Title: Heisenberg-Limited Quantum Metrology without Ancilla (VIRTUAL)

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Abstract: The asymptotic theory of quantum channel estimation has been well established, but in general noiseless and controllable ancilla is required for attaining the ultimate limit in the asymptotic regime. Little is known about the metrological performance without noiseless ancilla, which is more relevant in practical circumstances. In this work, we present a novel theoretical framework to address this problem, bridging quantum metrology and the asymptotic theory of quantum channels. Leveraging this framework, we prove sufficient conditions for achieving the Heisenberg limit with repeated application of the channel to estimate, both with and without applying interleaved unitary control operations. For the latter case, we design an algorithm to identify the control operation. Finally, we analyze several intriguing examples by our approach.

Zoom link

Heisenberg-Limited Quantum Metrology without Ancilla

Quantum Information and Computation Initiative

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Perimeter Institute Seminar

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Outline

- Background: what we know and why study ancilla-free quantum metrology
- Main results (informal)
- Some preliminary tools: quantum Fisher information, vectorization, spectral properties of quantum channels
- Theorem 1: sufficient condition for achieving HL without control, examples
- Theorem 2: sufficient condition for achieving HL with unitary control, examples
- Efficient algorithm in practice: tensor networks for optimizing metrology (if time permits)
- Summary and outlook

Quantum metrology: sensing better

[LIGO, PRL 123, 231107 (2019)]

Gravitational wave detection

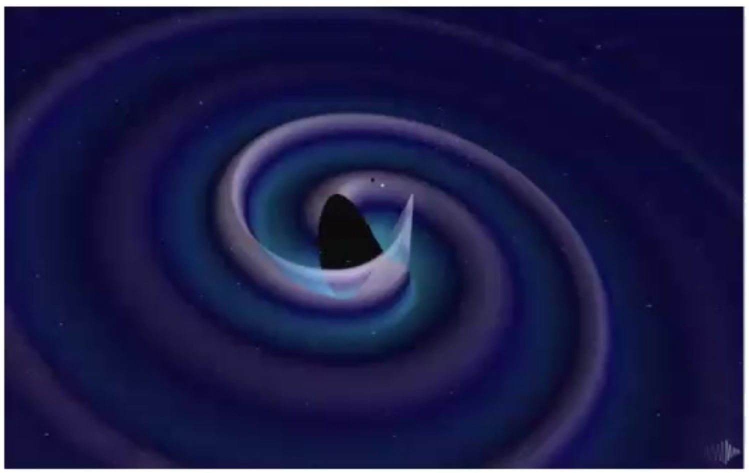
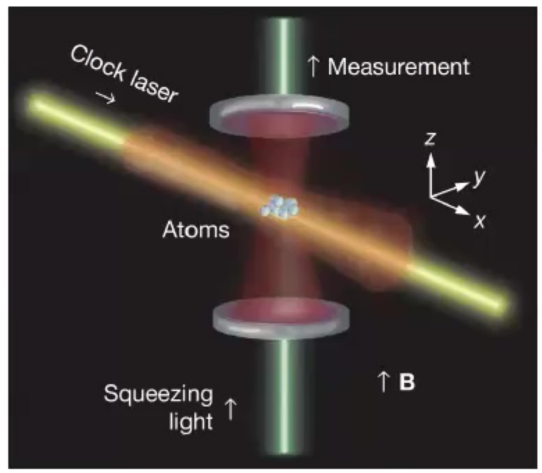


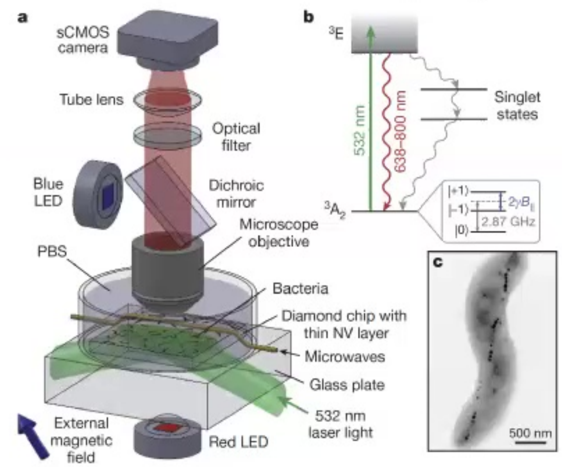
Image source: www.ligo.caltech.edu/MIT/image/ligo20210629c

Atomic clock



[Nature 588, 414–418 (2020)]

Biomedical imaging



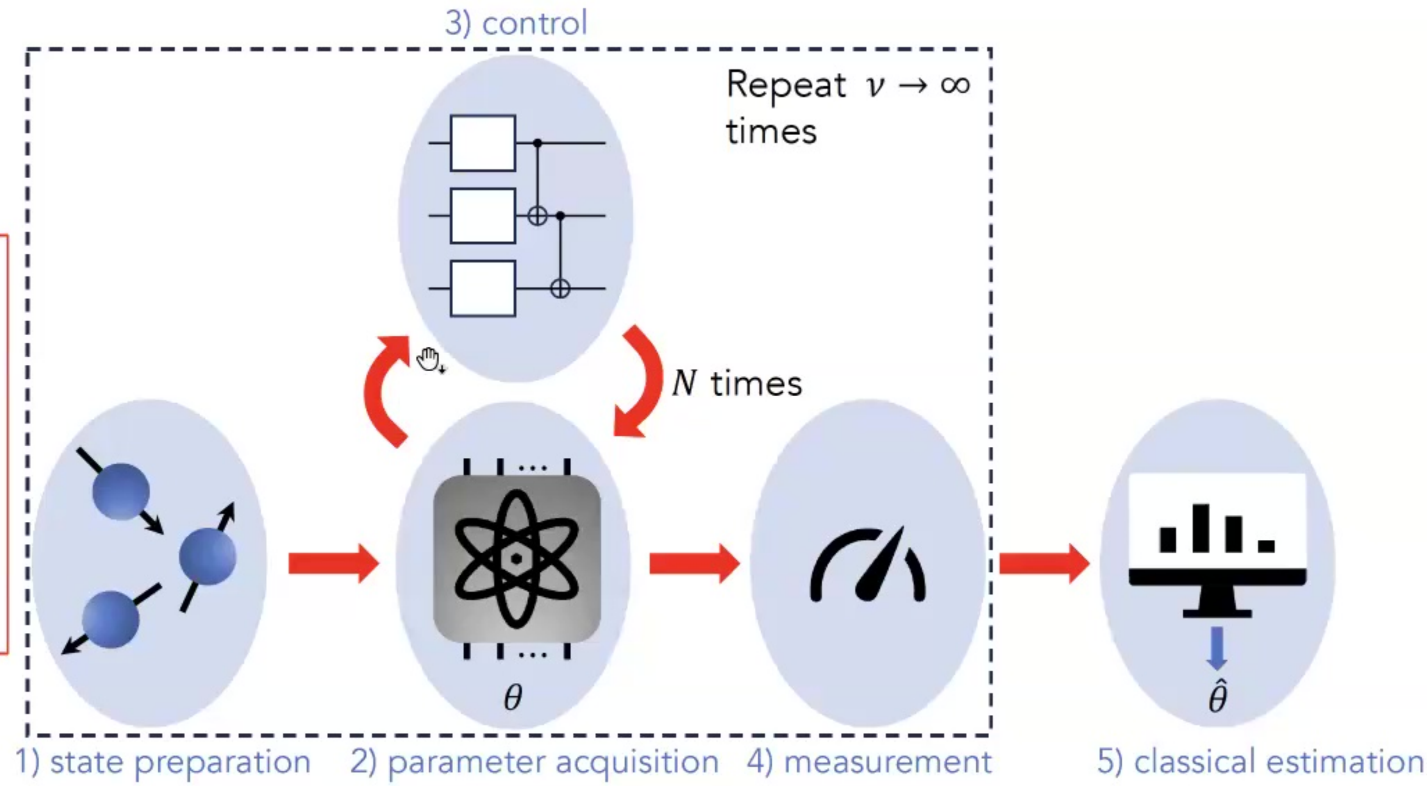
[Nature 496, 486–489 (2013)]

Quantum metrology: general framework

Goal: Estimate θ

Minimize the mean squared error (MSE):

$$\delta\theta^2 = \mathrm{E}\left[\left(\hat{\theta} - \theta\right)^2\right]$$

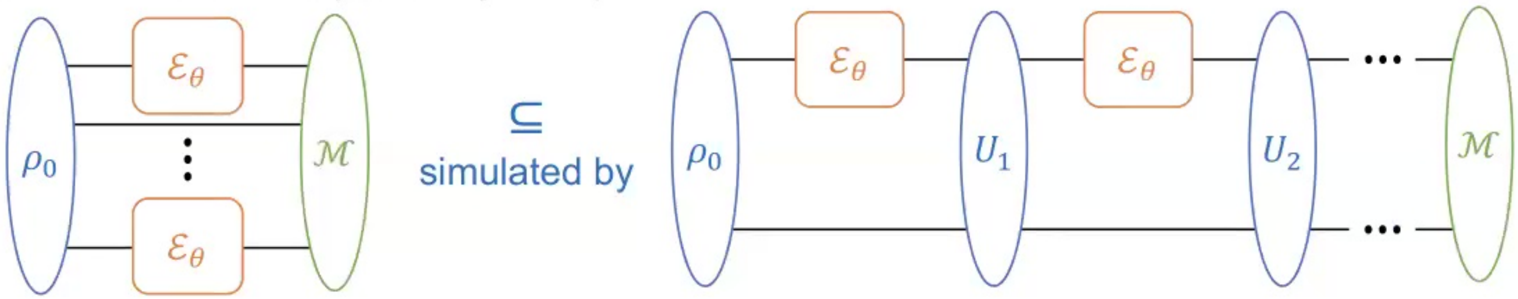


Quantum channel estimation

Estimate θ by N uses of unknown quantum channel $\mathcal{E}_\theta$

[Fujiwara & Imai JPA'08]
[Escher, de Matos Filho & Davidovich NP'11]
[Demkowicz-Dobrzański, Kołodyński & Guţă NC'12]

[Demkowicz-Dobrzański & Maccone PRL'14]



Parallel strategy

Sequential strategy

- (Tight) Quantum Cramér-Rao bound $\delta\theta^2 \geq \frac{1}{\nu F^Q(\rho_\theta)}$, $F^Q(\rho_\theta)$: Quantum Fisher information (QFI)
- Optimal asymptotic ($N \to \infty$) QFI: same for both strategies [Zhou & Jiang PRX Quantum'18] [Kurdziałek, Górecki, Albarelli & Demkowicz-Dobrzański PRL'23]
- Optimal QFI for small N: can be different [Liu, Hu, Yuan & Yang PRL'23]

Heisenberg limit vs Standard quantum limit

- "Hamiltonian-not-in-Kraus-span" (HNKS) condition:

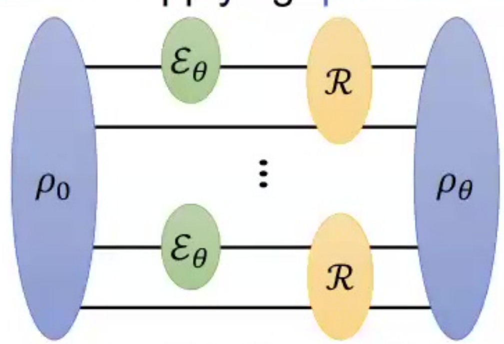
$$H := \mathrm{i}\sum_j K_j^\dagger \dot{K}_j \,, \qquad H \notin \mathcal{S} := \mathrm{Span}_{\mathrm{Herm}}\{K_i^\dagger K_j\}, \forall i,j$$

- For optimal estimation of $N \to \infty$ channels (parallel or sequential):

$$\lim_{N\to\infty} F^Q/N = \alpha \text{ (Constant)} \Leftrightarrow H \in \mathcal{S}$$ Standard quantum limit (SQL)

$$\lim_{N\to\infty} F^Q/N^2 = \min_{S\in\mathcal{S}} \|H - S\| \text{ (Constant)} \Leftrightarrow H \notin \mathcal{S}$$ Heisenberg limit (HL)

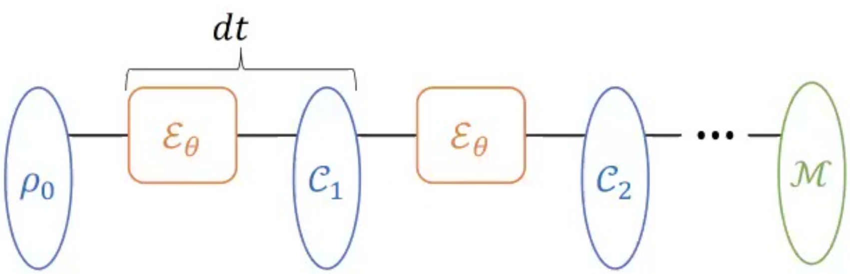
- Both scalings (and even coefficients) can be attained with parallel strategies, by introducing noiseless ancilla and applying quantum error correction (QEC) *[Zhou & Jiang PRX Quantum'18]*



Requires noiseless ancilla ☹

Ancilla-free QEC for metrology

- Ancilla-free metrology mainly studied using ancilla-free QEC for Hamiltonian parameter estimation under Markovian noise in previous work
- Ancilla-free QEC achieving HL with sequential strategies, when (i) "Hamiltonian-not-in-Lindblad-span" (HNLS) satisfied and (ii) signal commutes with noise *[Layden, Zhou, Cappellaro & Jiang PRL'19]*



Sequential strategy

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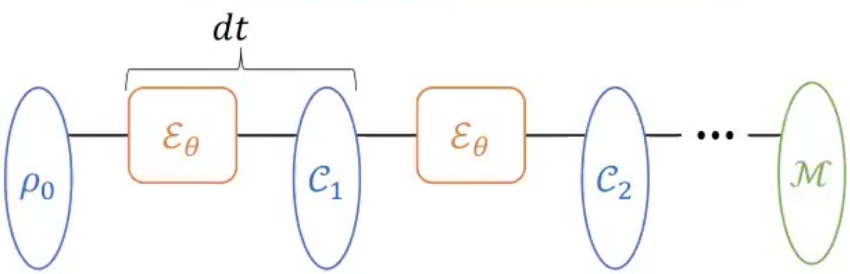
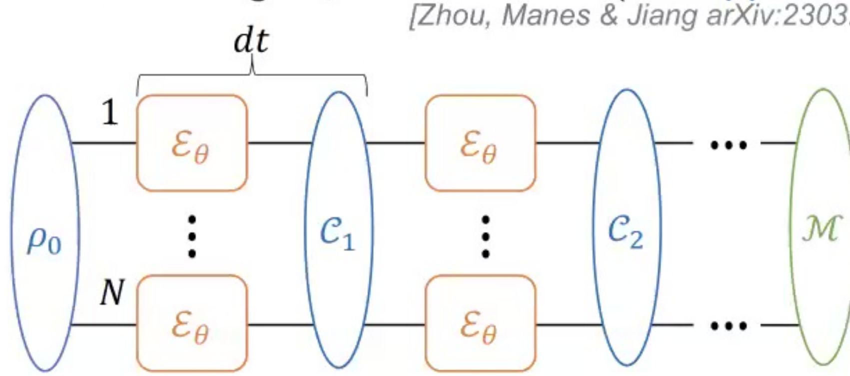
- Ancilla-free QEC achieving HL with sequential strategies, when (i) "Hamiltonian-not-in-Lindblad-span" (HNLS) satisfied and (ii) signal commutes with noise *[Layden, Zhou, Cappellaro & Jiang PRL'19]*
- Ancilla-free QEC achieving HL/SQL with parallel strategies, when $N \to \infty$ (not applicable to quantum channel estimation) *[Zhou, Manes & Jiang arXiv:2303.00881 (2023)]*



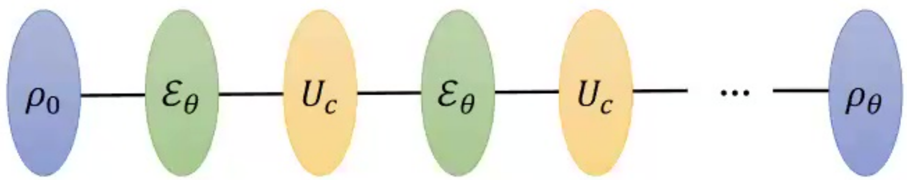
Sequential strategy



Parallel strategy

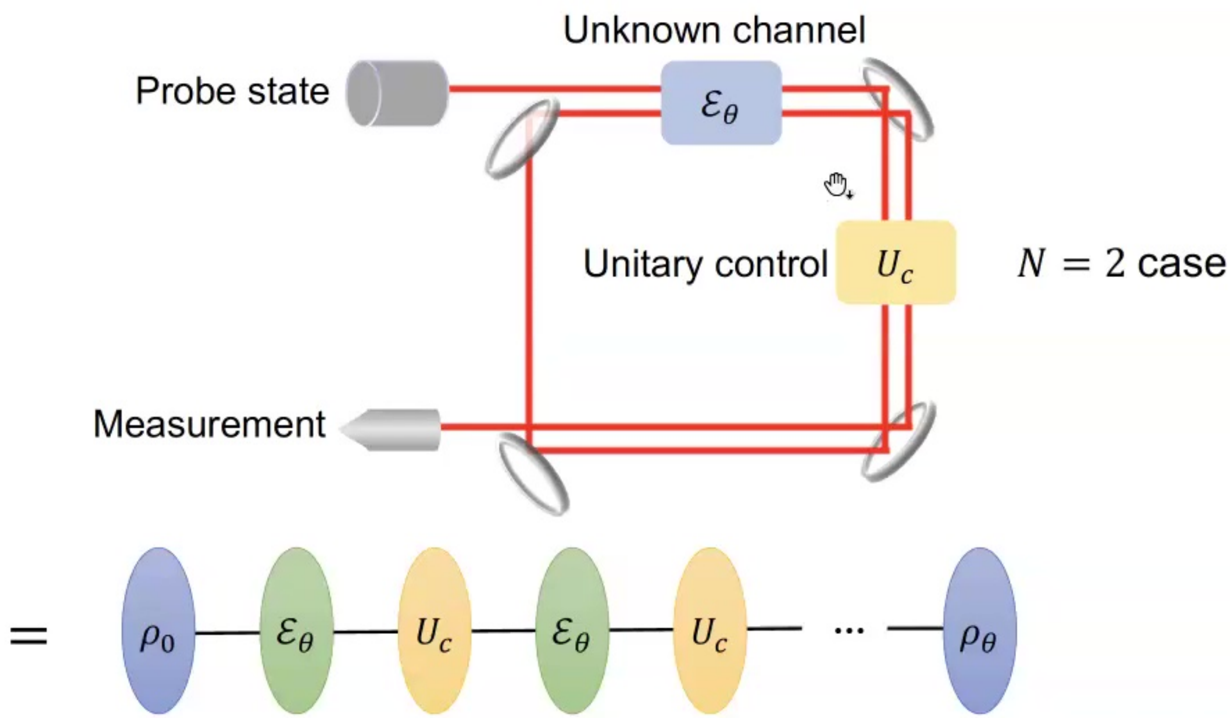
Ancilla-free quantum metrology

- Current experimental sensor demonstration has limited programmability
- For example, *[Hou et al, PRL 123, 040501 (2019)]* implemented estimation of $N = 8$ channels with the same unitary control in an optical loop



- Requirement of recovery operations and syndrome measurements even in "ancilla-free" QEC can still be challenging, requiring additional ancilla
- Very recently, *[Zhou arXiv:2402.18765 (2024)]* derived conditions for achieving SQL (as well as other sub-HL bounds) with limited control in single-qubit channel estimation
- A general framework for addressing ancilla-free metrology, especially for HL, is missing

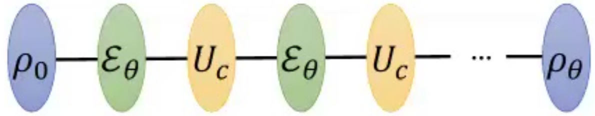
Experiment-friendly setup



Applicable to metrology with fixed-dimension noiseless ancilla: estimation of $\mathcal{E}_\theta \otimes \mathcal{I}_A$

Main results (informal)

- Quantum metrology (QFI) ↔ Asymptotics of quantum channels (spectral properties)
- HL/SQL could arise from nonzero derivatives of peripheral eigenvalues ($|\lambda| = 1$) of quantum channels
- Sufficient condition for achieving HL with repeated application of the channel to estimate without ancilla (control-free)
- Sufficient condition for achieving HL by inserting repeated unitary control operations without ancilla (control-enhanced)
- Applies to more resource-deficient scenarios, compared to QEC:
  - ➢ identical unitary control
  - ➢ no syndrome measurements
  - ➢ robust to state preparation error in some cases



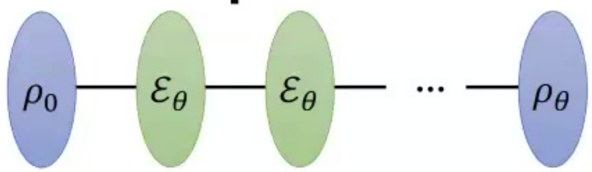
Associated QFI by vectorization

We study the QFI of the output state $F^Q(\rho_\theta) = F^Q[(\mathcal{U}_c \circ \mathcal{E}_\theta)^N(\rho_0)]$

- $F^Q(\rho_\theta) \coloneqq \mathrm{Tr}(\rho_\theta L_\theta^2)$, where L_θ is the SLD defined by $2\dot{\rho}_\theta = \rho_\theta L_\theta + L_\theta \rho_\theta$
- For pure state, simpler expression $F^Q(|\psi_\theta\rangle) = 4\left(\langle\dot{\psi}_\theta|\dot{\psi}_\theta\rangle - |\langle\psi_\theta|\dot{\psi}_\theta\rangle|^2\right)$
- Vectorize the state! Define $\tilde{\rho}_\theta \coloneqq |\rho_\theta\rangle\rangle\langle\langle\rho_\theta|/\mathrm{Tr}(\rho_\theta^2)$ and $2\dot{\tilde{\rho}}_\theta = \tilde{\rho}_\theta \tilde{L}_\theta + \tilde{L}_\theta \tilde{\rho}_\theta$
- Associated QFI $\tilde{F}^Q(\tilde{\rho}_\theta) \coloneqq \mathrm{Tr}(\tilde{\rho}_\theta \tilde{L}_\theta^2)$
- It has been shown $F^Q(\rho_\theta) \geq \frac{\mathrm{Tr}(\rho_\theta^2)}{4\lambda_{\max}(\rho_\theta)} \tilde{F}^Q(\tilde{\rho}_\theta)$ *[Alipour, Mehboudi & Rezakhani PRL'14]*
- It is sufficient to study the scaling of $\tilde{F}^Q(\tilde{\rho}_\theta)$ to lower bound $F^Q(\rho_\theta)$
- Lemma (generalization of *Alipour et al*'s result):

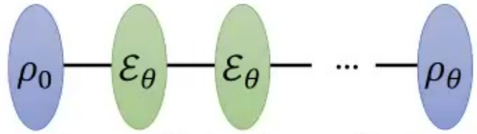
$$\tilde{F}^Q(\tilde{\rho}_\theta) = 4\left\{\frac{\langle\langle\dot{\rho}_\theta|\dot{\rho}_\theta\rangle\rangle}{\mathrm{Tr}(\rho_\theta^2)} - \left[\frac{\langle\langle\rho_\theta|\dot{\rho}_\theta\rangle\rangle}{\mathrm{Tr}(\rho_\theta^2)}\right]^2\right\}$$

Asymptotics of quantum channels 101



$$\mathcal{E}_\theta(\rho) = \sum_i K_i^\theta \rho K_i^{\theta\dagger}$$

- Liouville representation $T_\theta = \sum_i K_i^\theta \otimes K_i^{\theta *}$ ($d^2 \times d^2$ matrix for a d-dimensional system)
- Output state $|\rho_\theta\rangle\rangle = T_\theta^N |\rho_0\rangle\rangle$
- $T_\theta |R_i\rangle\rangle = \lambda_i |R_i\rangle\rangle$ with eigenvectors $|R_i\rangle\rangle$ (or eigenmatrices R_i)
- All the eigenvalues of T_θ satisfy $|\lambda| \leq 1$, and only those peripheral eigenvalues/eigenvectors (for $|\lambda| = 1$) take effect when $N \to \infty$
- Jordan blocks for peripheral eigenvalues are all one-dimensional *[Wolf '12]*
- Every channel has at least one fixed point state ρ_* (with $\lambda = 1$)
- Eigenvalues either are real or come in complex conjugate pairs, $T_\theta |R_i^\dagger\rangle\rangle = \lambda_i^* |R_i^\dagger\rangle\rangle$
- $\mathrm{Tr}(R_i) = 0$ if $\lambda_i \neq 1$ due to trace-preserving of T_θ



Asymptotic QFI of N channels

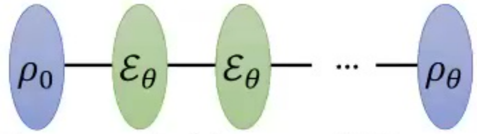
- Channel T_θ with peripheral eigenvalues $\{\lambda_i | i \in S\}$ and peripheral eigenvectors $\{|R_i\rangle\rangle | i \in S\}$
- Add additional linearly independent vectors $|R_i\rangle\rangle$ such that $\{R_i\}_{i=1}^{d^2}$ forms a Jordan basis
- Input state $|\rho_0\rangle\rangle = \sum_i a_i \, |R_i\rangle\rangle$ (a_i, R_i may be dependent on θ)
- Output state $|\rho_\theta\rangle\rangle = T_\theta^N |\rho_0\rangle\rangle$, with the associated state $\tilde{\rho}_\theta \coloneqq |\rho_\theta\rangle\rangle\langle\langle\rho_\theta|/\mathrm{Tr}(\rho_\theta^2)$
- Theorem 1 [sufficient condition for achieving HL/SQL]:

$$\frac{1}{4}\tilde{F}^Q(\tilde{\rho}_\theta) = N^2\left[\sum_{i,j\in S}\beta_{ij}\frac{\dot{\lambda}_i^*\dot{\lambda}_j}{\lambda_i^*\lambda_j} - \left(\sum_{i,j\in S}\beta_{ij}\frac{\dot{\lambda}_j}{\lambda_j}\right)^2\right] \quad \text{HL}$$

$$+ N\frac{1}{\mathrm{Tr}(\rho_\theta^2)}\sum_{i,j\in S}(\lambda_i^*\lambda_j)^N\left(2\frac{\dot{\lambda}_j}{\lambda_j} - \sum_{kl}\beta_{kl}\frac{\dot{\lambda}_k^*\dot{\lambda}_l}{\lambda_k^*\lambda_l}\right)\left(\dot{a}_i a_j\langle\langle R_i|R_j\rangle\rangle + a_i a_j\langle\langle \dot{R}_i|R_j\rangle\rangle\right) \quad \text{SQL}$$

$$+ o(N),$$

where $\beta_{ij} = (\lambda_i^*\lambda_j)^N a_i a_j \langle\langle R_i|R_j\rangle\rangle / \mathrm{Tr}(\rho_\theta^2)$ *satisfying* $\sum_{i,j\in S}\beta_{ij} = 1,\ \beta_{ii} = a_i^2 \geq 0$ *and* $\beta_{ij} = \beta_{ji}^*$.



Achieving HL without control

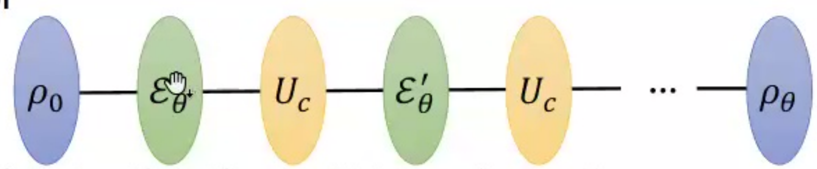
- Corollary 1 [sufficient condition for achieving HL, orthogonal peripheral eigenvectors]:

If all the peripheral eigenvectors $\left\{\lim_{\theta\to\theta_0} |R_i\rangle\rangle\right\}$ of T_θ are mutually orthogonal, and there exists some peripheral eigenvalue λ_j s.t. $\dot{\lambda}_j \neq 0$, then HL is achievable asymptotically ($N \to \infty$).

- Corollary 2 [sufficient condition for achieving HL, unital]:

For a unital channel T_θ, if there exists some peripheral eigenvalue λ_j s.t. $\dot{\lambda}_j \neq 0$, and the associated eigenmatrix $\lim_{\theta\to\theta_0} R_j$ either is Hermitian (up to a multiplicative factor) or satisfies $\lim_{\theta\to\theta_0} \mathrm{Tr}\left(R_j^2\right) = 0$, then HL is achievable asymptotically.

- These corollaries (especially Corollary 2) can yield simple conditions for achieving the HL with repeated unitary control



$\mathcal{E}_\theta = \mathcal{U}_c \circ \mathcal{E}_\theta'$

Example: Singularity of HNKS

- Estimate θ from N repeated applications of Z-rotation with dephasing noise:

$$K_1^\theta = \sqrt{1-p}\, e^{-\mathrm{i}\phi\sigma_z/2}, K_2^\theta = \sqrt{p}\,\sigma_z e^{-\mathrm{i}\phi\sigma_z/2}$$ $0 \le p \le 1, \phi$ are functions of θ

- $H = \mathrm{i}\sum_j K_j^{\theta\dagger}\dot{K}_j^\theta = \frac{\dot{\phi}\sigma_z}{2}$, $\mathcal{S} = \mathrm{Span}_{\mathrm{Herm}}\{K_i^{\theta\dagger}K_j^\theta\}$, HNKS condition *[Zhou & Jiang PRX Quantum'18]*

  so achieving HL (without ancilla constraints) $\Leftrightarrow H \notin \mathcal{S} \Leftrightarrow \dot{\phi} \neq 0$ and $p = 0$ or 1
- However, HNKS is ill-defined at $p = 0$ or 1
- T_θ has 4 eigenvalues: $\{\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = (1-2p)e^{-\mathrm{i}\phi}, \lambda_4 = (1-2p)e^{\mathrm{i}\phi}\}$
- $|\lambda_3| = |\lambda_4| = 1$ (peripheral eigenvalues) if $p = 0$ or 1
- $|\dot{\lambda}_3| = |\dot{\lambda}_4| \neq 0$ if $\dot{\phi} \neq 0$ or $\dot{p} \neq 0$, at $p = 0$ or 1
- Choose, e.g., $\rho_0 = \frac{I}{2} + \alpha(|0\rangle\langle 1| + |1\rangle\langle 0|)$ for achieving HL, even when $\dot{\phi} = 0$ but $\dot{p} \neq 0$ at $p = 0$ or 1

  $R_1 = \rho_*$ R_3 R_4

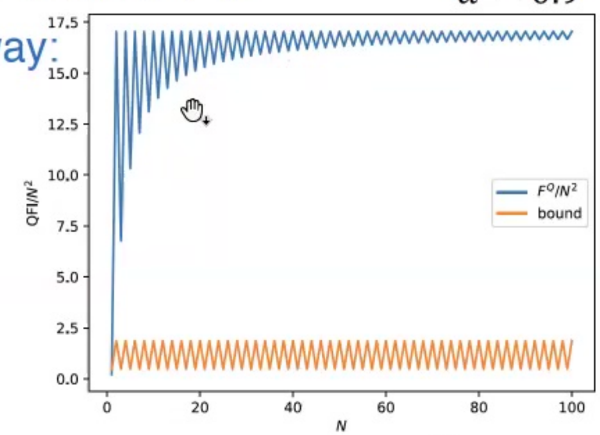
Example: HL beyond QEC

- Estimate θ from repeated applications of the qutrit channel $(0 \le \theta \le 1/2)$: *(Inspired by [Albert Quantum'19])*
  $K_1 = |2\rangle\langle 0|, K_2 = |2\rangle\langle 1|, K_3 = \sqrt{2\theta}|2\rangle\langle 2|, K_4 = \sqrt{1/2-\theta}|0\rangle\langle 2|, K_5 = \sqrt{1/2-\theta}|1\rangle\langle 2|$
- $T = \sum_i K_i^\theta \otimes K_i^{\theta *}$ has 2 nonzero eigenvalues: $\lambda_1 = 1,\ \lambda_2 = -1 + 2\theta$
- 2 nonorthogonal eigenmatrices: $R_1 = \mathrm{diag}\left\{\frac{1-2\theta}{4}, \frac{1-2\theta}{4}, \frac{1}{2}\right\}, R_2 = \mathrm{diag}\left\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{2}\right\}$
- At $\theta = 0$, choose an input state $\rho_0 = R_1 + \alpha R_2$ for some $-1 < \alpha < 1$ and $\alpha \neq 0$
- We have an Heisenberg scaling lower bounded in an oscillating way:

$$\lim_{N\to\infty} \frac{F^Q(\rho_\theta)}{N^2} \ge \frac{\mathrm{Tr}(\rho_\theta^2)}{4\lambda_{\max}(\rho_\theta)} \frac{\tilde{F}^Q(\tilde{\rho}_\theta)}{N^2} = \begin{cases} \frac{8\alpha^2}{(3\alpha^2+2\alpha+3)(1+\alpha)}, \text{odd } N \\ \frac{16\alpha^2}{(3\alpha^2-2\alpha+3)(1+\alpha)}, \text{even } N \end{cases}$$

(when $1/3 < \alpha < 1$)

$\alpha = 0.9$



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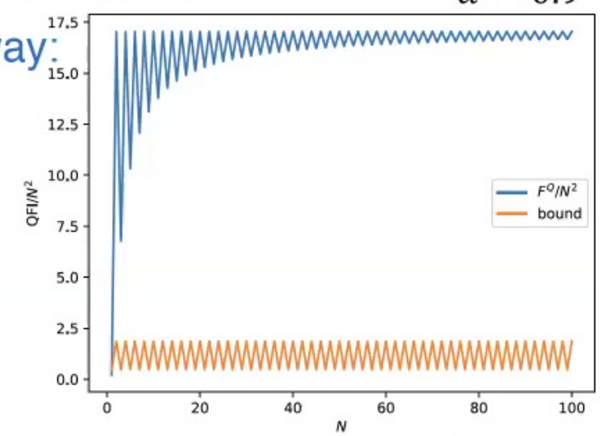
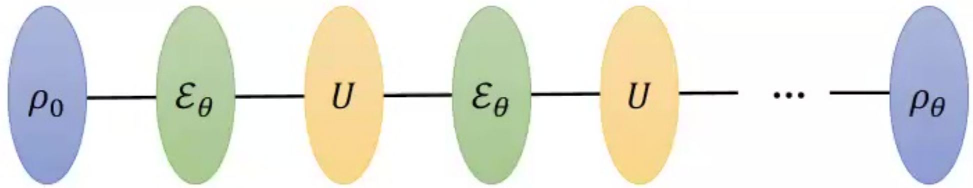


> ➢ No decoherence-free subspace (DFS)
> ➢ Singularity of HNKS

Achieving HL with unitary control



- Theorem 2 (sufficient condition for achieving HL with unitary control):

For a unital channel $T_\theta = \sum_i K_i^\theta \otimes K_i^{\theta *}$, let P be the projection on subspace of eigenmatrices of $T_\theta^\dagger T_\theta$ with eigenvalues 1. If at $\theta = \theta_0$,

(i) $P T_\theta^\dagger \dot{T}_\theta P \neq 0$ is a normal operator (non-vanishing signal condition)

(ii) there exists a unitary U_c such that for some eigenmatrix R_0 of $P T_\theta^\dagger \dot{T}_\theta P$,

$$\sum_k K_k^{\theta_0} R_0 K_k^{\theta_0 \dagger} = U_c^\dagger R_0 U_c, \quad \text{(unitary equivalence condition)}$$

then HL is achievable asymptotically by inserting U after each application of T_θ.

- Such U_c can be identified by Algorithm 1 whenever it exists

Identify the unitary control $\sum_k K_k^{\theta_0} R_0 K_k^{\theta_0\dagger} = U_c^\dagger R_0 U_c$

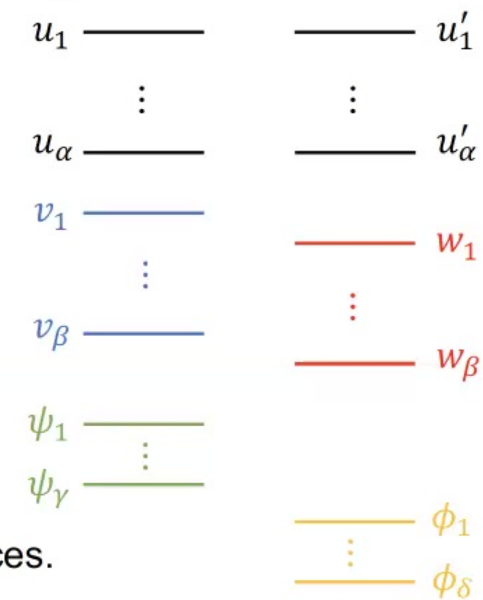
- Reformulate unitary equivalence condition:

$$\sum_k K_k^{\theta_0} R_i K_k^{\theta_0\dagger} = U_c^\dagger R_i U_c, \forall i = 1,2$$

for $R_1 = R_0 + R_0^\dagger, R_2 = \mathrm{i}(R_0 - R_0^\dagger)$ (except for trivial case where R_0 is proportional to a Hermitian matrix)

- Apply the spectral decomposition to R_1 and R_2, and obtain a set of subspaces
- Use principal angles between subspaces F and G to characterize the geometrical relations (arccosine values of the singular values of $\Pi_F \Pi_G$)
- Key idea: find U_c that preserves all these principal angles

Algorithm sketch: reduction of subspaces



- It is possible to choose two sets of orthonormal basis vectors

$V_F = \{|u_1\rangle, \ldots, |u_\alpha\rangle, |v_1\rangle, \ldots, |v_\beta\rangle, |\psi_1\rangle, \ldots, |\psi_\gamma\rangle\}$ for F, and

$V_G = \{|u'_1\rangle, \ldots, |u'_\alpha\rangle, |w_1\rangle, \ldots, |w_\beta\rangle, |\phi_1\rangle, \ldots, |\phi_\delta\rangle\}$ for G,

such that any vector in V_F and any vector in V_G are orthogonal, except for

$$\langle v_i|w_i\rangle = \cos(\eta_i), i = 1, \ldots, \beta$$
$$\langle u|u'_i\rangle = 1,, i = 1, \ldots, \alpha$$

- We can thus obtain a new set of subspaces with lower dimensions.
- We only need to characterize the principal angles between these new subspaces.
- After iterative reduction, subspaces either (i) have the same dimensions and have degenerate principal angles or (ii) mutually orthogonal.
- By choosing a uniquely determined canonical order of basis vectors for each subspace, we only need to find U_c that transforms between two sets of vectors.

Example: noisy Heisenberg model

- Estimate θ from N copies of two-qubit unital channel $T_\theta = T^{(\mathrm{noise})} U_t(\theta) \otimes U_t(\theta)^*$,

where $U_t(\theta) = e^{-\mathrm{i}tH_0(\theta)}, H_0(\theta) = \sigma_z \otimes I + I \otimes \sigma_z + \theta H_J, H_J = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$

- In our approach, T_θ is only required to be given in its matrix form, without the need to explicitly have a (non-unique) Kraus representation (suitable for experimental data)
- For simplicity of presentation, we present one possible Kraus representation for the noise $T^{(\mathrm{noise})}$:

$$K_1^{(\mathrm{noise})} = \sqrt{1 - p_1 - p_2 - p_3} W_1, \qquad K_2^{(\mathrm{noise})} = \sqrt{p_1} W_2 (\sigma_x \otimes \sigma_x),$$

$$K_3^{(\mathrm{noise})} = \sqrt{p_2} W_3 (\sigma_x \otimes \sigma_y), \qquad K_4^{(\mathrm{noise})} = \sqrt{p_3} W_4 (I \otimes \sigma_z),$$

$$W_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\mathrm{i}\phi_i} & 0 & 0 \\ 0 & 0 & 0 & e^{\mathrm{i}\phi_i} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \forall i = 1,2,3,4$$

Example: noisy Heisenberg model

- Estimate θ from N copies of two-qubit unital channel $T_\theta = T^{(\mathrm{noise})} U_t(\theta) \otimes U_t(\theta)^*$,

where $U_t(\theta) = e^{-\mathrm{i}tH_0(\theta)}, H_0(\theta) = \sigma_z \otimes I + I \otimes \sigma_z + \theta H_J, H_J = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$

- In our approach, T_θ is only required to be given in its matrix form, without the need to explicitly have a (non-unique) Kraus representation (suitable for experimental data)
- For simplicity of presentation, we present one possible Kraus representation for the noise $T^{(\mathrm{noise})}$:

$$K_1^{(\mathrm{noise})} = \sqrt{1 - p_1 - p_2 - p_3} W_1, \qquad K_2^{(\mathrm{noise})} = \sqrt{p_1} W_2 (\sigma_x \otimes \sigma_x),$$

$$K_3^{(\mathrm{noise})} = \sqrt{p_2} W_3 (\sigma_x \otimes \sigma_y), \qquad K_4^{(\mathrm{noise})} = \sqrt{p_3} W_4 (I \otimes \sigma_z),$$

$$W_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\mathrm{i}\phi_i} & 0 & 0 \\ 0 & 0 & 0 & e^{\mathrm{i}\phi_i} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \forall i = 1,2,3,4$$

> ➢ Signal and noise do not commute
> ➢ No subspace stabilized by the noise

Example: noisy Heisenberg model

- $PT_\theta^\dagger \dot{T}_\theta P = -\mathrm{i}tPT_\theta^\dagger T_\theta(H_J \otimes I - I \otimes H_J^T)P$ has two nonzero eigenvalues, with mutually orthogonal eigenmatrices

$$R_1 = U_t(\theta)^\dagger(|01\rangle\langle 11| + |10\rangle\langle 00|)U_t(\theta), \qquad R_2 = R_1^\dagger = U_t(\theta)^\dagger(|00\rangle\langle 10| + |11\rangle\langle 01|)U_t(\theta)$$

- Applying Algorithm 1 to $T_\theta = T^{(\mathrm{noise})}U_t(\theta) \otimes U_t(\theta)^*$, we can find a unitary control U_c numerically for achieving the HL: $T_{\theta_0}|R_i\rangle\rangle = U_c^\dagger \otimes U_c^T|R_i\rangle\rangle, \forall i = 1,2$
- To get a more understandable solution, apply Algorithm 1 to $T^{(\mathrm{noise})}$ (assume that we have prior knowledge about the signal and noise separately), i.e., find $\widetilde{U}_c$ s.t.

$$T^{(\mathrm{noise})}|\tilde{R}_i\rangle\rangle = \widetilde{U}_c^\dagger \otimes \widetilde{U}_c^T|\tilde{R}_i\rangle\rangle, \forall i = 1,2$$

for $\tilde{R}_1 = |01\rangle\langle 11| + |10\rangle\langle 00|, \tilde{R}_2 = |00\rangle\langle 10| + |11\rangle\langle 01|$

- Algorithm 1 outputs a simple solution: $\widetilde{U}_c = \mathrm{CNOT}$ ($= W_i$ if $\phi_i = 0$)
- Finally we can choose unitary control $U_c = U_t(\theta_0)^\dagger \widetilde{U}_c$ and input state

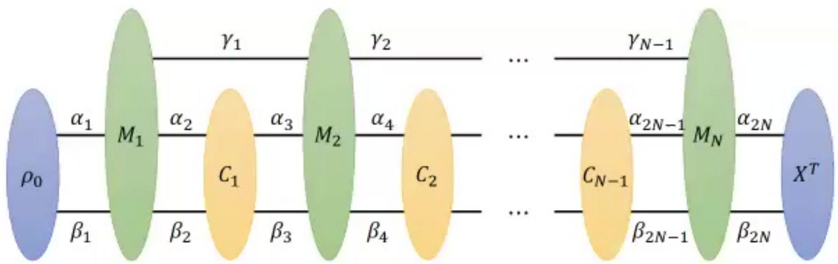
$$\rho_0 = \frac{I}{2} \otimes \frac{I}{2} + \alpha\left(R_1 + R_1^\dagger\right) = \frac{I}{2} \otimes \frac{I}{2} + \alpha U_t(\theta_0)^\dagger(\sigma_x \otimes I)U_t(\theta_0)$$

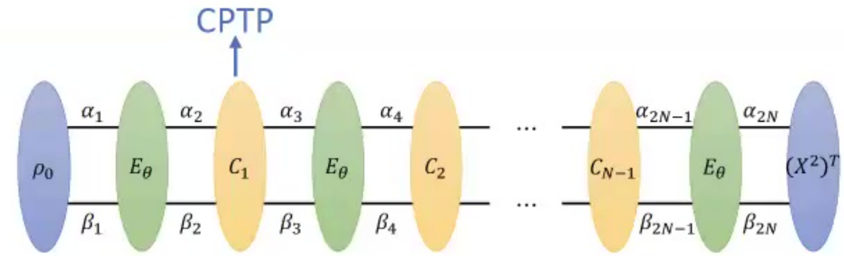
Implications: comparison with QEC

- The example above demonstrates three key distinctions between our approach and QEC:
  - We provide an algorithmic routine to find the probe state and identical unitary control for achieving the HL without ancilla, while such a routine is lacking for ancilla-free QEC.
  - We do not use syndrome measurement, which requires additional noiseless ancilla.
  - It is possible to input all the information into one qubit, while the other qubit can be arbitrarily initialized, and apply $U_t(\theta_0)^\dagger$ for preparing the input state for the HL.

See more details in arXiv:2403.09519

Tensor networks for sequential metrology





- Tensor-network approach has been used for large-scale parallel metrology *[Chabuda, Dziarmaga, Osborne & Demkowicz-Dobrzański NC'20]*
- Variational method for QFI: *[Macieszczak arXiv:1312.1356 (2013)]*

$$F^Q(\rho_\theta) = \sup_X \{\mathrm{Tr}[(\rho_0 * dE_\theta^{\otimes N}/d\theta * \otimes_i C_i)X] - \mathrm{Tr}[(\rho_0 * E_\theta^{\otimes N} * \otimes_i C_i)X^2]\}$$

- Tensor contraction with $dE_\theta^{\otimes N}/d\theta$ requires $O(N)$ operations, by representing $dE_\theta^{\otimes N}/d\theta$ as matrix product operator (MPO) with bond dimension of 2
- Search for the optimal QFI, probe state and control by an iterative optimization method: updating each tensor while fixing others in each step
- Computational complexity of each round of optimization: $O(N^2 d^4)$

See more details in arXiv:2403.09519

Improved metrology for finite N

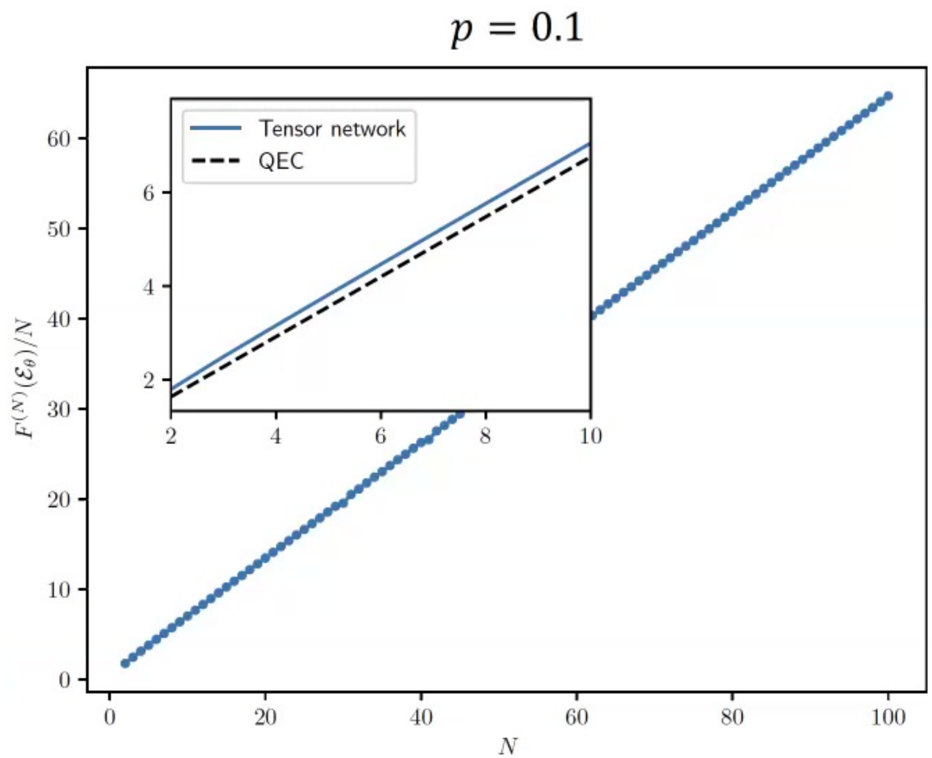
- Consider estimation of N channels

$$K_i^\theta = e^{-\mathrm{i}\theta\sigma_z/2}K_i, \text{ for}$$
$$K_1 = \sqrt{1-p}I, \qquad K_2 = \sqrt{p}\sigma_x$$

- Suppose we have one noiseless ancilla qubit
- HNKS satisfied for $p \neq 1/2$:

$$H = \mathrm{i}\sum_j K_j^\dagger \dot{K}_j = (1-2p)\sigma_z$$

- Our tensor network yields a higher QFI than the asymptotically optimal bound obtained by QEC for finite N up to 100



Summary & Outlook

arXiv:2403.04585

Summary

- A new perspective: bridging quantum metrology and asymptotic theory of quantum channels
- Sufficient conditions for achieving Heisenberg limits with/without identical unitary control
- Applicable to resource-deficient scenarios in sensors with limited programmability

Outlook

- Necessary conditions? Partly done by *[Zhou arXiv:2402.18765 (2024)]* (for single-qubit case)
- Conditions for other bounds such as SQL?
- More applications in real-world scenarios?