Title: Advanced General Relativity - 240403

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Collection: Advanced General Relativity (PHYS7840)

Date: April 03, 2024 - 10:30 AM

URL: https://pirsa.org/24040000

$$R_{\mu\nu} k^\mu k^\nu = -\partial_v \Theta + \kappa \Theta - \frac{1}{2}\Theta^2 - \sigma_{AB}\sigma^{AB}$$

$$R_{\mu\alpha} k^\mu e^\alpha_A = \partial_v \omega_A - \partial_A \kappa - \frac{1}{2}\partial_A \Theta + D_B \sigma_A{}^B + \Theta \omega_A$$

$$\Theta = \frac{1}{\sqrt{\Omega}} \partial_v \sqrt{\Omega} \qquad B_{AB} = \frac{1}{2}\Theta \Omega_{AB} + \sigma_{AB} = \partial_v \Omega_{AB}$$

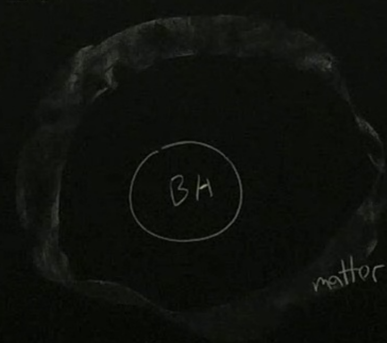
$$d\Sigma_\alpha = -k_\alpha \, dv \, \underbrace{dS}_{\sqrt{\Omega}\, d^2\theta}$$

Stationary BH : $K^\alpha = t^\alpha + \Omega_H \phi^\alpha$ null on horizon

$$\partial_v(\cdots) = 0$$

$$\Theta = 0 = \sigma_{AB}$$

$$T_{\alpha\beta} K^\alpha K^\beta = 0$$



$$\sqrt{\Omega}\,\partial^2\theta$$

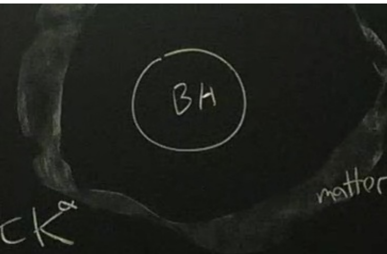
Stationary BH : $K^\alpha = t^\alpha + \Omega_H \phi^\alpha$ null on horizon

$$\partial_v(\dots) = 0$$

$$\Theta = 0 = \sigma_{AB}$$

$$T_{\alpha\beta}K^\alpha K^\beta = 0$$

$$K^\beta \nabla_\beta K^\alpha = \kappa K^\alpha$$



? $\partial_A \kappa = 0$

$$\Rightarrow \partial_A \kappa = -R_{\mu\alpha}K^\mu e^\alpha_A = -8\pi T_{\mu\alpha}K^\mu e^\alpha_A \Rightarrow \boxed{\partial_A \kappa = 8\pi j_\alpha e^\alpha_A}$$

$$\boxed{j^\alpha \equiv -T_\mu{}^\alpha K^\mu}$$

→ energy flux vector.

model: $T_{\alpha\beta} = \rho U_\alpha U_\beta$ (pressureless fluid = dust)

$$0 = \nabla_\beta T^{\alpha\beta} = \nabla_\beta(\rho U^\alpha U^\beta) = \nabla_\beta(\rho U^\beta)\,U^\alpha + \rho\,\underbrace{U^\beta \nabla_\beta U^\alpha}_{a^\alpha}$$

$$\begin{cases} 0 = \nabla_\beta(\rho U^\beta) \\ a^\alpha = U^\beta \nabla_\beta U^\alpha = 0 \end{cases}$$

Conserved quantities: $\left.\begin{aligned} \tilde{E} &= -u_\alpha t^\alpha = \text{energy per unit mass} \\ \tilde{L} &= +u_\alpha \ell = \text{angular momentum/mass} \end{aligned}\right\}$ constants of motion.

$$j^\alpha = -\rho(u_\beta K^\beta)\,u^\alpha = +\rho\,\underbrace{(-u_\beta K^\beta)}_{\geq 0 \text{ "boost factor"} = \tilde{E} - \Omega_H \tilde{J}}\,u^\alpha \equiv \text{energy flux vector.}$$

Dominant energy condition: $\boxed{j^\alpha \text{ must timelike or null}}$

conserved quantities: $\tilde{E} = -U_\alpha \xi^\alpha$ ...

$\tilde{L} = +U_\alpha \ell$ = angular momentum / mass

$$j^\alpha = -\rho (U_\mu K^\mu) U^\alpha = +\rho \underbrace{(-U_\mu K^\mu)}_{\geq 0 \ \text{"boost factor"} = \tilde{E} - \Omega_H \tilde{J}} U^\alpha \equiv \text{energy flux vector.}$$

Dominant energy condition: $\boxed{j^\alpha \text{ must timelike or null}}$

$$j_\alpha j^\alpha \leq 0$$

$$\Rightarrow (A K_\alpha + C^A e_{A\alpha})(A K^\alpha + C^B e_B^\alpha) \leq 0$$

$$\underbrace{C^A C^B \Omega_{AB}}_{\geq 0} \leq 0 \quad \Rightarrow C^A = 0$$

$$j^\alpha = A K^\alpha + \cancel{B N^\alpha} + C^A e^\alpha_A$$

$$j_\alpha K^\alpha = 0 = -B \quad \Rightarrow B = 0$$

$$\Rightarrow \boxed{j^\alpha = A K^\alpha}$$

$$d\Sigma_\alpha = -K_\alpha\, dv\, \underbrace{d^2S}_{\sqrt{\sigma}\, d^2\theta}$$

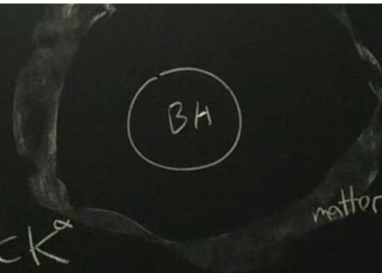
Stationary BH : $K^\alpha = t^\alpha + \Omega_H \phi^\alpha$ null on horizon

$\partial_v(\cdots) = 0$

$\Theta = 0 = \sigma_{AB}$

$$\boxed{T_{\alpha\beta} K^\alpha K^\beta = 0}$$

$$K^\beta \nabla_\beta K^\alpha = \kappa K^\alpha$$



Mass law

$$\boxed{M = \frac{\kappa}{8\pi} A + 2\Omega_H J_H - 2\int_\Sigma \left(T^\alpha{}_\beta - \frac{1}{2} T g^\alpha{}_\beta\right) t^\beta\, d\Sigma_\alpha}$$

↳ total mass, measured at ∞.

What is total mass M?

coordinate answer: $g_{tt} = -1 + 2M/r + O(r^{-2})$

geometric answer: $\boxed{M = -\frac{1}{8\pi}\oint_\infty \nabla^\alpha t^\beta \, dS_{\alpha\beta}}$

$dS_{\alpha\beta} = 2\, r_{[\alpha} n_{\beta]}\, dS$

$-\frac{1}{8\pi} \nabla_\alpha t^\beta \left(2 r^\alpha n_\beta\right) dS = -\frac{1}{4\pi} \underbrace{\left(\nabla_\alpha t^\beta\right) r^\alpha n_\beta}_{} \, r^2 d\Omega$

$= \frac{1}{4\pi} M \, d\Omega$

$\left(\partial_\alpha t^\beta + \Gamma^\beta_{\alpha\gamma} t^\gamma\right) r^\alpha n_\beta = -\Gamma^t_{tr}$

$= -\frac{1}{2}(-1)\left(\cancel{\partial_t g_{tr}} + \partial_r g_{tt} - \cancel{\partial_t g_{tr}}\right) = \frac{1}{2}\left(\frac{-2M}{r^2}\right) = \frac{-M}{r^2}$

geometric answer:

$$\frac{}{8\pi}\oint_\infty$$

$$-\frac{1}{8\pi}\nabla_\alpha t^\beta\,(2r^\alpha n_\beta)\,dS = -\frac{1}{4\pi}\underbrace{(\nabla_\alpha t^\beta)\,r^\alpha n_\beta}\,r^2\,d\Omega$$

$$= \frac{1}{4\pi}M\,d\Omega$$

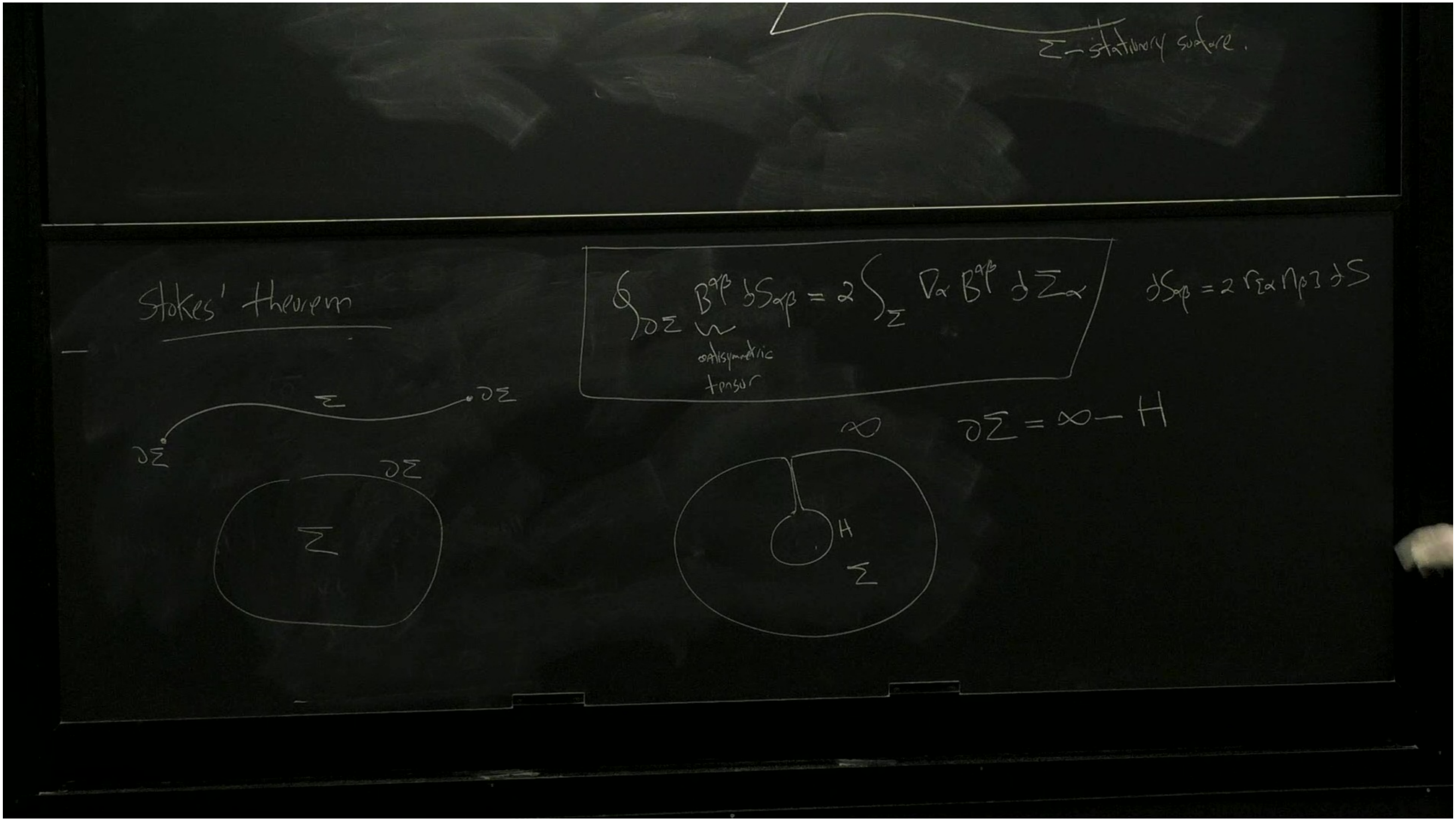
$$(\partial_\alpha t^\beta + \Gamma^\beta_{\alpha\gamma}t^\gamma)\,r^\alpha n_\beta = -\Gamma^t_{tr}$$

$$= -\frac{1}{2}(-1)\left(\cancel{\partial_t g_{tr}} + \partial_r g_{tt} - \cancel{\partial_t g_{tr}}\right) = \frac{1}{2}\left(\frac{-2M}{r^2}\right) = -\frac{M}{r^2}$$

What is total angular momentum J ?

$$g_{t\varphi} = -\frac{2J}{r}\sin^2\theta + O(r^{-2})$$

$$\boxed{J = \frac{1}{16\pi}\oint_\infty \nabla^\alpha \varphi^\beta\, dS_{\alpha\beta}}$$



Σ – stationary surface.
Stokes' theorem
$\oint_{\partial\Sigma} B^{\alpha\beta}\, dS_{\alpha\beta} = 2\int_{\Sigma} \nabla_\alpha B^{\alpha\beta}\, d\Sigma_\alpha$
antisymmetric tensor
$dS_{\alpha\beta} = 2\, r_{[\alpha} n_{\beta]}\, dS$
Σ
∂Σ
∂Σ
∂Σ
Σ
∞
$\partial\Sigma = \infty - H$
H
Σ

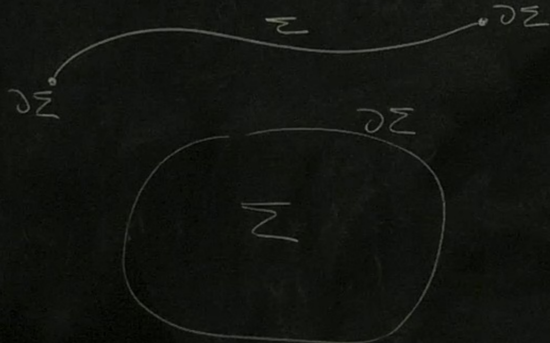
Σ – stationary surface.

Stokes' theorem

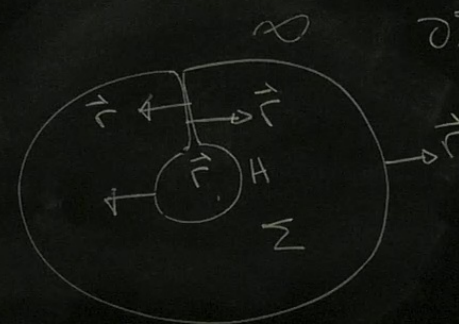
$$\boxed{\oint_{\partial\Sigma} \underbrace{B^{\alpha\beta}}_{\text{antisymmetric tensor}} dS_{\alpha\beta} = 2\int_{\Sigma} \nabla_\alpha B^{\alpha\beta}\, d\Sigma_\alpha}$$

$dS_{\alpha\beta} = 2\, r_{[\alpha} n_{\beta]}\, dS$

$r^\alpha \equiv$ outward normal to $\partial\Sigma$



$\partial\Sigma = \infty - H$



$$\oint_{\partial\Sigma} B^{\alpha\beta}\, dS_{\alpha\beta} = \int_{\infty} B^{\alpha\beta}\, dS_{\alpha\beta} - \int_{H} B^{\alpha\beta}\, dS_{\alpha\beta}$$

$$\boxed{\oint_{\infty} B^{\alpha\beta}\, dS_{\alpha\beta} = \oint_{H} B^{\alpha\beta}\, dS_{\alpha\beta} + 2\int_{\Sigma} \nabla_\beta B^{\alpha\beta}\, d\Sigma_\alpha}$$

$$B^{\alpha\beta} = -\frac{1}{8\pi}\nabla^\alpha t^\beta \longrightarrow M = M_H + 2\int_\Sigma \nabla_\beta\left(\nabla^\alpha t^\beta\right) d\Sigma_\alpha$$

$$\left[\begin{array}{l} M_H \equiv -\dfrac{1}{8\pi}\displaystyle\oint_H \nabla^\alpha t^\beta\, dS_{\alpha\beta} \\ J_H = +\dfrac{1}{16\pi}\displaystyle\oint_H \nabla^\alpha \phi^\beta\, dS_{\alpha\beta} \end{array}\right.$$

$$\begin{cases} M_H \equiv -\dfrac{1}{8\pi}\oint_H \nabla^\alpha t^\beta \, dS_{\alpha\beta} \\[2ex] J_H = +\dfrac{1}{16\pi}\oint_H \nabla^\alpha \phi^\beta \, dS_{\alpha\beta} \end{cases}$$

$$\boxed{M_H - 2\Omega_H J_H = \frac{\kappa}{8\pi} A} = -\frac{1}{8\pi}\oint_H \underbrace{\nabla^\alpha \left(t^\beta + \Omega_H \phi^\beta\right)}_{\nabla^\alpha k^\beta} \underbrace{dS_{\alpha\beta}}_{+2 k_\alpha N_\beta \, dS}$$

$$= -\frac{1}{4\pi}\oint_H \underbrace{\left(k^\alpha \nabla_\alpha k^\beta\right)}_{\kappa k^\beta} N_\beta \, dS = \frac{1}{4\pi}\oint_H \kappa \, dS$$

$$\overset{0\text{th}}{=} \frac{\kappa}{4\pi}\int_H dS = \frac{\kappa}{4\pi} A$$

Σ – stationary surface.

$$\nabla_\beta (\nabla^\alpha t^\beta) = R^\alpha{}_\beta t^\beta = 8\pi \left(T^\alpha{}_\beta - \tfrac{1}{2} T g^\alpha{}_\beta\right) t^\beta$$

Smarr law

$$\boxed{M = \frac{\kappa}{8\pi} A + 2\Omega_H J_H - 2\int_\Sigma \left(T^\alpha{}_\beta - \tfrac{1}{2} T g^\alpha{}_\beta\right) t^\beta \, d\Sigma_\alpha}$$