Title: Quantum toroidal algebras and spiralling branes

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Series: Mathematical Physics

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Abstract: After briefly reviewing the dictionary between branes of Type IIB string theory and representations of the simplest quantum toroidal algebra, I will present a new class of brane setups which I call spiralling branes. Partition functions of these remarkable configurations reproduce K-theoretic vertex partition function, while on the algebraic side they correspond to Drinfeld twists of R-matrices. Time permitting, I will also discuss other applications of spiralling branes.

Zoom link

Quantum toroidal algebras (& spiralling branes)

① Introduction

Toy model

3d Chern-Simons theory $\Rightarrow$ field theory

$$S = \# \int \mathrm{tr}\left(A\,dA + \tfrac{2}{3}A^3\right)$$

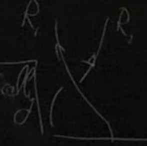
$\Downarrow$

quantum algebra

$$\mathcal{U}_q(\mathfrak{sl}_N)$$

Properties of the algebra

Wilson lines



$= \mathcal{R}$

Type IIB string theory on

$$\mathbb{C}_q \times \mathbb{C}_{t^{-1}} \times \mathbb{C}_{t/q} \times \mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_\tau \times S^1$$

$S \circlearrowleft D3_q$ — —

$S \circlearrowleft D5_{q,t^{-1}}$ — — — —

$NS5_{q,t^{-1}}$ — — — —

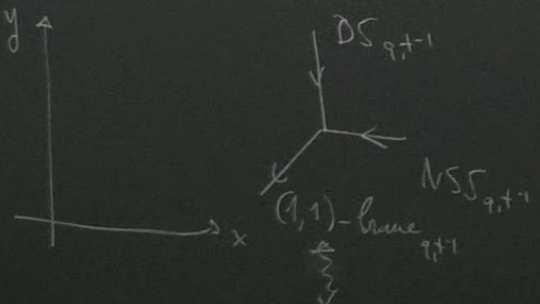
$\Rightarrow$ ① $(q, t^{-1}, t/q)$ enter symmetrically

② $SL(2,\mathbb{Z})$ automorphism

any brane ⇝ a representation of $\mathcal{A} = U_{q,t}(\hat{\hat{gl}}_1)$

Symmetries of the theory ⇝ automorphisms of $\mathcal{A}$.

A brane junction / crossing ⇝ intertwining operators between representations of $\mathcal{A}$



$\Longrightarrow$

$\mathcal{A} \ni e_{n,m} \quad (n,m) \in \mathbb{Z}^2 \supset SL(2,\mathbb{Z})$

Reps corresponding to branes:

① $D3_q$ ⇝ rep on functions of x

$$\rho(e_{n,m}) = x^n q^{m x \frac{\partial}{\partial x}}$$

② 5-branes ⇝ rep on $\mathcal{F}^{(n_1,n_2)}_{q,t^{-1}}(u)$

$\{ a_{-n_1} \cdots a_{-n_k} |\emptyset\rangle \} = \{ |\lambda\rangle \}$

u: spectral parameter

$e_{n,m}$ act as vertex operators on $\mathcal{F}$

$$\Psi : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$$

intertwining operators between representations of $\mathcal{A}$

Refined topological vertex

Awata, Feigin, Shiraishi

$DS_{q,t^{-1}}$

$NS5_{q,t^{-1}}$

$(1,1)$-brane

$\langle \nu | \Psi | \lambda \rangle \otimes | \mu \rangle = C_{\lambda\mu\nu}(q,t)$

$|\lambda\rangle \quad |\mu\rangle \quad |\nu\rangle$

$\mathcal{F}^{(1,0)}_{q,t^{-1}} \otimes \mathcal{F}^{(0,1)}_{q,t^{-1}} \xrightarrow{\Psi} \mathcal{F}^{(1,1)}_{q,t^{-1}}$

$\boxed{\tilde{x} + \tilde{y} - 1 = 0}$

$\Longrightarrow$ explicit expression, exact in q, t/q, etc

$\mathcal{A} \ni e_{n,m} \quad (n,m) \in \mathbb{Z}^2 \circlearrowleft SL(2,\mathbb{Z})$

Reps corresponding to branes

① $D3_q \rightsquigarrow$ rep on functions of x

$\rho(e_{n,m}) = x^n q^{m x \frac{\partial}{\partial x}}$

② 5-branes $\rightsquigarrow$ rep on $\mathcal{F}^{(n_1,n_2)}_{q,t^{-1}}$

$e_{n,m}$ act as vertex operators on $\mathcal{F}$

$\mathcal{F}^{(u)}_{q,t^{-1}}\{ a_{-k_1} \cdots a_{-k_l} |\emptyset\rangle \} = \{ |\lambda\rangle \}$

spectral parameter

Not toy example

$$S_{5dCS} = \frac{1}{\epsilon_2} \int \omega^{(2,0)} \wedge \left(A\,dA + \frac{2}{3} A \star A \star A \right)$$

non-commutative

Ψ can be obtained perturbatively in $\epsilon_2 \sim \log(t/q)$

$\mathbb{R}_t \times \boxed{\mathbb{C}^\times_x \times \mathbb{C}^\times_y} \circlearrowleft SL(2,\mathbb{Z})$

Complexification of $\mathbb{R}_x \times \mathbb{R}_y$

$\omega^{(2,0)} = \frac{d\tilde{x}}{\tilde{x}} \wedge \frac{d\tilde{y}}{\tilde{y}} \qquad \left(\hat{x}\hat{y} = \frac{q}{t} \hat{y}\hat{x} \right)$

$D3_q \longleftrightarrow$ Wilson lines along $\mathbb{R}_t$

5-branes $\longleftrightarrow$ defects lying on curves in $\mathbb{C}^\times_x \times \mathbb{C}^\times_y$

$$\mathbb{C}_q \times \mathbb{C}_{t^{-1}} \times \mathbb{C}_{t/q} \times \boxed{\mathbb{R}_x \times \mathbb{R}_y} \times \mathbb{R}_\tau \times S^1$$

$D3_q$ — —

$D5_{q,t^{-1}}$ — — — —

$NS5_{q,t^{-1}}$ — — — —

(n_1, n_2) 5 brane

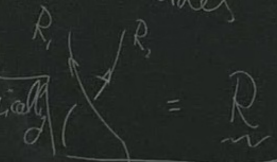
$\Rightarrow$

Properties of the algebra

① $(q, t^{-1}, t/q)$ enter symmetrically

② $SL(2,\mathbb{Z})$ automorphism

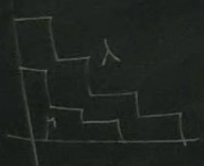
Wilson lines



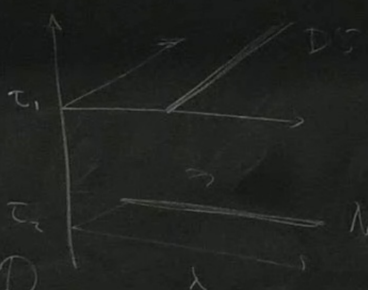
The zoo of branes is very rich, so is the rep theory of A

① Brane crossings

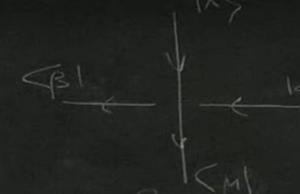
heights of columns of λ



$(\lambda_1 \geq \mu_1) \geq (\lambda_2 \geq \mu_2) \geq (\lambda_3 \geq \mu_3) \geq$



$\Rightarrow$



$\leadsto \mathcal{F}^{(1,0)}_{q,t^{-1}} \otimes \mathcal{F}^{(0,1)}_{q,t/q} \longrightarrow$

$\longrightarrow \mathcal{F}^{(0,1)}_{q,t/q} \otimes \mathcal{F}^{(1,0)}_{q,t^{-1}}$

① Crossing corresponds to R-matrix of A

② R-matrix in these reps is explicitly calculable

(2a) $R_{(q,t^{-1}),(q,t^{-1})}{}^{\lambda}_{\mu} \neq 0$ only when $\lambda \supseteq \mu$

(2b) $R_{(q,t^{-1})(q,t/q)}{}^{\lambda}_{\mu} \neq 0$ only when $\lambda \succeq \mu$

② More brane junctions (all explicitly calculable) (e.g. D3-NS5)

Refined topological vertex

Awata, Feigin, Shiraishi

$D5_{q,t^{-1}}$

$NS5_{q,t^{-1}}$

$(1,1)$-brane

$\langle \nu | \Psi | \lambda \rangle \otimes | \mu \rangle = C_{\lambda\mu\nu}(q,t)$

$|\lambda\rangle \quad |\mu\rangle \quad |\nu\rangle$

$\mathcal{F}^{(1,0)}_{q,t^{-1}}(u) \otimes \mathcal{F}^{(0,1)}_{q,t^{-1}}(v) \longrightarrow \mathcal{F}^{(1,1)}_{q,t^{-1}}(uv)$

$\boxed{\tilde{x} + \tilde{y} - 1 = 0}$

$\Longrightarrow$ explicit expression, exact in q, t/q, etc

$\mathcal{A} \ni e_{n,m} \quad (n,m) \in \mathbb{Z}^2 \circlearrowright SL(2,\mathbb{Z})$

Reps corresponding to branes

① $D3_q \rightsquigarrow$ rep on functions of x

$n \quad m \, x\frac{\partial}{\partial x}$

$\rho(e_{n,m}) = x^n q^{m x \frac{\partial}{\partial x}}$

② 5-branes $\rightsquigarrow$ rep on $\mathcal{F}^{(n_1,n_2)}_{q,t^{-1}}(u)$

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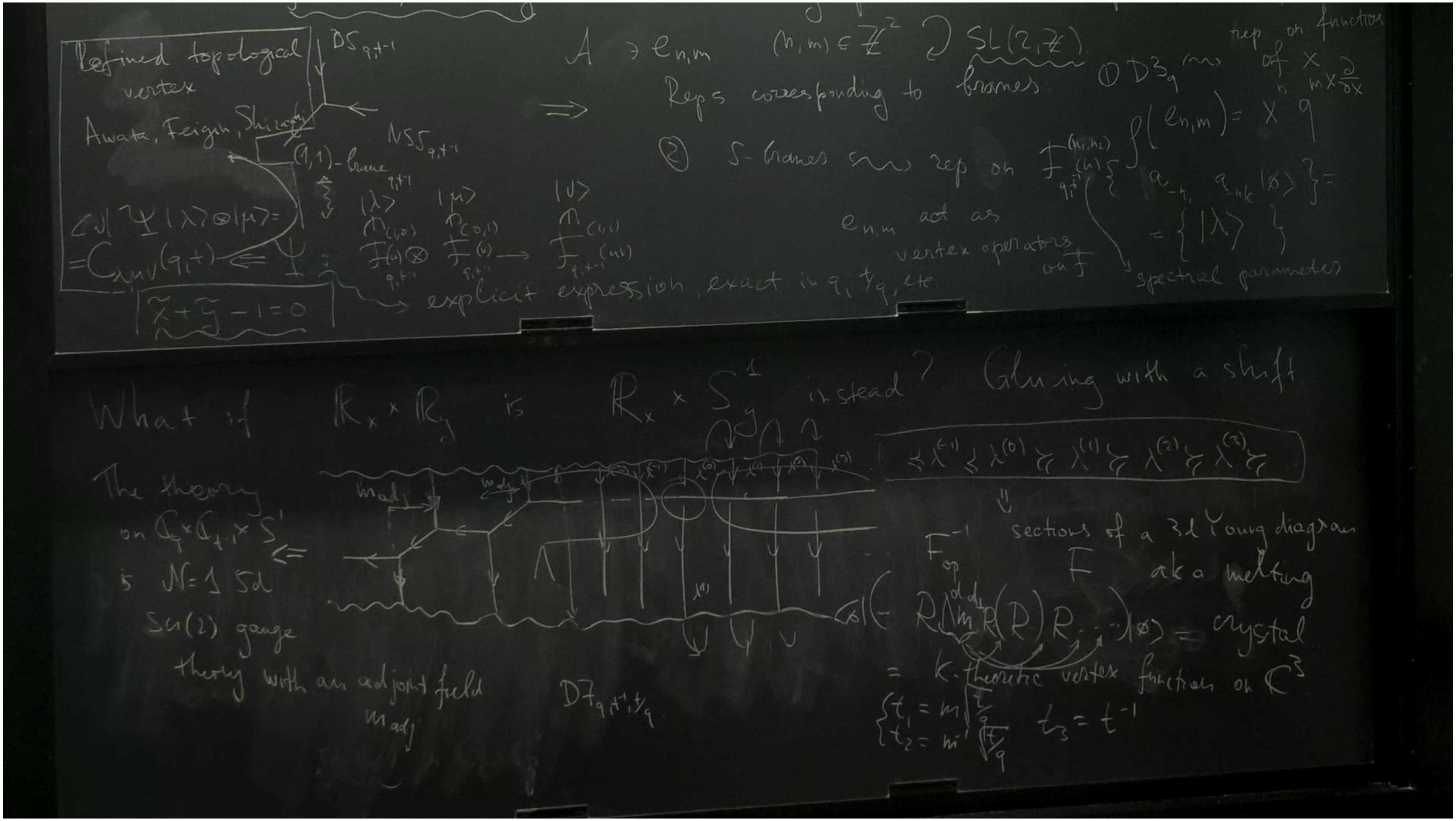
5-branes $\leftrightarrow$ defects lying on curves in $\mathbb{C}^\times_x \times \mathbb{C}^\times_y$


Refined topological vertex
Awata, Feigin, Shiraishi
(1,1)-brane
explicit expression, exact in q, t/q, etc
Reps corresponding to branes
S-branes
rep on functions
e_{n,m} act as vertex operators
spectral parameters
What if R_x × R_y is R_x × S^1_y instead? Gluing with a shift
The theory on C_q × C_{t^-1} × S^1 is N=1 5d SU(2) gauge theory with an adjoint field
m_adj
sections of a 3d Young diagram
aka a melting crystal
= K-theoretic vertex function on C^3