Title: The Delannoy Category

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Series: Mathematical Physics

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Abstract: The prototypical example of a symmetric tensor category is Rep(G) for G a compact group. The other main source of symmetric tensor categories are Deligne's interpolation categories (S_t, GL_t, and O_t) which extend a family of group representation categories defined at positive integers to all complex numbers. Harman and Snowden have developed a theory of symmetric tensor categories coming from Oligomorphic permutation groups together with a well-behaved measure on G-sets. This includes Deligne's S_t which comes from the infinite symmetric group together with a measure where the usual permutation G-set has volume t. The simplest new example of their theory is the group of order-preserving bijections of the real line with the measure given by Euler characteristic. In joint work with Harman and Snowden (arXiv:2211.15392) we give a detailed description of this new symmetric tensor category, which has a number of novel properties. We call this category the Delannoy category because the dimensions of Hom spaces are given by Delannoy numbers. In this talk I'll outline our main results, including the classification of simple objects, the tensor product rules, and a combinatorial model for the category using Delannoy paths.

Zoom link

Pre-Tannakian Categories

"Looks like" Rep(G) Algebraic Group

- Symmetric monoidal
- Has duals "rigid"

usually $\mathbb{C}$ in this talk

- Finite dimensional k-linear Hom spaces
- Abelian. Objects have finite length.
- $\mathbb{1}$ is absolutely simple $\mathrm{End}(\mathbb{1}) = k$

Problem 1

Classify pre-Tannakian categories.

Additional Source: "essentially Tannakian"

Ex Representations of supergroups

Ex Algebraic groups in Ver_p Ostrik et al.

Thm (Deligne) If $\mathcal{C}$ is pre-Tannakian over $\mathbb{C}$ and has moderate growth then $\mathcal{C} \cong \mathrm{Rep}(G)$

length$(X^{\otimes n}) \leq C^n$

supergroup

Any Others? (Deligne, Brauer)

Yes! GL_t, O_t, S_t, etc.

Interpolate between group categories

Concreteness

Concrete: Rep(G), R-mod-R, etc.

Objects have internal mathematical structure

Abstract GL_t, O_t, S_t, TLJ, HOMFLY

Objects are formal. Only use morphisms.

Problem 2

Can Deligne categories be made concrete?

Harman-Snowden:

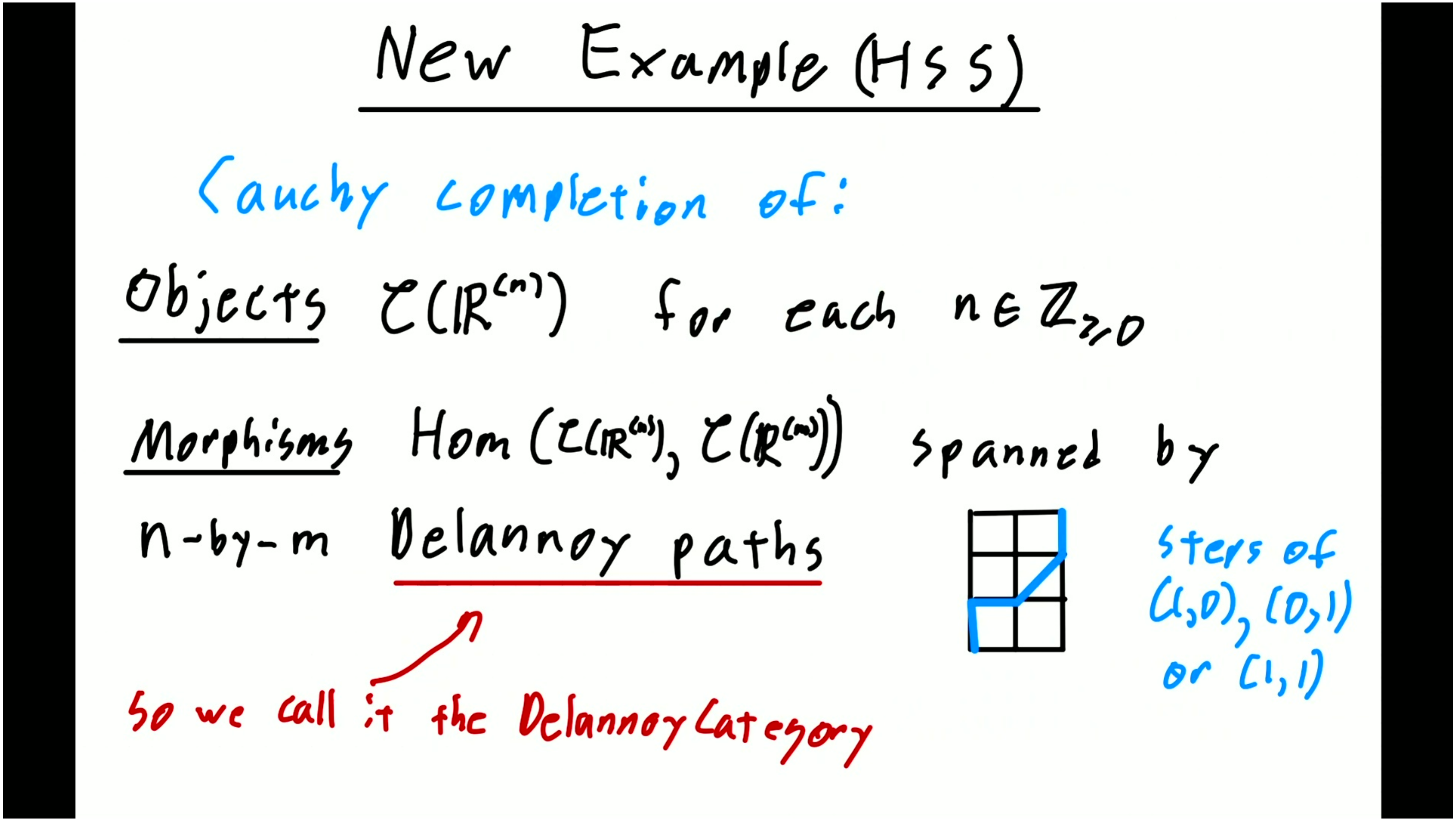
Answer 1: "Discrete" pTCs are almost classified by certain (infinite) groups with a kind of measure. S_t comes from S_∞.

Answer 2: Such $\mathrm{Rep}(G,\mu)$ are concrete

Bonus: This yields many novel examples

Outline

- Rep(G)-like $\otimes$-categories
- The Delannoy Category HSS
- $\otimes$-categories from measures on Oligomorphic groups HS
- Examples HS + HSS



New Example (HSS)

Cauchy completion of:

Objects $\mathcal{C}(\mathbb{R}^{(n)})$ for each $n \in \mathbb{Z}_{\geq 0}$

Morphisms $\mathrm{Hom}(\mathcal{C}(\mathbb{R}^{(n)}), \mathcal{C}(\mathbb{R}^{(m)}))$ spanned by n-by-m Delannoy paths

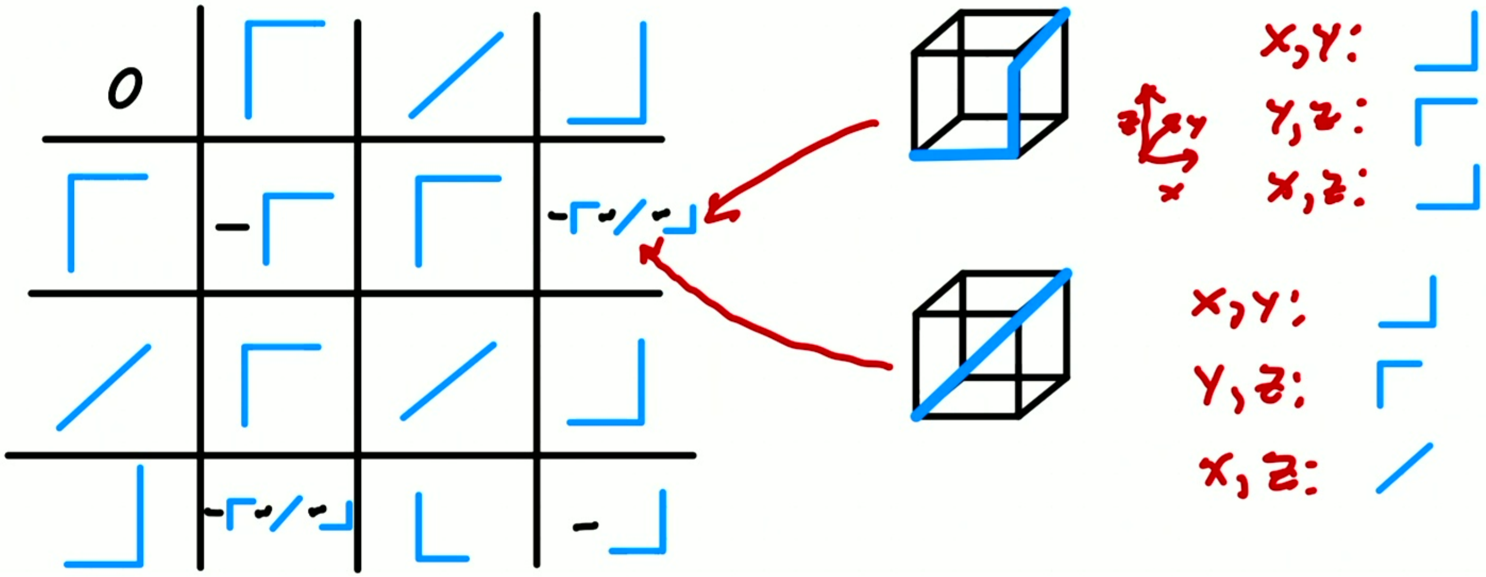
Steps of (1,0), (0,1) or (1,1)

So we call it the Delannoy Category

Composition: $p \circ q = \sum_{\gamma} (-1)^{\cdots} r$

Where γ is a 3-dim Delannoy path which projects down to p, q, and r

Ex



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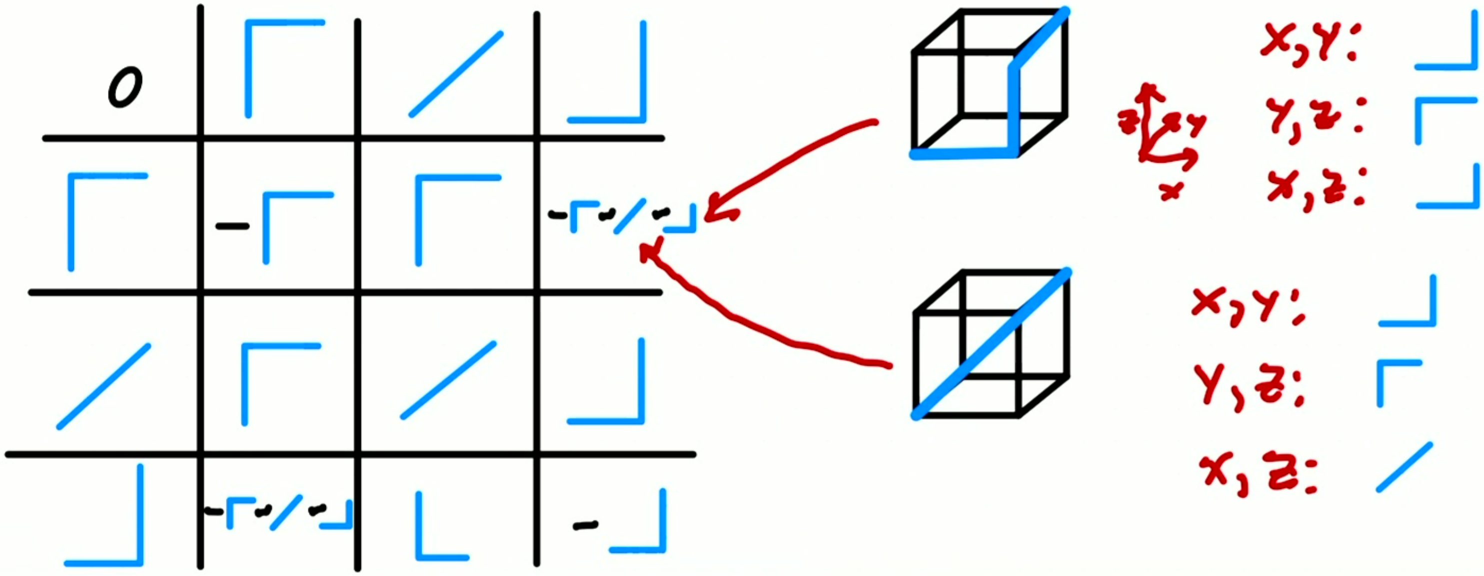


Tensoring objects:

$$\mathcal{C}(\mathbb{R}^{(a)}) \otimes \mathcal{C}(\mathbb{R}^{(b)}) := \bigoplus_{\lambda:(a,b)\text{-Delannoy}} \mathcal{C}(\mathbb{R}^{(\ell(\lambda))})$$

Associator is straightforward

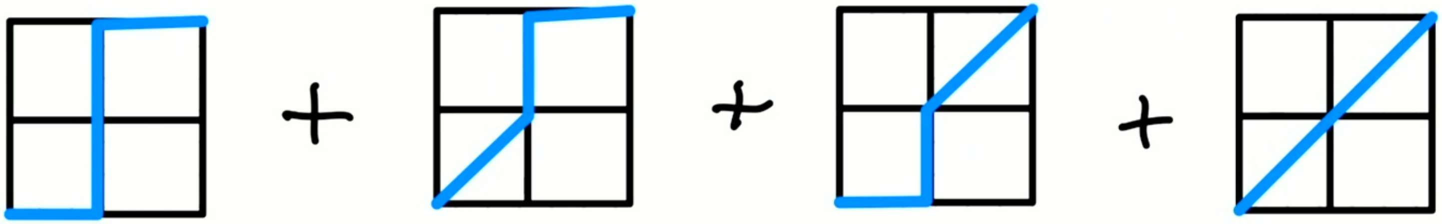
but tensoring morphisms is harder

λ a word in $\bullet, \circ$. Can build objects L_λ:

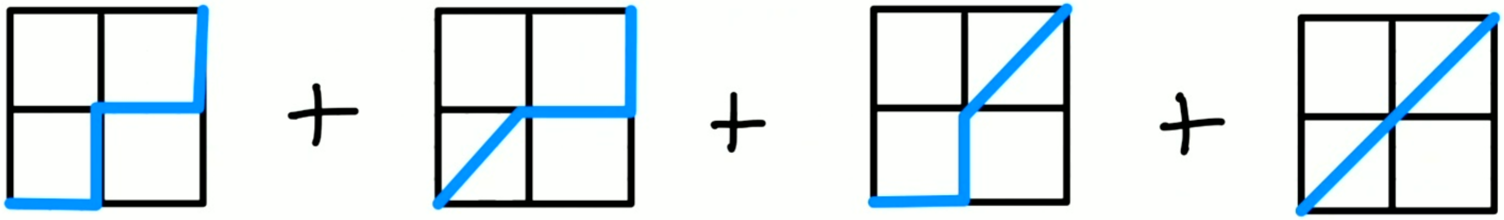
Thm

- L_λ is simple
- $L_\lambda \cong L_\mu$ iff $\lambda = \mu$
- Every simple obj is isom. to some L_λ
- $J_+ \mathrm{Ker}\left(\mathcal{C}(\mathbb{R}^n) \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \mathcal{C}(\mathbb{R}^{n-1})\right) \cong \bigoplus_{\lambda : \mathrm{len}(\lambda) = n} L_\lambda$

__Ex__ $L_{\bullet\circ}$ is the image of the projection



$L_{\bullet\bullet}$ is the image of the projection



Fusion Rules

Thm Fusion rules given by "ruffle product"

$$L_{\bullet} \otimes L_{\bullet\circ} \cong L_{\bullet\bullet\circ} \oplus L_{\bullet\bullet\circ} \oplus L_{\bullet\circ\bullet} \quad \text{"shuffles"}$$

$$\underbrace{\oplus L_{\bullet\circ}}_{\bullet \,\&\, \bullet \text{ collide}} \underbrace{\oplus L_{\bullet\circ} \oplus L_{\bullet\bullet} \oplus L_{\bullet}}_{\bullet \,\&\, \circ \text{ collide}} \quad \text{"ruffles"}$$

Grothendieck Ring

K has basis a_x

Hopf algebra via:

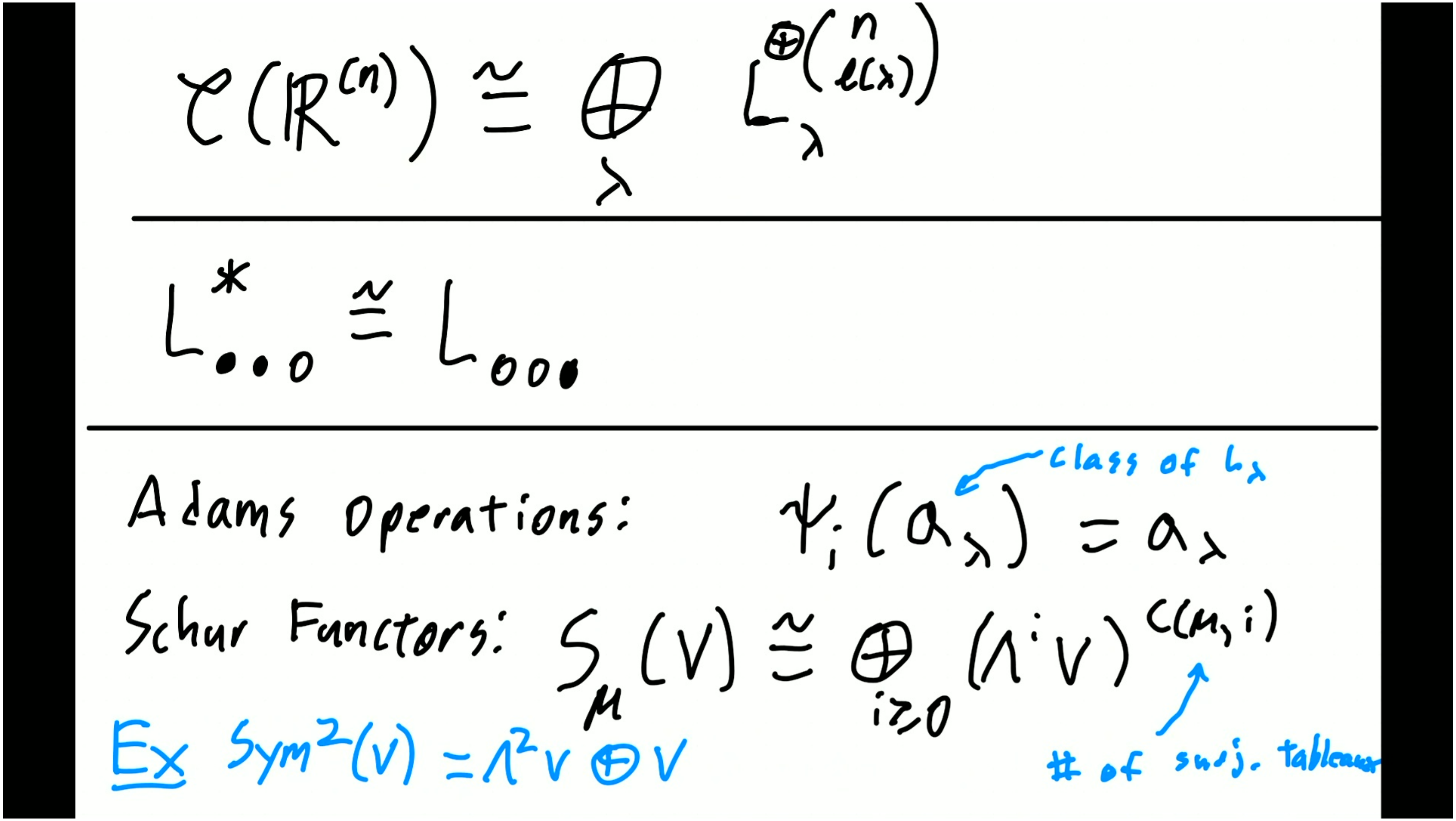
$$\mathrm{res}: K \longrightarrow K \otimes K$$

$$\dim: K \longrightarrow \mathbb{Z}$$

$$a_x \longmapsto (-1)^{\mathrm{len}(x)}$$

Lyndon Word

$\mathbb{Q} \otimes K$ is a polynomial algebra in a_x



$$\mathcal{C}(\mathbb{R}^{(n)}) \cong \bigoplus_{\lambda} L_{\lambda}^{\oplus\binom{n}{\ell(\lambda)}}$$

$$L^{*}_{\bullet\bullet\circ} \cong L_{\circ\circ\bullet}$$

Adams Operations: $\psi_i(a_\lambda) = a_\lambda$

class of L_λ

Schur Functors: $S_\mu(V) \cong \bigoplus_{i\geq 0} (\Lambda^i V)^{c(\mu,i)}$

# of surj. tableau

Ex $\mathrm{Sym}^2(V) \simeq \Lambda^2 V \oplus V$

Novel Properties (HS)

- Doesn't come from (super)groups or from Deligne interpolation.
- In characteristic p it's the first exampl of a semisimple pre-Tannakian cat that does not have finite growth.

But where does it come from?

Idea:

1) Start with a group   Exs S_∞ or $\mathrm{Aut}(\mathbb{R},>)$

2) Define a category $\mathrm{Perm}(G,\mu)$ of permutation representations. Uses a measure on G.

3) Take an abelian envelope

Sometimes this envelope is $\mathrm{Rep}(G,\mu)$ ← Concrete!

Most general version still open.

Under strong conditions can just take Cauchy completion

Oligomorphic Permutation Groups

Group G acting on a set Ω.

G acts on $\underset{\nearrow \text{n-tuples}}{\Omega^n}$ with $\underbrace{\text{finitely many orbits}}_{\text{finitary}}$

G has a topology where the basic open subgroups are stabilizers of pts. in Ω^n

Key idea: Want to also understand restrictions to open subgroups.

Def A $\hat{G}$-set is a set with an action of some open subgroup. Shrinking subgroup doesn't change the $\hat{G}$ set.

Perm(G, μ)

Objects are Vec_X ← formal symbol where X is a finitary and smooth G-set

finitary ← finitely many orbits

smooth ← stabilizers are open subgroups

Ex Ω^n, $\Omega^{(n)}$ ← n element subsets

Morphisms $\mathrm{Hom}(X,Y)$ are G-covariant

"$X\times Y$ matrices" ← Schwartz functions on $X\times Y$

smooth and finitary support

How to compose?

$$B\circ A(x,z) = \int_Y B(x,y)\,A(y,z)\,dy$$

needs a measure

not required to be positive!

Skip?

Schwartz functions

Def $A(G,\mu) = \varprojlim_{U} \mathcal{C}(G/U) = \varprojlim_{U} \varinjlim_{V} \mathcal{C}(V\backslash G/U)$

is an algebra via convolution product.

Def $\underline{\mathrm{Rep}}(G,\mu)$ is the category of smooth A-modules.

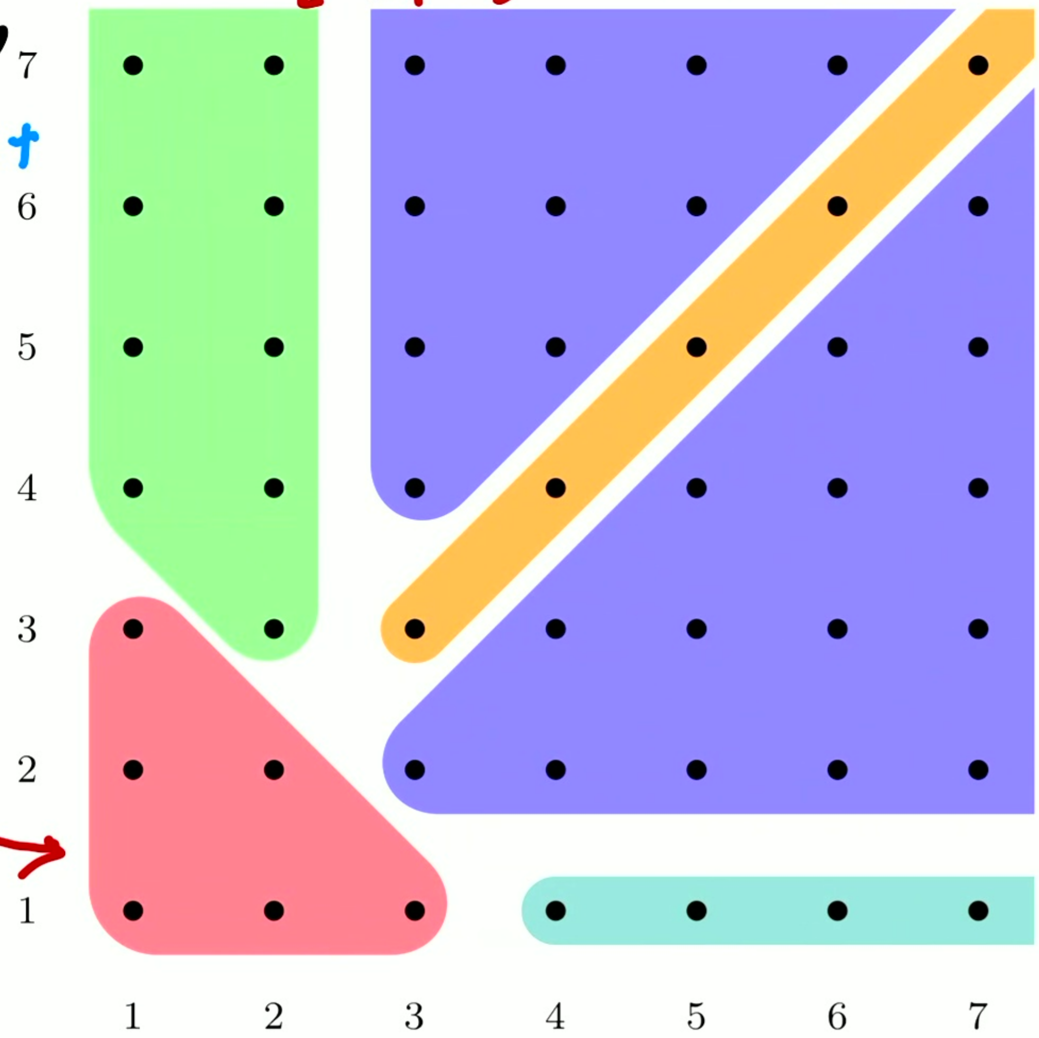
Thm If μ is well-behaved

$\underline{\mathrm{Perm}}(G,\mu) \longrightarrow \underline{\mathrm{Rep}}(G,\mu)$ is an abelian envelop

$X \longmapsto \mathcal{C}(X)$

Ex S_∞

$M(\Omega) = t$



$\leftarrow 2t - 5$

$\leftarrow t - 2$

isom. inv.

mult in fibers

$(t-2)\cdot(t-1)$

$- (t-2)$

$= (t-2)^2$

$6 \rightarrow$

$\leftarrow t - 3$

additivity

Thm (HS) $\exists!$ μ_t with $\mu_t(\Omega) = t$ given by $\mu_t(X) = P_X(t)$ where

$$P_X(n) = \# \underbrace{X^{S(n)}}_{\text{fixed pts}} \quad \text{for } n \gg 0.$$

Thm (HS) If $\text{char}(K) = 0$ then $\underline{\text{Rep}}(S_t)$ recovers Deligne's S_t

Composition

Hom(Ω,Ω) has basis $\delta_{x=y}$, $\delta_{x\neq y}$

	$\delta_{x=y}$	$\delta_{x\neq y}$
$\delta_{y=z}$	$\delta_{x=z}$	$\delta_{x\neq z}$
$\delta_{y\neq z}$	$\delta_{x\neq z}$	$\delta_{x=z}\cdot\overbrace{\mu(\Omega-\{x\})}^{t-1}+\delta_{x\neq z}\cdot\overbrace{\mu(\Omega-\{x,z\})}^{t-2}$

<u>Proj</u> $\frac{1}{t}\left(\delta_{x=y}+\delta_{x\neq y}\right)$

$\frac{t-1}{t}\delta_{x=y}-\frac{1}{t}\delta_{x\neq y}$

Ex $\mathcal{C}(\Omega)$ has submodules

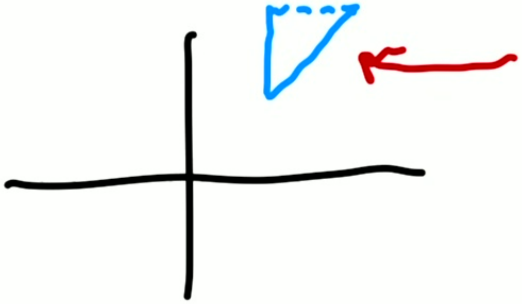
- Constant functions ← image of $\delta_{x=y}$
- Functions with integral 0.

Check that the projections act by 1 and 0 on these subspaces

__Ex__ $\mathrm{Aut}(\mathbb{R}, >)$

Open subsets are $G(A)$ fixing a finite $A \subseteq \mathbb{R}$

There's a measure give by Euler char.



measure $1-2+1 = 0.$

Has nice properties so again get a pre-Tamak

Concrete category $\underline{\mathrm{Rep}}(G)$.

Ex $\mathrm{Hom}(\mathrm{Vec}_{\mathbb{R}}, \mathrm{Vec}_{\mathbb{R}}) = \mathbb{C}^3$

Spanned by $\chi_{x<y}^{:=A}, \chi_{x=y}^{:=I}, \chi_{x>y}^{:=B}$

↑ Characteristic function

These correspond combinatorially to $\lrcorner$, $/$, $\ulcorner$

Check $A \circ B(x,z) = \int_Y A(x,y) \cdot B(y,z)\, dy$

$= \mu(\{y : y > x, z\}) = \mu(\circ\!-\!\!-\!\!-) = 0 - 1 + 0 = -1$

$= -A - B - I.$

Composition: $p \circ q = \sum_{\gamma} (-1)^{\cdots} r$

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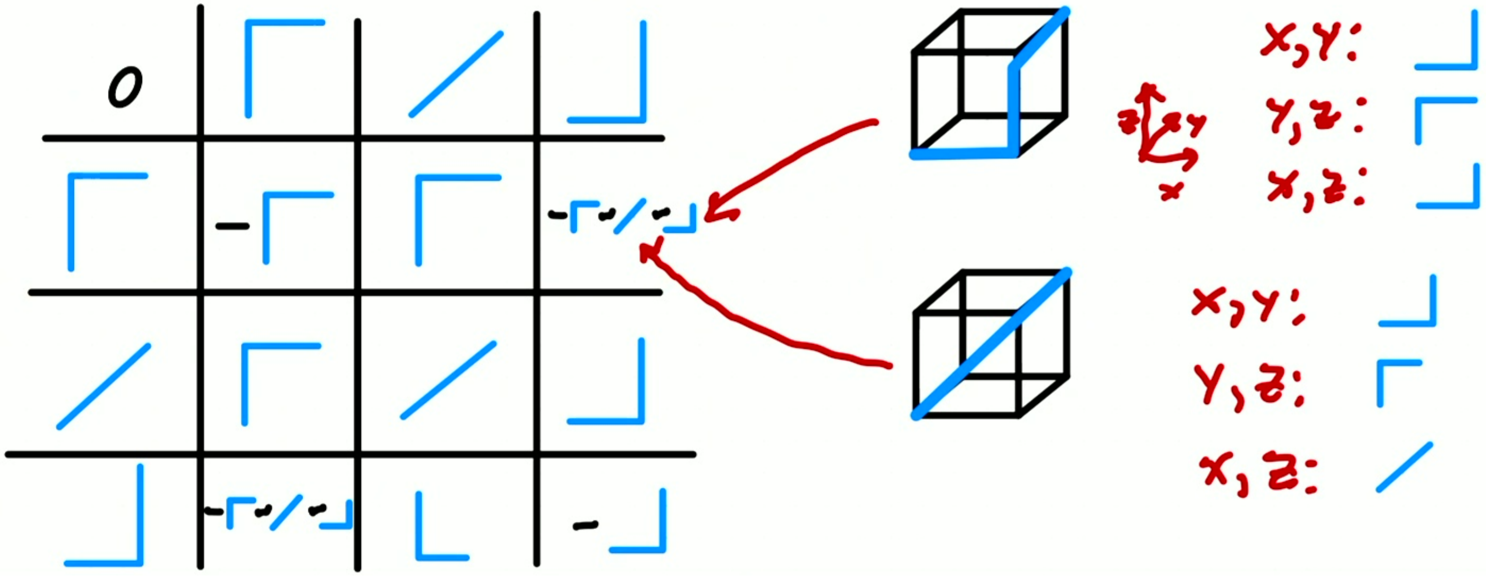
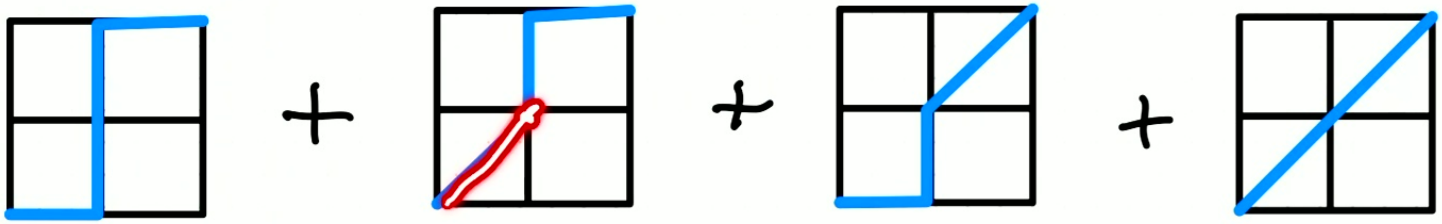
Ex



Ex $L_{\bullet\circ}$ is the image of the projection



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