelerated traveler has horizons:

- can't influence events below the line $x^0 = -x^1$, i.e., with $x^0 + x^1 \le 0$
- can't be influenced by events above the line $x^0 = x^1$, i.e., with $x^0 - x^1 \ge 0$

} (A)

Inertial light cone coordinate system:

The inertial cartesian coordinates are fine to describe particle motion.
For wave equations, often light cone coordinates have advantages. (Esp. in 1+1D):

$$\tilde{x}^\mu(x^0,x^1) := (u(x^0,x^1), v(x^0,x^1))$$

$$\left.\begin{aligned} \text{with: } u(x^0,x^1) &:= x^0 - x^1 \\ v(x^0,x^1) &:= x^0 + x^1 \end{aligned}\right\} (B)$$

The metric: In inertial, cartesian cds x^μ: $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}$

$$\boxed{\text{In inertial light cone cds } \tilde{x}^\mu: \quad g_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0, & 1/2 \\ 1/2, & 0 \end{pmatrix} \quad \text{i.e.: } ds^2 = du\,dv}$$

Exercise: Check this, using $g_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x)$.

The trajectory above in inertial light cone cds:

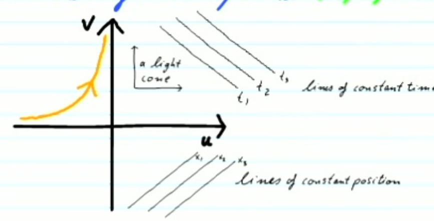
$$\tilde{x}(\tau) = (u(\tau), v(\tau))$$

$$\text{with } u(\tau) = t(\tau) - x(\tau) = -\frac{1}{a} e^{-a\tau}$$

$$v(\tau) = t(\tau) + x(\tau) = \frac{1}{a} e^{a\tau}$$

Notice: From (A) $\wedge$ (B): the traveller

- can't influence events (u,v) with $v \le 0$
- can't be influenced by events (u,v) with $u \ge 0$

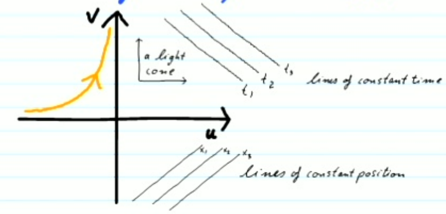


The metric: In inertial, cartesian cds x^μ: $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix}1,0\\0,-1\end{pmatrix}$

$$\boxed{\text{In inertial light cone cds } \tilde{x}^\mu: \quad g_{\mu\nu}(\tilde{x}) = \begin{pmatrix}0, 1/2\\ 1/2, 0\end{pmatrix} \quad \text{i.e.: } ds^2 = du\,dv}$$

Exercise: Check this, using $g_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x)$.

□ can't be influenced by events (u,v) with $u \geq 0$



A coordinate system that is comoving with the traveler

We want a coordinate system ξ^μ so that our traveler's trajectory is:

$$\xi^\mu(\tau) = (\tau, 0)$$

But this fixes the new cds only on the trajectory!

Q: How to continue the new cds to elsewhere?

A: We can require (in 1+1 dimensions) that the light cones are still at 45°, i.e., that

$$g_{\mu\nu}(\xi) = f(\xi)\begin{pmatrix}1,0\\0,-1\end{pmatrix},$$

i.e. $ds^2 = f(\xi)\left(d\xi^{0\,2} - d\xi^{1\,2}\right)$, i.e., $\underbrace{ds^2 = 0}_{\text{condition for light-like tangent.}} \Rightarrow d\xi^1 = \pm d\xi^0$

Proposition:

Under the change of coordinates

$$\left.\begin{aligned} x^0(\xi) &= a^{-1} e^{a\xi^1} \sinh(a\xi^0) \\ x^1(\xi) &= a^{-1} e^{a\xi^1} \cosh(a\xi^0) \end{aligned}\right\}(T)$$

we have that the trajectory $\xi^\mu(\tau) = (\tau, 0)$ is indeed the trajectory of our traveler:

$$x^\mu(\tau) = \left(a^{-1}\sinh(a\tau), a^{-1}\cosh(a\tau)\right)$$

And in addition: The Minkowski metric $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ reads in the ξ coordinates:

$$\boxed{g_{\alpha\beta}(\xi) = e^{2a\xi^1}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$\Rightarrow$ In this cds, light travels still at 45°.

A: We can require (in 1+1 dimensions) that the light cones are still at 45°, i.e., that

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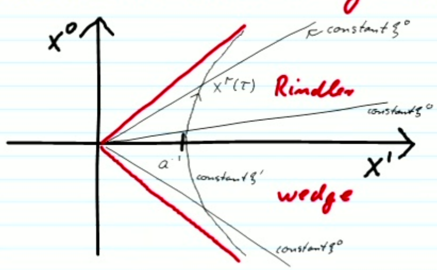
$\Rightarrow$ In this cds, light travels still at 45°.

In (T), why did we map the new cds to the old: $\xi^\mu \to x^\mu$?

There is no inverse $x^\mu \to \xi^\mu$!

Why? Because all of $(\xi^0, \xi^1) \in \mathbb{R}^2$ maps only on to the Rindler wedge $x^1 > |x^0|$



From (T):

For each ξ^1, obtain a hyperbola within the Rindler wedge.

Together they cover exactly only the Rindler wedge.

We knew that the traveler has horizons.

His comoving cds ξ^μ reaches only as far as to his horizons.

Accelerated light cone coordinates.

In ξ^μ cds, light still travels at 45°.

$\Rightarrow$ It will be useful for wave equations to introduce accelerated light cone coordinates:

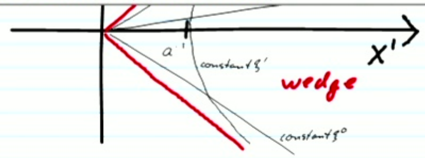
$$\tilde{\xi}^\mu(\xi) = \left(\tilde{\xi}^0(\xi), \tilde{\xi}^1(\xi)\right) = \left(\bar{u}(\xi), \bar{v}(\xi)\right)$$

where: $\bar{u}(\xi) = \xi^0 - \xi^1$

$\bar{v}(\xi) = \xi^0 + \xi^1$

In the cds $\tilde{\xi}^\mu$ we have:

$$\boxed{\begin{gathered} g_{\mu\nu}(\tilde{\xi}) = e^{a(\bar{v}-\bar{u})}\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix} \\ \text{i.e.: } ds^2 = e^{a(\bar{v}-\bar{u})}\, d\bar{u}\, d\bar{v} \end{gathered}}$$



hyperbola within the Rindler wedge.

Together they cover exactly only the Rindler wedge.

We knew that the traveler has horizons.
His comoving cds ξ^μ reaches only as far as to his horizons.

$$\bar{v}(\xi) = \xi^0 + \xi^1$$

In the cds $\tilde{\xi}^\mu$ we have:

$$\boxed{g_{\mu\nu}(\tilde{\xi}) = e^{a(\bar{v}-\bar{u})}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \quad \text{i.e.: } ds^2 = e^{a(\bar{v}-\bar{u})}\, d\bar{u}\, d\bar{v}}$$

Remark: We can also directly map the accelerated light cone cds $\tilde{\xi} = (\bar{u}, \bar{v})$ into the inertial light cone coordinates $\tilde{x} = (u, v)$: (Exercise: show this)

Important later! → $\boxed{u(\bar{u},\bar{v}) = -\frac{1}{a}\, e^{-a\bar{u}}}$ $\quad v(\bar{u},\bar{v}) = \frac{1}{a}\, e^{a\bar{v}}$

Summary:

	Coordinate system	Form of the metric
	$x = (x^0, x^1)$	$g_{\mu\nu}(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
	$\tilde{x} = (u, v)$	$g_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$
These cds cover only the Rindler wedge	$\xi = (\xi^0, \xi^1)$	$g_{\mu\nu}(\xi) = e^{2a\xi^1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
	$\tilde{\xi} = (\bar{u}, \bar{v})$	$g_{\mu\nu}(\tilde{\xi}) = e^{a(\bar{v}-\bar{u})}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$

Observation: These metrics are pairwise conformally related:

$$g_{\mu\nu}(x) \longrightarrow \bar{g}_{\mu\nu}(x) := \Omega^2(x)\, g_{\mu\nu}(x)$$

Proposition: In 2 dimensions, the K.G. action is invariant:

$$S_g[\phi] = \frac{1}{2}\int_{\mathbb{R}^2} g^{\alpha\beta}\, \phi_{,\alpha}\, \phi_{,\beta} \sqrt{|g|}\, d^2x$$

$$= S_{\bar{g}}[\phi]$$

Proof: We have $g^{\mu\nu}(x) \to \bar{g}^{\mu\nu}(x) = \Omega^{-2}(x)\, g^{\mu\nu}(x)$

and $\sqrt{|g|} \to \sqrt{|\bar{g}|} = \Omega^2(x)\sqrt{|g|}$ in 2 dimensions. ✓

$$\tilde{x} = (u, v) \qquad g_{\mu\nu}(\tilde x) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

These cds cover only the Rindler wedge

$$\begin{cases} \xi = (\xi^0, \xi^1) & g_{\mu\nu}(\xi) = e^{2a\xi^1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \tilde\xi = (\tilde u, \tilde v) & g_{\mu\nu}(\tilde\xi) = e^{a(\tilde v - \tilde u)}\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \end{cases}$$

Proof: We have $g^{\mu\nu}(x) \to \bar g^{\mu\nu}(x) = \Omega^{-2}(x)\, g^{\mu\nu}(x)$

and $\sqrt{|g|} \to \sqrt{|\bar g|} = \Omega^2(x)\sqrt{|g|}$ in 2 dimensions. ✓

⇒ The Klein Gordon action

$$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\,\sqrt{|g|}\,d^2x \qquad \text{general cds}$$

$$= \frac{1}{2}\int_{\mathbb{R}^2} (\partial_0\phi)^2 - (\partial_1\phi)^2\, dx^0 dx^1 \qquad \text{inertial cartesian cds}$$

$$= 2\int_{\mathbb{R}^2} (\partial_u\phi)(\partial_v\phi)\, du\, dv \qquad \text{inertial light cone cds}$$

On Rindler Wedge: (easy to see because of conformal invariance)

$$S_{RW}[\phi] = \frac{1}{2}\int_{\mathbb{R}^2} (\partial_{\xi^0}\phi)^2 - (\partial_{\xi^1}\phi)^2\, d\xi^0 d\xi^1 \qquad \text{accelerated cartesian cds}$$

$$= 2\int_{\mathbb{R}^2} (\partial_{\tilde u}\phi)(\partial_{\tilde v}\phi)\, d\tilde u\, d\tilde v \qquad \text{accelerated light cone cds}$$

because massive action is not conformal

Remark: A massive field would have a different equation motion in accelerated frames.

i.e.: accelerated observer can find out he's accelerating using masses.

The Klein Gordon equation:

In inertial light cone coordinates:

$$\frac{\delta S}{\delta\phi} = \partial_u \frac{\delta S}{\delta \partial_u\phi} + \partial_v \frac{\delta S}{\delta\partial_v\phi} \qquad \text{i.e.}\quad \partial_u\partial_v\phi(u,v) = 0$$

Easily solved: $\phi(u,v) = A(u) + B(v)$, with A, B arbitrary functions.

For example: $\phi(u,v) = e^{-i\omega u} = e^{-i\omega(t-x)} = e^{-i\omega(x^0 - x^1)}$

is a right-moving positive frequency solution.

The usual Minkowski space quantum field solution $\hat\phi(x^0, x^1)$ can be written this way:

$$\hat\phi(u,v) = \int_0^\infty \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\Big(\underbrace{e^{-i\omega u}\hat a_k + e^{i\omega u}\hat a_k^+}_{\text{right movers}} + \underbrace{e^{-i\omega v}\hat a_{-k} + e^{i\omega v}\hat a_{-k}^+}_{\text{left movers}}\Big) \quad \text{and } \omega = |k|$$

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$$S^{\prime}_{Rw}[\phi] = \frac{1}{2}\int_{\mathbb{R}^2} (\partial_{\xi^0}\phi)^2 - (\partial_{\xi^1}\phi)^2 \, d\xi^0 d\xi^1 \qquad \text{accelerated cartesian cds}$$

$$= 2\int_{\mathbb{R}^2} (\partial_{\bar u}\phi)(\partial_{\bar v}\phi)\, d\bar u\, d\bar v \qquad \text{accelerated light cone cds}$$

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The Klein Gordon equation in the accelerated frame:

In accelerated light cone coordinates: (covering only the Rindler wedge)

$$\frac{\delta S^{\prime}_{Rw}}{\delta\phi} = \partial_{\bar u}\frac{\delta S^{\prime}_{Rw}}{\delta\partial_{\bar u}\phi} + \partial_{\bar v}\frac{\delta S^{\prime}_{Rw}}{\delta\partial_{\bar v}\phi} \qquad \text{i.e.} \quad \partial_{\bar u}\partial_{\bar v}\phi(\bar u,\bar v) = 0$$

Easily solved: $\phi(\bar u,\bar v) = A(\bar u) + B(\bar v)$, with A, B arbitrary functions.

For example: $\phi(\bar u,\bar v) = e^{-i\omega\bar u} = e^{-i\omega(\xi^0-\xi^1)}$

is a right-moving positive frequency solution.

In the accelerated frame, the quantum field in the Rindler wedge is:

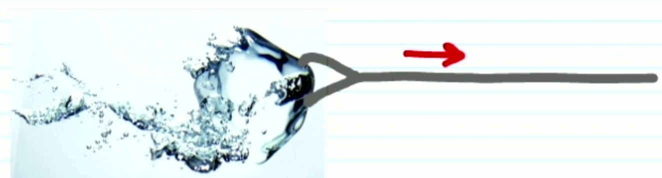
$$\hat\phi(\bar u,\bar v) = \int_0^\infty \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\Big(\underbrace{e^{-i\omega\bar u}\hat b_k + e^{i\omega\bar u}\hat b_k^\dagger}_{\text{right movers}} + \underbrace{e^{-i\omega\bar v}\hat b_{-k} + e^{i\omega\bar v}\hat b_{-k}^\dagger}_{\text{left movers}}\Big) \quad \text{and } \omega = |k|$$

Notice: hermiticity conditions, K.G. eqn and CCRs obeyed.

For the inertial observer, the vacuum state obeys: $a_k|0_M\rangle = 0$

But for the accelerated observer, the vacuum state obeys: $b_k|0_R\rangle = 0$

We will assume that the state of the system is $|\psi\rangle = |0_M\rangle$.



Will acceleration melt ice?

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For example: $\phi(\bar{u},v) = e \quad = e$

is a right-moving positive frequency solution.

In the accelerated frame, the quantum field in the Rindler wedge is:

$$\hat{\phi}(\bar{u},\bar{v}) = \int_0^\infty \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\Big(\underbrace{e^{-i\omega\bar{u}}\hat{b}_k + e^{i\omega\bar{u}}\hat{b}_k^+}_{\text{right movers}} + \underbrace{e^{-i\omega\bar{v}}\hat{b}_{-k} + e^{i\omega\bar{v}}\hat{b}_{-k}^+}_{\text{left movers}}\Big) \text{ and } \omega = |k|$$

We arrived at a typical situation:

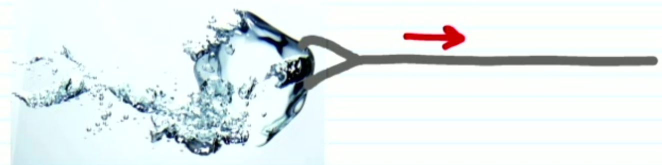
$$\boxed{\begin{aligned}\hat{\phi}(u,v) &= \int_0^\infty \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\left(e^{-i\omega u}\hat{a}_k + e^{i\omega u}\hat{a}_k^+ + e^{-i\omega v}\hat{a}_{-k} + e^{i\omega v}\hat{a}_{-k}^+\right)\\ &= \int_0^\infty \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\left(e^{-i\omega\bar{u}}\hat{b}_k + e^{i\omega\bar{u}}\hat{b}_k^+ + e^{-i\omega\bar{v}}\hat{b}_{-k} + e^{i\omega\bar{v}}\hat{b}_{-k}^+\right)\end{aligned}} \quad (A)$$

There must exist a Bogolubov transformation linking the a_k, a_k^+ and b_k, b_k^+!

Observation:

The left and right movers won't mix.

$\rightsquigarrow$ For simplicity we'll consider only the right movers.

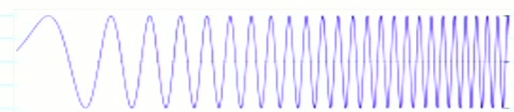


Will acceleration melt ice?

Observation: Among right movers all frequencies may mix:

$$\boxed{b_\Omega = \int_0^\infty d\omega\,\left(\alpha_{\Omega\omega}\,a_\omega - \beta_{\Omega\omega}\,a^+_\omega\right) \text{ with } \omega = k} \quad (B)$$

Intuition: To the traveller, any monochromatic wave sounds like a chirp.



Exercise: Check that $[a_k, a^+_{k'}] = \delta(k-k')$ and $[b_k, b^+_{k'}] = \delta(k-k')$ imply:

$$\boxed{\int_0^\infty d\omega\,\left(\alpha_{\Omega\omega}\,\alpha^*_{\Omega'\omega} - \beta_{\Omega\omega}\,\beta^*_{\Omega'\omega}\right) = \delta(\Omega-\Omega')} \quad (C)$$

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15

The left and right movers won't mix.

$\rightsquigarrow$ For simplicity we'll consider only the right movers.

Calculation of $\alpha_{\Omega\omega}$ and $\beta_{\Omega\omega}$: (lengthy, for more details, see Mukhanov & Winitzki text.)

- Substitute (B) into (A) and collect coefficients of a_ω

$$\Rightarrow \quad \omega^{-1/2} e^{-i\omega u} = \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}} \left(\alpha_{\Omega'\omega} e^{-i\Omega'\bar{u}} - \beta^*_{\Omega'\omega} e^{i\Omega'\bar{u}} \right)$$

- Act with $\int_{-\infty}^{\infty} d\bar{u}\, e^{\pm i\Omega\bar{u}}$ on the equation

and then use that $\int_{-\infty}^{\infty} e^{i(\Omega-\Omega')\bar{u}} d\bar{u} = 2\pi\,\delta(\Omega-\Omega')$.

$$\Rightarrow \quad \begin{matrix} + \text{case:} \\ - \text{case:} \end{matrix} \quad \left.\begin{matrix} \alpha_{\Omega\omega} \\ \beta_{\Omega\omega} \end{matrix}\right\} = \pm \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} e^{\mp i\omega u + i\Omega\bar{u}} d\bar{u}$$

Exercise: Check that $[a_k, a^+_{k'}] = \delta(k-k')$ and $[b_k, b^+_{k'}] = \delta(k-k')$ imply:

$$\boxed{\int_0^\infty d\omega \left(\alpha_{\Omega\omega}\alpha^*_{\Omega'\omega} - \beta_{\Omega\omega}\beta^*_{\Omega'\omega} \right) = \delta(\Omega-\Omega')} \quad (C)$$

Recall: $\boxed{u(\bar{u},\bar{v}) = -\frac{1}{a} e^{-a\bar{u}}}$ (encoding the chirping)

and, therefore: $\frac{du}{d\bar{u}} = e^{-a\bar{u}} \Rightarrow d\bar{u} = (-au)^{-1} du$

$$\Rightarrow \quad \left.\begin{matrix} \alpha_{\Omega\omega} \\ \beta_{\Omega\omega} \end{matrix}\right\} = \pm \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} e^{\mp i\omega u + i\Omega\bar{u}} d\bar{u} = \pm \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{0} e^{\mp i\omega u} (-au)^{-\frac{i\Omega}{a}-1} du$$

- Now, using $\Gamma(r) = \int_0^\infty s^{r-1} e^{-s} ds$

$$\Rightarrow \quad \boxed{\left.\begin{matrix} \alpha_{\Omega\omega} \\ \beta_{\Omega\omega} \end{matrix}\right\} = \pm \frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}\, e^{\pm\frac{\pi\Omega}{2a}}\, e^{i\left(\frac{\Omega}{a}\ln\frac{\omega}{a}\right)}\, \Gamma\!\left(-\frac{i\Omega}{a}\right)} \quad !$$

and then use that $\int_{-\infty}^{\infty} e^{i(\Omega-\Omega')\bar{u}}\, d\bar{u} = 2\pi\,\delta(\Omega-\Omega')$.

$\Rightarrow$

+ case: $\alpha_{\Omega\omega}$
− case: $\beta_{\Omega\omega}$

$$\left.\begin{matrix}\alpha_{\Omega\omega}\\ \beta_{\Omega\omega}\end{matrix}\right\} = \pm\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} e^{\mp i\omega u + i\Omega\bar{u}}\, d\bar{u}$$

Now, using $\Gamma(r) = \int_0^\infty s^{r-1}e^{-s}\,ds$

$$\Rightarrow \boxed{\left.\begin{matrix}\alpha_{\Omega\omega}\\ \beta_{\Omega\omega}\end{matrix}\right\} = \pm\frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}\; e^{\pm\frac{\pi\Omega}{2a}}\; e^{i\left(\frac{\Omega}{a}\ln\frac{\omega}{a}\right)}\,\Gamma\!\left(-\frac{i\Omega}{a}\right)} \;!$$

Observation: $\Rightarrow |\beta_{\Omega\omega}|^2 = e^{-\frac{2\pi\Omega}{a}}|\alpha_{\Omega\omega}|^2$ (D)

So for acceleration $a\to 0$ we have $|\beta_{\Omega\omega}|\to 0$, i.e. then no particles observed in travelers frame.

How many particles does an accelerated observer see if $a\neq 0$?

$$\langle\psi_i|\hat{N}_\Omega|\psi_i\rangle = \langle 0_M|\hat{N}_\Omega|0_M\rangle$$
$$= \langle 0_M|\hat{b}_\Omega^+\hat{b}_\Omega|0_M\rangle$$
$$= \langle 0_M|\left(\int_0^\infty \alpha^*_{\omega\Omega}\hat{a}^+_\omega - \beta^*_{\omega\Omega}\hat{a}_\omega\, d\omega\right)\left(\int_0^\infty \alpha_{\omega'\Omega}\hat{a}_{\omega'} - \beta_{\omega'\Omega}\hat{a}^+_{\omega'}\, d\omega'\right)|0_M\rangle$$
$$= \int_0^\infty d\omega\, |\beta_{\omega\Omega}|^2$$

Using (C) and (D) $\Rightarrow$

$$\langle\psi_i|\hat{N}_\Omega|\psi_i\rangle = \int_0^\infty d\omega\,|\beta_{\omega\Omega}|^2 = \frac{1}{e^{\frac{2\pi\Omega}{a}}-1}\,\delta(\Omega-\Omega)$$

$\uparrow$ Divergent

Observation:

With infrared cutoff through (accelerating) box of size V we have discrete k, discrete $\Omega(k)$ and $\delta(\Omega-\Omega')$ in (C) becomes $V\delta_{\Omega,\Omega'}$.

Then:

$$\langle\psi_i|\hat{N}_\Omega|\psi_i\rangle = \int_0^\infty d\omega\,|\beta_{\omega\Omega}|^2 = \frac{1}{e^{\frac{2\pi\Omega}{a}}-1}\,V\,\delta_{\Omega,\Omega}$$

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$$= \langle 0_M | b_\Omega b_\Omega | 0_M \rangle$$

$$= \langle 0_M | \left(\int_0^\infty \alpha^*_{\omega\Omega} \hat{a}^+_\omega - \beta^*_{\omega\Omega} \hat{a}_\omega \, d\omega \right) \left(\int_0^\infty \alpha_{\omega'\Omega} \hat{a}_{\omega'} - \beta_{\omega'\Omega} \hat{a}^+_{\omega'} \, d\omega' \right) | 0_M \rangle$$

$$= \int_0^\infty d\omega \, |\beta_{\omega\Omega}|^2$$

$\Rightarrow$ Number density:

$$\boxed{\bar{n}_\Omega := \frac{1}{V} \langle \psi_i | \hat{N}_\Omega | \psi_i \rangle = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1}}$$

Compare: If a harmonic oscillator of energy levels $E_n = \Omega(n + \frac{1}{2})$ is exposed to a heat bath of temperature T, then its expected excitation number is

$$\bar{n} = \frac{1}{e^{\Omega/T} - 1}$$

$\Rightarrow$ The traveler's mode oscillators are excited as if exposed to a heat bath of the Unruh temperature:

$$\boxed{T = \frac{a}{2\pi}}$$

Then:

$$\langle \psi_i | \hat{N}_\Omega | \psi_i \rangle = \int_0^\infty d\omega \, |\beta_{\omega\Omega}|^2 = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} V \delta_{\Omega,\Omega}$$

Observation:

- Could the quantum field be in the state $|0_R\rangle$?
- We'd expect that then inertial observers would see particles!
- But $|0_R\rangle$ is not a physically implementable state, even in principle! Why?
- $|0_R\rangle$ is a state with regions of diverging energy density!

Why? If $\hat{\phi}$ is in state $|0_R\rangle$ then, in accelerated cds, energy density is constant throughout that cds.

But this cds piles up at the horizons!

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$$\bar{n} = \frac{1}{e^{\Omega/T} - 1}$$

$\Rightarrow$ The traveler's mode oscillators are excited as if exposed to a heat bath of the **Unruh temperature**:

$$\boxed{T = \frac{a}{2\pi}}$$

$|0_R\rangle$ is a state with regions of diverging energy density!

Why? If $\hat{\phi}$ is in state $|0_R\rangle$ then, in accelerated cds, energy density is constant throughout that cds.

But this cds piles up at the horizons!

Recall: $T_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - \eta_{\mu\nu}(\partial_\alpha \phi)(\partial^\alpha \phi)$

$\Rightarrow$ Need to study terms of the form $\langle 0_R | (\partial\phi)^2 | 0_R \rangle$.

Calculate: $\underbrace{\langle 0_R | (\partial_u \hat{\phi})^2 | 0_R \rangle}_{\text{enters } \langle 0_R | T_{\mu\nu}(u,v) | 0_R \rangle \text{ calculation of inertial observer!}} = \langle 0_R | \left(\frac{\partial \bar{u}}{\partial u}\right)^2 (\partial_{\bar{u}} \hat{\phi})^2 | 0_R \rangle$

Recall: $\boxed{u(\bar{u}, \bar{v}) = -\frac{1}{a} e^{-a\bar{u}}}$

$\Rightarrow \frac{du}{d\bar{u}} = -a u \Rightarrow$

$$= \frac{1}{(a u)^2} \langle 0_R | (\partial_{\bar{u}} \hat{\phi})^2 | 0_R \rangle$$

same b/c calculated exact same way from (A)

$$= \frac{1}{(a u)^2} \underbrace{\langle 0_M | (\partial_u \hat{\phi})^2 | 0_M \rangle}_{\text{Finite after renormalization.}}$$

But: $u^{-1} \to \infty$ at the traveler's horizon!

$\Rightarrow$ In states $|\psi\rangle = |0_R\rangle$, or $|\psi\rangle = b_n^+ |0_R\rangle$ etc.,

$\langle \psi | T_{\mu\nu}(u,v) | \psi \rangle \to \infty$ as $u \to 0$ (future horizon)

and similarly also for $v \to 0$.

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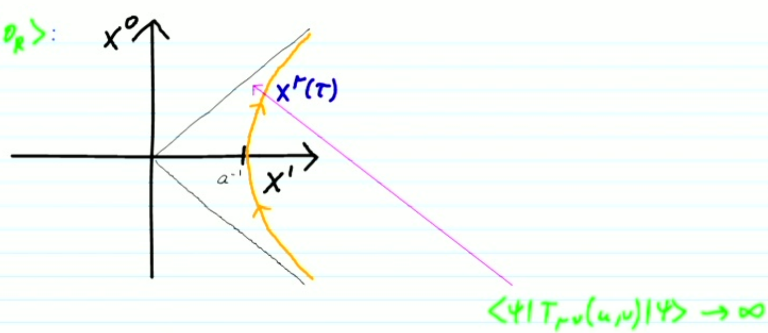
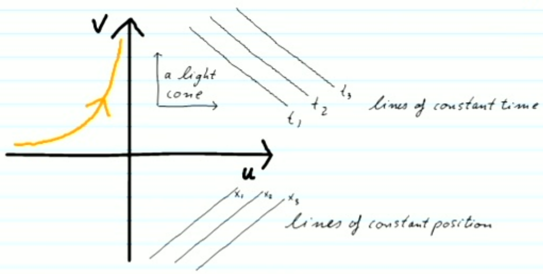
calculation of inertial observer!

Recall: $\boxed{u(\tilde{u},\tilde{v}) = -\frac{1}{a}\, e^{-a\tilde{u}}}$

$\Rightarrow \quad \frac{du}{d\tilde{u}} = -a\,u \quad \Rightarrow$

If $\hat{\phi}$ is in state $|0_R\rangle$:





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