Title: Rotation symmetry protected boundary modes in Abelian topological phases

Speakers: Abhinav Prem

Series: Quantum Matter

Date: March 12, 2024 - 11:00 AM

URL: https://pirsa.org/24030108

Abstract: Spatial symmetries can enrich the topological classification of interacting quantum matter and endow additional ``weak" topological indices upon systems with non-trivial strong topological invariants (protected by internal symmetries). In this talk, I will discuss the boundary physics of charge conserving systems with a non-zero shift invariant, which is protected by either a continuous U(1) or discrete CN rotation symmetry. In particular, I will discuss an interface between two systems with the same Chern number but distinct shift invariants and show that the interface hosts protected gapless edge modes. For general Abelian topological orders in 2D, I will prove sufficient conditions for gapless edge states protected by continuous rotation symmetry. For the case of discrete rotation symmetries, I will show that the Chern-Simons field theory for systems with gappable edges predicts fractional corner charges. These can also be computed when the system is placed on the two-dimensional surface of a Platonic solid, which relates to the fractional charge bound at disclination defects. Time permitting, I will discuss recent results regarding the relation between the shift and many-body real-space invariants for 2D systems with crystalline symmetry.

Zoom link

Rotation Symmetry Protected Boundary Modes
(in Abelian topological phases)

Abhinav Prem
Institute for Advanced Study

Quantum Matter Seminar, Perimeter Institute

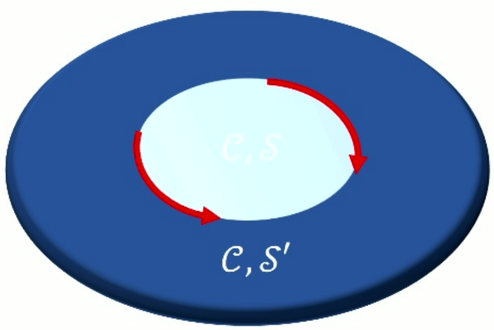
12th March 2024

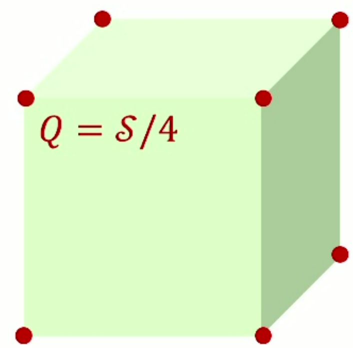
Phys. Rev. B 107, 195130

Motivation

Topological Shift $\mathcal{S}$: new invariant for 2+1D symmetry-enriched topological (SET) orders with rotation symmetry

Find sufficient conditions for protected boundary modes in Abelian SETs with (continuous or discrete) rotation symmetry.



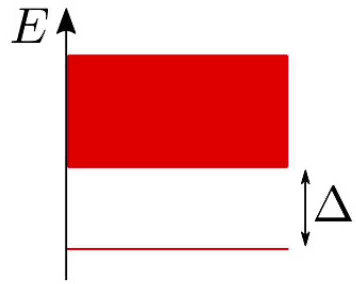


Outline

- General background
- Explicit Example: QHE
- General Theory for Abelian SETs
  - Gapless edge modes for continuous rotation symmetry
  - Fractional corner charges for discrete rotation symmetry
- Open questions

Quantum Many-Body Topology

Quantum many-body system



Gapped

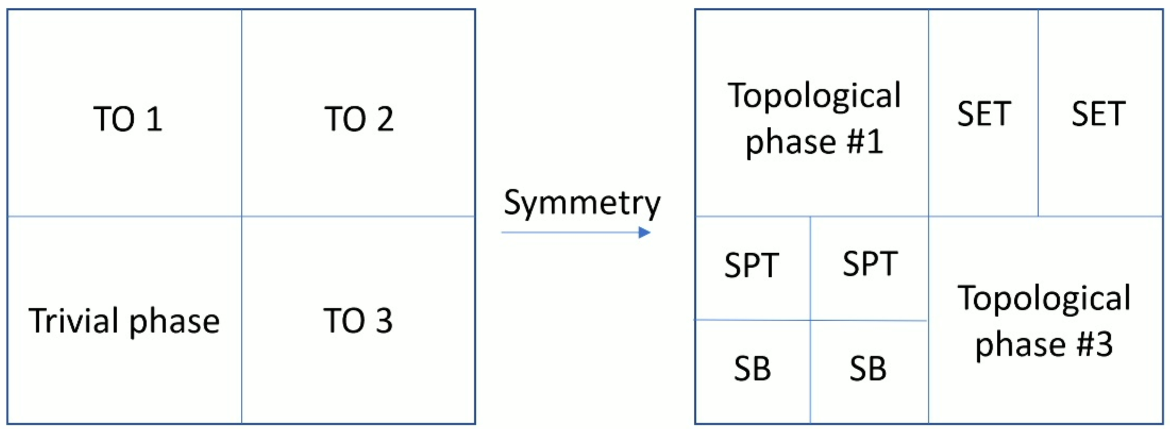
Resource States for
Quantum Info Tasks

Topological phases: structured entanglement in the many-body ground state

Quantized responses
e.g. Hall conductance

Fractionalized excitations

Topological Phases of Matter with Symmetries



TO = topological order

SB = symmetry breaking phases
SPT = symmetry-protected topological phases
SET = symmetry-enriched topological phases

For example:

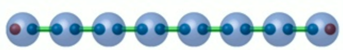
- Topological insulators [protected by time-reversal symmetry]
- Z2 gauge theory [enriched by Z2 symmetry]

Symmetry Protected Topological Phases

T = 0 gapped phases of matter characterised by:

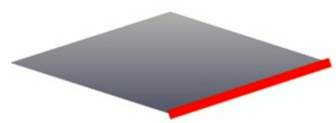
- Symmetry G not spontaneously broken
- Unique short-range entangled ground state
- Only local excitations in the bulk
- Non-trivial boundary signatures
- Classification well-understood for internal G symmetries

d=1



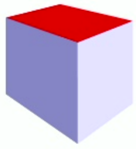
Kitaev chain

d=2



Quantum spin-hall insulator

d=3

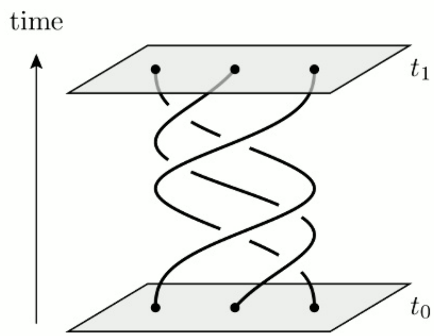


Topological band insulator

Symmetry Enriched Topological Phases

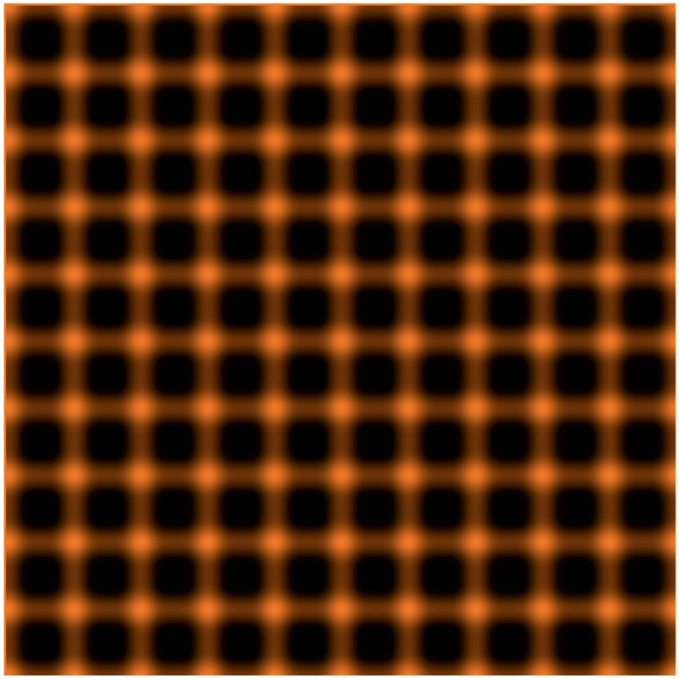
T = 0 gapped phases of matter characterised by:

- Long-range entangled (topologically ordered) ground state(s)
- Enriched by unbroken symmetry G
- Deconfined fractionalized excitations (anyons) in the bulk
- Anyons can carry fractional G-charge
- In 2+1D, classified by G-crossed braided tensor category



- Eg. Quantum spin liquids, FQHE

Topological Crystalline Phases



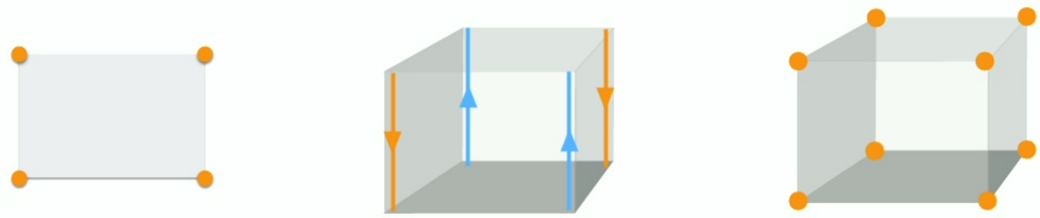
Quantum many-body system

Crystals possess space-group symmetry

- Crystals have *exact* lattice translation symmetry
- Translation symmetry protects topological phase *e.g. weak topological insulators*

Topological Crystalline Phases

Inclusion of crystalline symmetries allows for rich new phases



- Topological crystalline insulators: $\mathrm{Pb}_{1-x}\mathrm{Sn}_x\mathrm{Te}$ and $\mathrm{Pb}_{1-x}\mathrm{Sn}_x\mathrm{Se}$

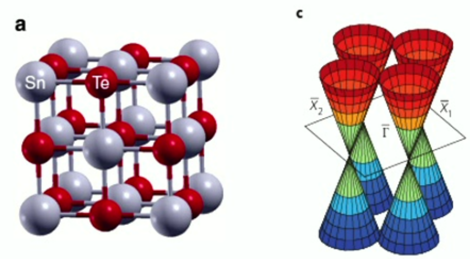
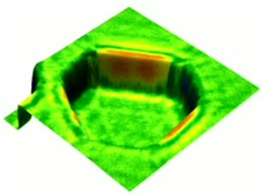


Image:
Greg Fiete, *Nature Materials* **11**, 1003 (2012),

TH. Hsieh, et. al., Nature Commun. 3:982 (2012).

- Higher order topological insulator: Bismuth

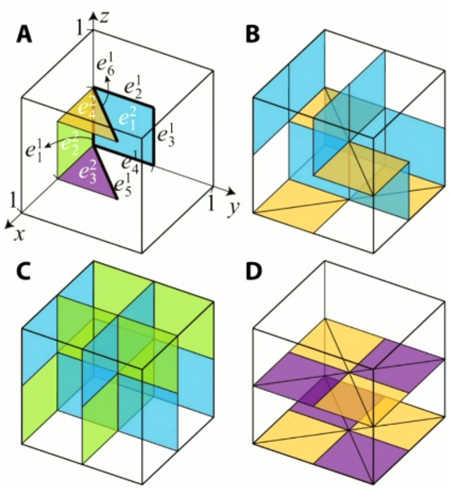


Nature Phys **14,** 918–924 (2018)

Topological Crystalline Phases

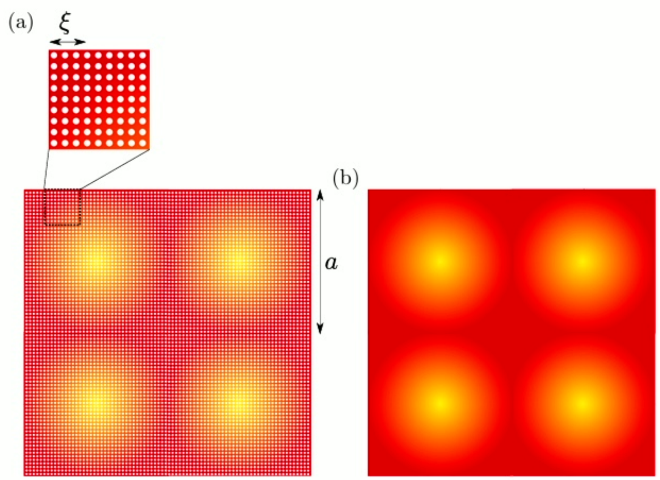
General approaches:

Topological crystals (defect networks)

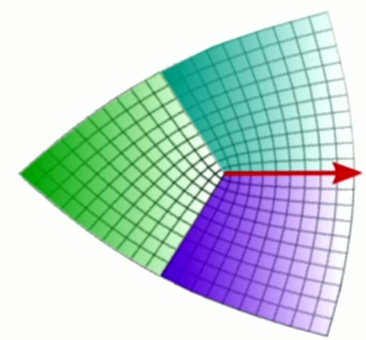


Song, Huang, Fu, Hermele, PRX 2017
Huang, Song, Huang, Hermele, PRB 2017
Song, Huang, Qi, Fang, Hermele, Sci. Adv. 2019

Smooth states



"Gauging" crystalline symmetry



Thorngren and Else, PRX 2018
Else and Thorngren, PRB 2019

Crystalline Equivalence Principle

Crystalline topological phases with space group G

$\longleftrightarrow$

Topological phases with *internal* symmetry G

Thorngren, Else PRX (2018)

For Fermionic theories: fCEP

Fractionalization of G_b on fermions encoded in $\omega_2 \in H^2(G_b, \mathbb{Z}_2)$

A. Debray (2021)
Barkeshli, Chen, Hsin, Manjunath (2022)

$$1 \to \mathbb{Z}_2^f \to G_f \to G_b \to 1$$

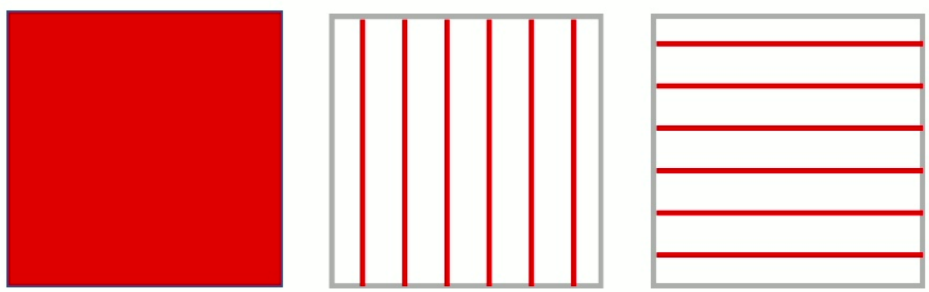
General classification of translation SPTs in crystals

$$\mathcal{Q}_d^{\text{trans}} = \bigoplus_{k=0}^{d} \mathcal{Q}_k^{\times \binom{d}{k}}$$

$\mathcal{Q}_k$ is the classification of "strong" SPTs in k spatial dimensions

For example in d=2:

$$\mathcal{Q} = \mathcal{Q}_2 \bigoplus \mathcal{Q}_1 \bigoplus \mathcal{Q}_1 \bigoplus \mathcal{Q}_0$$



Topological Crystalline Phases

TABLE I. The classification of bosonic SPT phases in (2 + 1)-D protected by space-group symmetries, for each of the 17 2D space groups (sometimes known as "wallpaper groups").

Number	Name	Classification
1	p1	0
2	p2	$\mathbb{Z}_2^{\times 4}$
3	pm	$\mathbb{Z}_2^{\times 2}$
4	pg	0
5	cm	$\mathbb{Z}_2$
6	p2mm	$\mathbb{Z}_2^{\times 8}$
7	p2mg	$\mathbb{Z}_2^{\times 3}$
8	p2gg	$\mathbb{Z}_2^{\times 2}$
9	c2mm	$\mathbb{Z}_2^{\times 5}$
10	p4	$\mathbb{Z}_2 \times \mathbb{Z}_4^{\times 2}$
11	p4mm	$\mathbb{Z}_2^{\times 6}$
12	p4gm	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_4$
13	p3	$\mathbb{Z}_3^{\times 3}$
14	p3m1	$\mathbb{Z}_2$
15	p31m	$\mathbb{Z}_2 \times \mathbb{Z}_3$
16	p6	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_3^{\times 2}$
17	p6mm	$\mathbb{Z}_2^{\times 4}$

Thorngren, Else PRX (2018)

TABLE III. The interacting classification of crystalline TI for 2D interacting fermionic systems. The results for both spinless and spin-1/2 fermions are summarized together. We note that the classifications are the same for those wall paper groups with only one reflection axis. We label the classification indices with fermionic/bosonic root phases with red/blue.

G_b	Spinless	Spin-1/2
$p1$	$\mathbb{Z}$	$\mathbb{Z}$
$p2$	$\mathbb{Z} \times \mathbb{Z}_4^3 \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_4^3 \times \mathbb{Z}_2$
pm	$\mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2$
pg	$\mathbb{Z}$	$\mathbb{Z}$
cm	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$
pmm	$\mathbb{Z} \times \mathbb{Z}_4^3 \times \mathbb{Z}_2^4$	$2\mathbb{Z} \times \mathbb{Z}_2^8$
pmg	$\mathbb{Z} \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$
pgg	$\mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2$
cmm	$\mathbb{Z} \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^2$	$2\mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2^4$
$p4$	$\mathbb{Z} \times \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$p4m$	$\mathbb{Z} \times \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^3$	$2\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2^6$
$p4g$	$\mathbb{Z} \times \mathbb{Z}_8 \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2^2$
$p3$	$\mathbb{Z} \times \mathbb{Z}_3^2 \times \mathbb{Z}_3^3$	$\mathbb{Z} \times \mathbb{Z}_3^2 \times \mathbb{Z}_3^3$
$p3m1$	$\mathbb{Z} \times \mathbb{Z}_3^2 \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_3^2 \times \mathbb{Z}_2$
$p31m$	$\mathbb{Z} \times \mathbb{Z}_3 \times \mathbb{Z}_6$	$\mathbb{Z} \times \mathbb{Z}_3 \times \mathbb{Z}_6$
$p6$	$\mathbb{Z} \times \mathbb{Z}_{12} \times \mathbb{Z}_6 \times \mathbb{Z}_3$	$\mathbb{Z} \times \mathbb{Z}_{12} \times \mathbb{Z}_6 \times \mathbb{Z}_3$
$p6m$	$\mathbb{Z} \times \mathbb{Z}_{12} \times \mathbb{Z}_2^2$	$2\mathbb{Z} \times \mathbb{Z}_6 \times \mathbb{Z}_2^3$

Zhang, Yang, Qi, Gu PRR (2018)

Rich set of invariants in interacting SPTs with lattice symmetry

Topologically Ordered States with Spatial Symmetry

CEP/fCEP extends to SETs, but must be studied case by case.

Recent work on Abelian TOs (incl. FQHE) with

$$G_b = U(1) \times G_{space}, G_{space} = \mathbb{Z} \rtimes \mathbb{Z}_M \ (M = 1,2,3,4,6)$$

TQFT description in terms of "crystalline gauge fields"

- Predicts lattice versions of known continuum invariants & new invariants specific to the lattice
- Non-perturbative (goes beyond band theory)

Manjunath, Barkeshli arXiv (2020), PRR (2021)

Topologically Ordered States with Spatial Symmetry

CEP/fCEP extends to SETs, but must be studied case by case.

Recent work on Abelian TOs (incl. FQHE) with

$$G_b = U(1) \times G_{space}, G_{space} = \mathbb{Z} \rtimes \mathbb{Z}_M \; (M = 1,2,3,4,6)$$

$$\mathcal{L} = -\frac{1}{4\pi} a^I \cup K_{IJ} da^J + \mathcal{L}_{\text{frac}} + \mathcal{L}_{\text{SPT}},$$

$$\mathcal{L}_{\text{frac}} = \frac{1}{2\pi} a^I \cup (q_I dA + s_I dC + \vec{t}_I \cdot d\vec{\mathcal{R}} + m_I A_{XY}),$$

$$\mathcal{L}_{\text{SPT}} = \frac{k_1}{2\pi} A \cup dA + \frac{k_2}{2\pi} A \cup dC + \frac{k_3}{2\pi} C \cup dC + \frac{1}{2\pi} A \cup (\vec{k}_4 \cdot d\vec{\mathcal{R}}) + \frac{1}{2\pi} C \cup (\vec{k}_5 \cdot d\vec{\mathcal{R}}) + \left(\frac{k_6}{2\pi} A + \frac{k_7}{2\pi} C\right) \cup A_{XY}.$$

Manjunath, Barkeshli arXiv (2020), PRR (2021)

Topologically Ordered States with Rotation Symmetry

- What response properties do these new invariants encode?
  - In the bulk, these imply non-trivial charge bound at lattice defects (disclinations/dislocations)

- What non-trivial boundary physics do they encode for generic interacting systems?

Topologically Ordered States with Rotation Symmetry

Consider only systems with $U(1)_c \times U(1)_r$ or $U(1)_c \times \mathbb{Z}_M \quad (M = 1,2,3,4,6)$

Effective Response Theory:

$$\mathcal{L} = \frac{\sigma_{xy}}{4\pi} A \wedge dA + \frac{\mathcal{S}}{2\pi} A \wedge d\omega + \frac{\ell_s}{2\pi} \omega \wedge d\omega + \mathcal{L}_{anomaly} + \cdots$$

A: $U(1)_c$ background gauge field
ω: $U(1)_r$ *or* $\mathbb{Z}_M$ *internal* background gauge field

Topologically Ordered States with Rotation Symmetry

Consider only systems with $U(1)_c \times U(1)_r$ or $U(1)_c \times \mathbb{Z}_M$ $(M = 1,2,3,4,6)$

Effective Response Theory:

$$\mathcal{L} = \frac{\sigma_{xy}}{4\pi} A \wedge dA + \frac{\mathcal{S}}{2\pi} A \wedge d\omega + \frac{\ell_s}{2\pi} \omega \wedge d\omega + \mathcal{L}_{anomaly} + \cdots$$

σ_{xy}: Hall conductance

$\mathcal{S}$: Shift

ℓ_s: ~ OAM

$\frac{\mathcal{S}}{2\pi} A \wedge d\omega$: Wen-Zee term

A: $U(1)_c$ background gauge field
ω: $U(1)_r$ *or* $\mathbb{Z}_M$ *internal* background gauge field

Wen, Zee PRL (1992)

Topologically Ordered States with Rotation Symmetry

Consider only systems with $U(1)_c \times U(1)_r$ or $U(1)_c \times \mathbb{Z}_M$ $\quad (M = 1,2,3,4,6)$

Effective Response Theory:

$$\mathcal{L} = \frac{\sigma_{xy}}{4\pi} A \wedge dA + \frac{\mathcal{S}}{2\pi} A \wedge d\omega + \frac{\ell_s}{2\pi} \omega \wedge d\omega + \cdots$$

Non-invertible states: $\sigma_{xy}, \mathcal{S}$ fractional

Invertible states: $\sigma_{xy} \in \mathbb{Z}\ (2\mathbb{Z}), \mathcal{S} \in \frac{\mathbb{Z}}{2}(\mathbb{Z})$ for fermionic (bosonic) theories.

Topologically Ordered States with Rotation Symmetry

Consider only systems with $U(1)_c \times U(1)_r$ or $U(1)_c \times \mathbb{Z}_M \quad (M = 1,2,3,4,6)$

Effective Response Theory:

$$\mathcal{L} = \frac{\sigma_{xy}}{4\pi} A \wedge dA + \frac{\mathcal{S}}{2\pi} A \wedge d\omega + \frac{\ell_s}{2\pi} \omega \wedge d\omega + \cdots$$

Non-invertible states: $\sigma_{xy}, \mathcal{S}$ fractional

Invertible states: $\sigma_{xy} \in \mathbb{Z}\ (2\mathbb{Z}), \mathcal{S} \in \frac{\mathbb{Z}}{2}(\mathbb{Z})$ for fermionic (bosonic) theories.

On the lattice: $\mathcal{S}$ is only defined mod M (and is origin dependent)

Shift Invariant: Physical Interpretation

$$\mathcal{L} = \frac{\sigma_{xy}}{4\pi} A \wedge dA + \frac{\mathcal{S}}{2\pi} A \wedge d\omega + \frac{\ell_s}{2\pi} \omega \wedge d\omega + \cdots$$

$$\frac{\delta S}{\delta A_0} = \rho = \frac{\sigma_{xy}}{2\pi} dA + \frac{\mathcal{S}}{2\pi} d\omega$$

"Shifts" particle density

$$\frac{\delta S}{\delta \omega_0} = L_z = \frac{\mathcal{S}}{2\pi} dA$$

$U(1)_c$ flux carries non-trivial angular momentum

Shift Invariant: Continuum Limit

$$\mathcal{L} = \frac{\sigma_{xy}}{4\pi}\, A \wedge dA + \frac{\mathcal{S}}{2\pi} A \wedge d\omega + \frac{\ell_s}{2\pi} \omega \wedge d\omega + \cdots$$

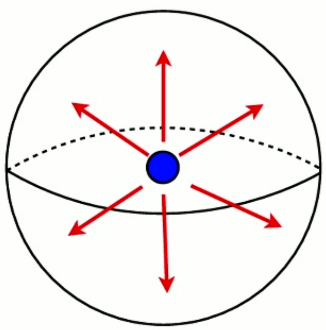
$$\frac{\delta S}{\delta A_0} = \rho = \frac{\sigma_{xy}}{2\pi}\, dA + \frac{\mathcal{S}}{2\pi} d\omega$$

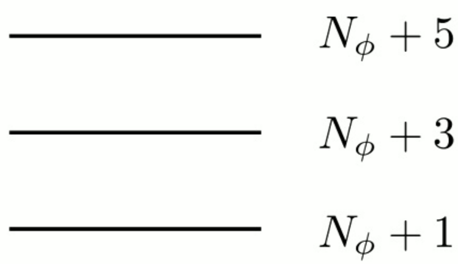
$$\frac{\mathcal{S}}{2\pi} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu \omega_\lambda = \frac{\mathcal{S}}{4\pi} \sqrt{g} A_0 R + \cdots$$

$$Q = \int d^2x \sqrt{g}\, \rho = \int d^2x \sqrt{g} \left(\frac{\sigma_{xy}}{2\pi} B + \frac{\mathcal{S}}{4\pi} R\right) = \sigma_{xy} N_\varphi + S\chi = \sigma_{xy} N_\varphi + 2S\,(1 - g)$$

Shift Invariant: Continuum Landau levels

$$\mathcal{L} = \frac{\sigma_{xy}}{4\pi} A \wedge dA + \frac{\mathcal{S}}{2\pi} A \wedge d\omega + \frac{\ell_s}{2\pi} \omega \wedge d\omega + \cdots$$





<u>IQH States</u>

$\sigma_{xy} = n$ in n^{th} Landau Level

Fill C LLs on the sphere: $Q = \mathcal{C}N_\varphi + \mathcal{C}^2 \Rightarrow \mathcal{S} = \frac{\mathcal{C}^2}{2}$

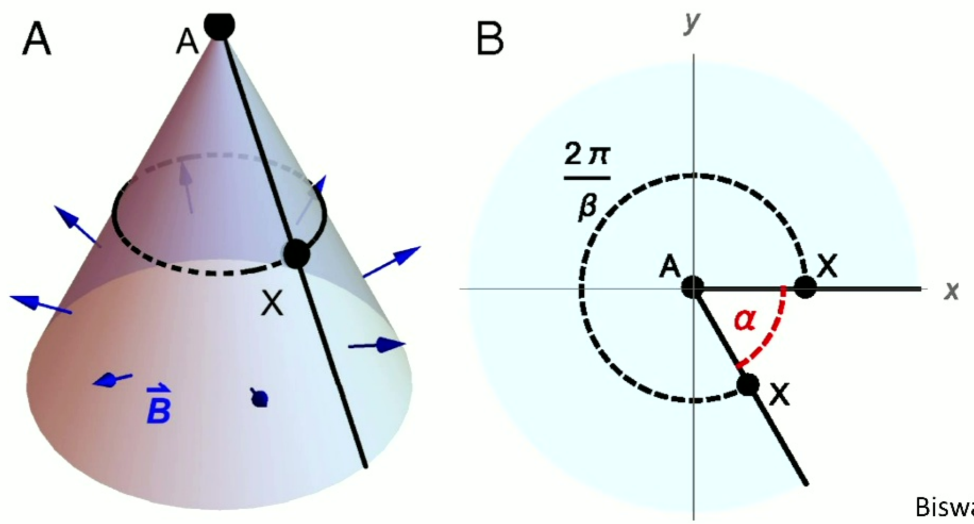
$\mathcal{S}$ within n^{th} Landau Level = $n + \frac{1}{2}$

Shift Invariant: Continuum Limit

- FQH states with $U(1)_r$: Hall viscosity $\eta_H = \frac{\hbar}{4}\mathcal{S}\rho$

Read PRB (2009)

- Implies fractional charge bound to conical defects

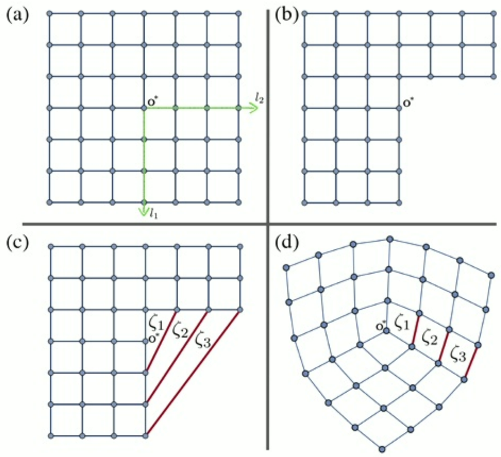


Biswas, Son PNAS (2016)

Discrete Shift

$$\mathcal{L} = \frac{\mathcal{S}}{2\pi} A \wedge d\omega + \cdots$$

- $2\mathcal{S}$ excess charge on surface of cube vs torus at fixed filling
- Assigns fractional charge to elementary disclination of angle $\frac{2\pi}{M}$



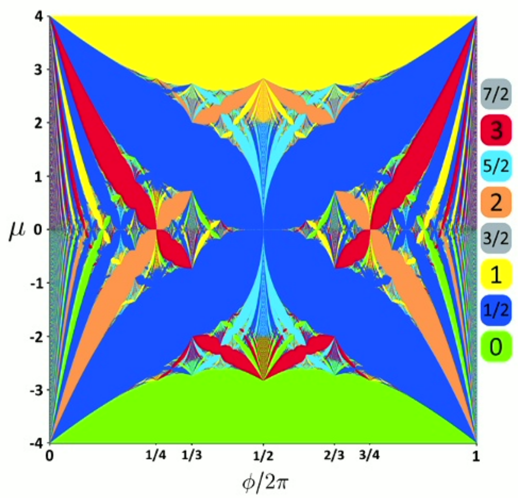


FIG. 1. $\mathcal{S}$ for Hofstadter model, from Eq. (5).

Li, Zhu, Benalcazar, Hughes PRB (2020)
Lu, Vishwanath, Khalaf PRX (2019)
Zhang, Manjunath, Nambiar, Barkeshli PRL (2022), PRX (2023)

Explicit Example: Integer Quantum Hall

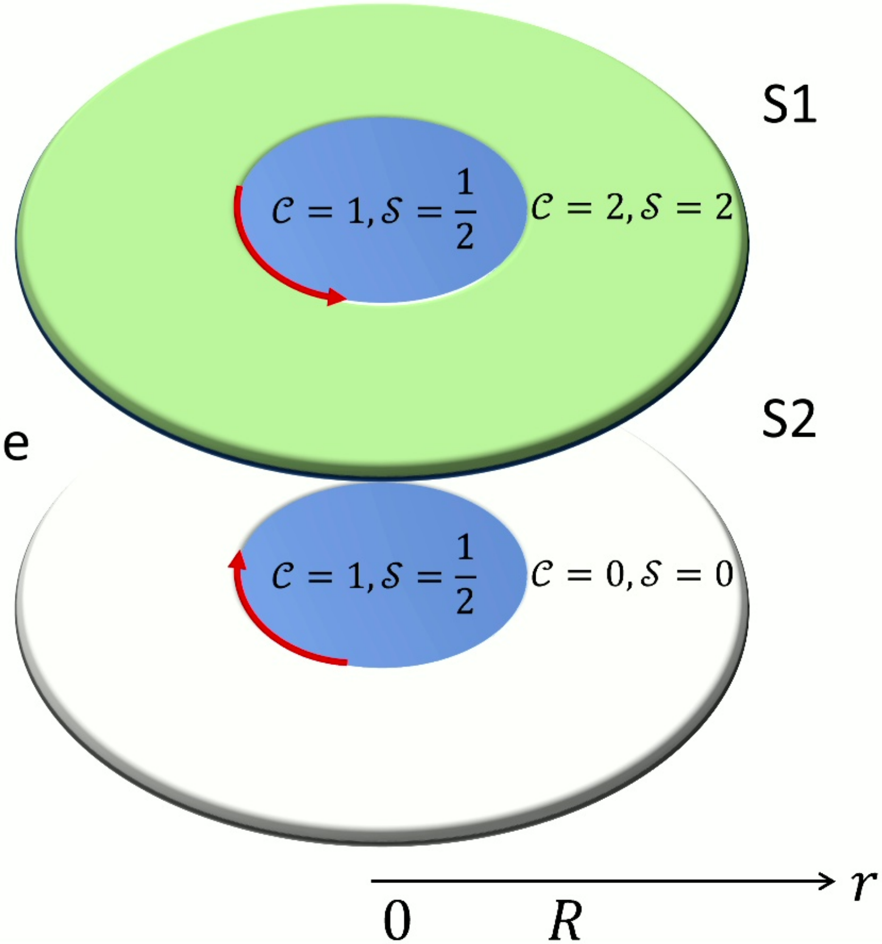
- Can non-zero $\mathcal{S}$ protect edge states?
- Consider interface b/w two systems with same $\mathcal{C}$ but distinct $\mathcal{S}$
- Find $U(1)_c \times U(1)_r$ symmetry protected counter-propagating modes
- Note: Wen-Zee term does not give extra boundary modes

Explicit Example: Integer Quantum Hall

Stack of IQH states: $(\mathcal{C}, \mathcal{S}) = (2,1), \ \ r \ll R$

$(\mathcal{C}, \mathcal{S}) = (2,2), \ \ r \gg R$

- $\mathcal{C}$ increases in S1: chiral edge mode
- $\mathcal{C}$ decreases in S2: anti-chiral edge mode



Explicit Example: Integer Quantum Hall

Stack of IQH states: $(\mathcal{C},\mathcal{S}) = (2,1),\ \ r \ll R$

$(\mathcal{C},\mathcal{S}) = (2,2),\ \ r \gg R$

$$H_i = \frac{(\mathbf{p}_i + e\mathbf{A}_i)^2}{2m} - \mu_i(\mathbf{r}_i) \qquad \mathbf{A}_i = \tfrac{B}{2}(-y_i, x_i, 0)$$

$$|n,m\rangle_i := \frac{(a_i^\dagger)^n (b_i^\dagger)^m}{\sqrt{n!m!}} |0,0\rangle_i$$

$$H = \sum_i H_i = \sum_i \left[\left(a_i^\dagger a_i + \frac{1}{2}\right)\omega_c - \mu_i(\mathbf{r}_i)\right]$$

$$J_i = b_i^\dagger b_i - a_i^\dagger a_i$$

$$\ell_B = 1/\sqrt{eB} = 1/\sqrt{\omega_c}$$

Express radial potential in this basis:

$$\frac{\hat{x} + i\hat{y}}{\sqrt{2}\ell_B} = i(a - b^\dagger)$$

$$\frac{\hat{x} - i\hat{y}}{\sqrt{2}\ell_B} = -i(a^\dagger - b)$$

$$\hat{r}^2 = 2\ell_B^2(1 + a^\dagger a + b^\dagger b - ab - a^\dagger b^\dagger)$$

$$\frac{\hat{r}^2}{\ell_B^2}|n,m\rangle = 2(n+m)|n,m\rangle - 2\sqrt{mn}\,|n-1,m-1\rangle - 2\sqrt{(n+1)(m+1)}\,|n+1,m+1\rangle$$

$$\ell = m - n$$

Explicit Example: Integer Quantum Hall

Stack of IQH states: $(\mathcal{C},\mathcal{S}) = (2,1), \;\; r \ll R$

$(\mathcal{C},\mathcal{S}) = (2,2), \;\; r \gg R$

$$H_i = \frac{(\mathbf{p}_i + e\mathbf{A}_i)^2}{2m} - \mu_i(\mathbf{r}_i) \qquad \mathbf{A}_i \;=\; \tfrac{B}{2}(-y_i, x_i, 0)$$

$$|n,m\rangle_i := \frac{(a_i^\dagger)^n (b_i^\dagger)^m}{\sqrt{n!m!}} \, |0,0\rangle_i$$

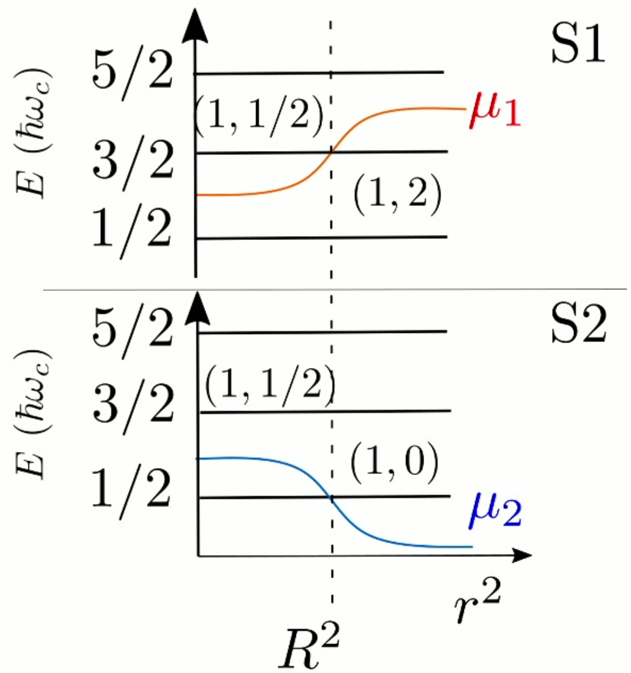
$$H = \sum_i H_i = \sum_i \left[\left(a_i^\dagger a_i + \frac{1}{2} \right) \omega_c - \mu_i(\mathbf{r}_i) \right]$$

$$J_i = b_i^\dagger b_i - a_i^\dagger a_i$$

$$\ell_B = 1/\sqrt{eB} = 1/\sqrt{\omega_c}$$

$$\mu_i(\mathbf{r}_i) = \frac{2 + (-1)^{i+1}}{2} \omega_c + (-1)^{i+1} K \tanh \frac{\hat{r}_i^2 - R_i^2}{\xi^2}$$

$$0 < K < \tfrac{1}{2}\omega_c \text{ and } R_1, R_2 \approx R \gg \ell_B$$



Explicit Example: Integer Quantum Hall

$$H = \sum_i H_i = \sum_i \left[\left(a_i^\dagger a_i + \frac{1}{2}\right)\omega_c - \mu_i(\mathbf{r}_i)\right]$$

$$\mu_i(\mathbf{r}_i) = \frac{2+(-1)^{i+1}}{2}\omega_c + (-1)^{i+1} K \tanh\frac{\hat{r}_i^2 - R_i^2}{\xi^2}$$

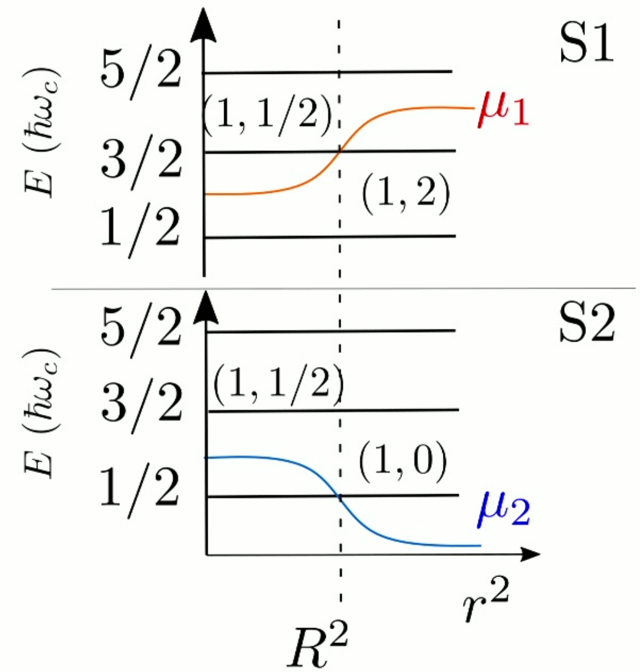
$$0 < K < \tfrac{1}{2}\omega_c \text{ and } R_1, R_2 \approx R \gg \ell_B$$

$r \ll R$: Only LLL below zero energy in S1 and S2

$r \gg R$: n = 0,1 LL below E = 0 (S1) and no LLs below E = 0 (S2)

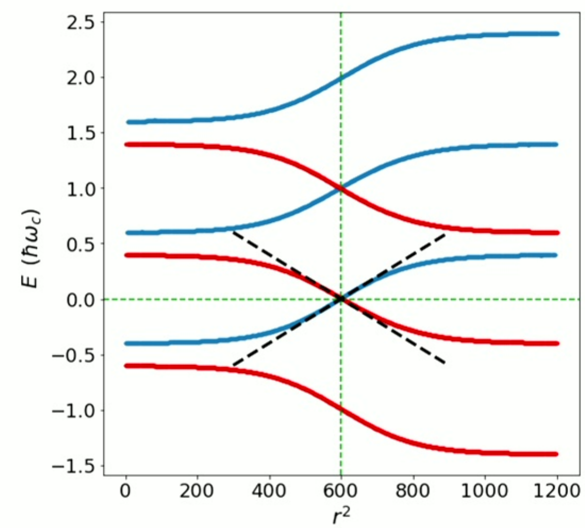
Recall $\mathcal{S}$ within n^{th} Landau Level = $n + \frac{1}{2}$

$$\mathcal{S}^1_{r\ll R} = \mathcal{S}^2_{r\ll R} = \frac{1}{2}; \mathcal{S}^1_{r\gg R} = \frac{1}{2} + \frac{3}{2} = 2; \mathcal{S}^1_{r\gg R} = 0$$

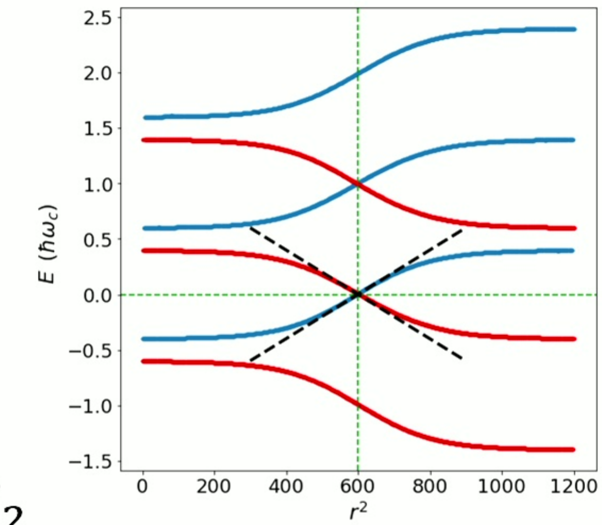


Explicit Example: Integer Quantum Hall

- Non-interacting case:
  - Two zero-energy states localized at $r \sim R$
  - Each state ψ_j has $\ell_j = \langle \psi | J_j | \psi \rangle$



Explicit Example: Integer Quantum Hall



- Non-interacting case:
  - Two zero-energy states localized at $r \sim R$
  - Each state ψ_j has $\ell_j = \langle\psi|J_j|\psi\rangle$

- If $\ell_1 = \ell_2$: can be symmetrically gapped out.

- Numerics: for $R^2 = \langle\psi|r_1^2|\psi\rangle = \langle\psi|r_2^2|\psi\rangle$, $\ell_1 \neq \ell_2$

  for $\langle\psi|r_1^2|\psi\rangle \neq \langle\psi|r_2^2|\psi\rangle$, $\ell_1 = \ell_2$

BUT spatial separation >> localization length: no local term can gap out.

Explicit Example: Integer Quantum Hall

- Heuristic argument for $\xi \gg \ell_B$:

- ψ_i has well-defined quantum numbers n_i^*, m_i^*: $\ell_i^* = m_i^* - n_i^*$

- For any |n,m>

$$_i \langle n,m| \left(\frac{\hat{r}_i^2}{2\ell_B^2} - J_i - 1 \right) |n,m\rangle_i = 2n$$

- This + $\langle \psi_i | \hat{r}_i^2 | \psi_i \rangle = R^2$:

$$\ell_1 - \ell_2 = \langle \psi_1 | J_1 | \psi_1 \rangle - \langle \psi_2 | J_2 | \psi_2 \rangle = 2(n_2^* - n_1^*) = -2.$$

- Solve radial TISE w/in WKB:

$$\psi(r,\theta) = \frac{u(r)e^{i\ell\theta}}{\sqrt{r}}$$

$$\partial_r^2 u(r) - Q_\ell(r) u(r) = 0$$

$$Q_\ell(r) = \left(\frac{m^2\omega_c^2 r^2}{4} + \frac{m\ell\omega_c}{2} - \frac{1/4 - \ell^2}{r^2} - 2m(E + \mu_i(r)) \right)$$

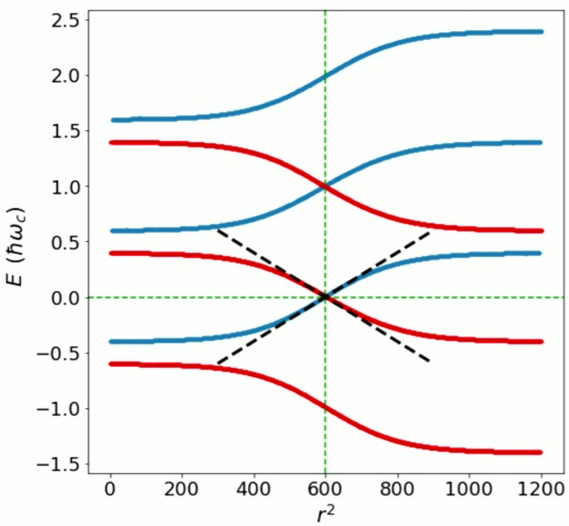
Work in large flux limit: $R \gg \xi \gg \ell_B \sim N_\varphi = BR_i^2 \gg 1$

$$E_{n_0,\ell}^{(i)} \approx \left(n_0 - \frac{1 + (-1)^{i+1}}{2} \right) \omega_c + (-1)^{i+1} \frac{KR_i^2}{\xi^2} \left(\frac{2n_0 + \ell + 1}{N_\phi(R_i)} - 1 \right)$$

exp. localized chiral mode near $r \sim R_i$

$n_0 = 1$ (S1)
$n_0 = 0$ (S2)

Explicit Example: Integer Quantum Hall



- Solve radial TISE w/in WKB:

$$\psi(r,\theta) = \frac{u(r)e^{i\ell\theta}}{\sqrt{r}}$$

$$\partial_r^2 u(r) - Q_\ell(r)u(r) = 0$$

$$Q_\ell(r) = \left(\frac{m^2\omega_c^2 r^2}{4} + \frac{m\ell\omega_c}{2} - \frac{1/4-\ell^2}{r^2} - 2m(E+\mu_i(r))\right)$$

Work in large flux limit: $R \gg \xi \gg \ell_B \sim N_\varphi = BR_i^2 \gg 1$

$$E^{(i)}_{n_0,\ell} \approx \left(n_0 - \frac{1+(-1)^{i+1}}{2}\right)\omega_c + (-1)^{i+1}\frac{KR_i^2}{\xi^2}\left(\frac{2n_0+\ell+1}{N_\phi(R_i)} - 1\right)$$

exp. localized chiral mode near $r \sim R_i$

$n_0 = 1\ (S1)$
$n_0 = 0\ (S2)$

Ang. Mom. of E = 0 modes:

$$\ell_i = \frac{R_i^2}{2\ell_B^2} - 2 + (-1)^i$$

Explicit Example: Integer Quantum Hall

Ang. Mom. of E = 0 modes: $$\ell_i = \frac{R_i^2}{2\ell_B^2} - 2 + (-1)^i$$

If $R_1 = R_2$: $$\ell_1 - \ell_2 = -2$$

If $R_1 \neq R_2$ & $\ell_1 = \ell_2 = \ell$: obtain minimal separation b/w wavefunctions $R_1^2 - R_2^2 = 4\ell_B^2$

If wfn overlap non-zero at this separation, can be gapped out by local symmetric perturbation.

$$\psi^i_{n_0,\ell}(r) \sim \frac{1}{\sqrt{R_i}} H_{n_0}\left(\frac{r - R_i}{\ell_B}\right) \exp\left(-\frac{(r - R_i)^2}{2\ell_B^2}\right)$$

Different principal quantum number n_0 for edge modes (due to different shift): wfn overlap vanishes.

Explicit Example: Integer Quantum Hall

Ang. Mom. of E = 0 modes: $\ell_i = \frac{R_i^2}{2\ell_B^2} - 2 + (-1)^i$

If $R_1 = R_2$: $\ell_1 - \ell_2 = -2$

If $R_1 \neq R_2$ & $\ell_1 = \ell_2 = \ell$: obtain minimal separation b/w wavefunctions $R_1^2 - R_2^2 = 4\ell_B^2$

If wfn overlap non-zero at this separation, can be gapped out by local symmetric perturbation.

$$\psi^i_{n_0,\ell}(r) \sim \frac{1}{\sqrt{R_i}} H_{n_0}\left(\frac{r - R_i}{\ell_B}\right) \exp\left(-\frac{(r - R_i)^2}{2\ell_B^2}\right)$$

Different principal quantum number n_0 for edge modes (due to different shift): wfn overlap vanishes.

Abelian Topological Order: General Theory

- Prior argument relied on specific model/set-up and microscopic details.
- Use Chern-Simons field theory to study interacting Abelian Tos with continuous/discrete rotation symmetry.
- Following CEP, treat crystalline gauge field as internal.
- Sufficient constraints from algebraic considerations and (separately) from flux threading argument.

Abelian CS theory with Global Symmetries

$$\mathcal{L}_{\text{bulk}} = -\frac{\epsilon^{\mu\nu\lambda}}{4\pi} K_{I,J} a^I_\mu \partial_\nu a^J_\lambda + \frac{\epsilon^{\mu\nu\lambda}}{2\pi} t_I A_\mu \partial_\nu a^I_\lambda + \frac{\epsilon^{i\mu\nu}}{2\pi} s_I \omega_i \partial_\mu a^I_\nu,$$

K: $N_K \times N_K$ symmetric $\mathbb{Z}$ matrix, invertible.
t: $\mathbb{Z}$ charge vector, s: $\mathbb{Z}$ $(\frac{1}{2}\mathbb{Z})$ spin vector
A: $U(1)_c$ background gauge field
ω: $U(1)_r$ *or* $\mathbb{Z}_M$ *internal* background gauge field

$$\text{GSD} = |\det \mathbf{K}|^g$$

$$\mathcal{L}_{\text{eff}} = \frac{\sigma_{xy}}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{S}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu \omega_\lambda + \frac{\ell_s}{4\pi} \epsilon^{\mu\nu\lambda} \omega_\mu \partial_\nu \omega_\lambda,$$

$$\sigma_{xy} = t^T K^{-1} t; \quad S = s^T K^{-1} t; \quad \ell_s = s^T K^{-1} s.$$

Wen, Zee PRB (1992)

Abelian CS theory with Global Symmetries

$$\mathcal{L}_{\text{bulk}} = -\frac{\epsilon^{\mu\nu\lambda}}{4\pi} K_{I,J} a^I_\mu \partial_\nu a^J_\lambda + \frac{\epsilon^{\mu\nu\lambda}}{2\pi} t_I A_\mu \partial_\nu a^I_\lambda + \frac{\epsilon^{i\mu\nu}}{2\pi} s_I \omega_i \partial_\mu a^I_\nu,$$

Bulk quasi-particles labelled by integer vector $\vec{\ell} = (\ell_1, \dots, \ell_N)^T$, $\ell_j \in \mathbb{Z}$

Self-statistics $\theta_{\mathbf{l}} = \pi \mathbf{l}^T \mathbf{K}^{-1} \mathbf{l}, \quad \mathbf{l} \in \mathbb{Z}^N.$

Mutual statistics $\tilde{\theta}_{\mathbf{l},\mathbf{l}'} = 2\pi \mathbf{l}^T \mathbf{K}^{-1} \mathbf{l}', \quad \mathbf{l}, \mathbf{l}' \in \mathbb{Z}^N.$

Gauge inv. microscopic dofs $\tilde{\mathbf{l}} = \mathbf{K}\mathbf{l}, \quad \mathbf{l} \in \mathbb{Z}^N.$

Lu, Vishwanath PRB (2016)

Edge Theory for Abelian SETs

$$\mathcal{L}_{\text{bulk}} = -\frac{\epsilon^{\mu\nu\lambda}}{4\pi} K_{I,J} a_\mu^I \partial_\nu a_\lambda^J + \frac{\epsilon^{\mu\nu\lambda}}{2\pi} t_I A_\mu \partial_\nu a_\lambda^I + \frac{\epsilon^{i\mu\nu}}{2\pi} s_I \omega_i \partial_\mu a_\nu^I,$$

Edge theory on open disc with radius R $(x = R\theta), \theta \in S^1$: bulk constraint $\epsilon^{ij}\partial_i a_j^I = 0 \;\rightarrow\; a_j^I = \partial_j \phi^I$

$$\mathcal{L}_{\text{edge}} = -\frac{K_{I,J}}{4\pi} \partial_x \phi^I \partial_t \phi^J + \frac{t_I \epsilon^{\mu\nu}}{2\pi} A_\mu \partial_\nu \phi^I + \frac{s_I}{2\pi} \omega_x \partial_t \phi^I + \cdots$$

$$[\phi^I(x), \partial_y \phi^J(y)] = -2\pi i \delta(x-y) K_{I,J}^{-1}.$$

$$\rho_c = \frac{t_I}{2\pi} \partial_x \phi^I, \qquad \rho_s = \frac{s_I}{2\pi} \partial_x \phi^I.$$

Edge Theory for Abelian SETs

- Symmetry transformations of edge excitations $V_{\vec{\ell}} \sim e^{i\,\ell_I \phi^I(x,t)}$
  - Under $U(1)_C$ phase rotation by α: $e^{i\alpha\int\rho(x)dx} V_{\vec{l}} e^{-i\alpha\int\rho(x)dx} = e^{-i\alpha l_I K^{-1}_{I,J} t_J} V_{\vec{l}}.$
  - Charge of excitation: $Q_{\vec{l}} = -l_I K^{-1}_{I,J} t_J.$
  - Under $U(1)_r$ spatial rotation by α: $e^{i\alpha L_z} V_{\vec{l}}(x) e^{-i\alpha L_z} = e^{-i\alpha l_I K^{-1}_{I,J} s_J} V_{\vec{l}}(x + R\alpha).$

    $$L_z = -i\,\partial_\theta + \int \rho_s(x)dx.$$

  - For rotation invariant edge excitation: $L_z(\vec{l}, n) = n - l_I K^{-1}_{I,J} s_J.$

    $$V_{\vec{l},n} \equiv \int_0^{2\pi} d\theta\, e^{-in\theta} V_{\vec{l}}(x = \theta R)$$

Gappability conditions: Lagrangian subgroup

- Local bosonic d.o.f.s can condense in bulk w/o changing TO → certain backscattering (or Higgs) terms are allowed on boundary

- If net chirality ($n_+ \neq n_-$), not all edge modes can be gapped.

Haldane PRL (1995)
Levin PRX (2013)

Gappability conditions: Lagrangian subgroup

- Local bosonic d.o.f.s can condense in bulk w/o changing TO → certain backscattering (or Higgs) terms are allowed on boundary
- If net chirality ($n_+ \neq n_-$), not all edge modes can be gapped.
- If no net chirality ($n_+ = n_-$), edge states of 2+1D Abelian TOs gapped ⟺ ∃ a Lagrangian subgroup $M = \{m_i\}(m_i \in \mathbb{Z})$:
  - $m_i^T K^{-1} m_j = 0 \ (mod\ 1)\ \forall\ m_i, m_j \in M$
  - For any qp $\vec{\ell}$, either $\vec{\ell} \in M$ ($\vec{\ell} = \sum c_i m_i$) or $\exists\ m_i \in M: m_i^T K^{-1} \ell \neq 0 \ \ (mod\ 1)$

Haldane PRL (1995)
Levin PRX (2013)

Gappability conditions: Lagrangian subgroup

- Lagrangian subgroup $M = \{m_i\}(m_i \in \mathbb{Z})$:
  - $m_i^T K^{-1} m_j = 0 \;\; mod\; 1 \; \forall\; m_i, m_j \in M$
  - For any qp $\vec{\ell}$, either $\vec{\ell} \in M$ ($\vec{\ell} = \sum c_i m_i$) or $\exists\; m_i \in M$: $m_i^T K^{-1} \ell \neq 0 \;\; (mod\; 1)$

- Physically, $\vec{m} \in M \sim$ local boson $V_{\vec{m}} \sim e^{im^I \phi^I}$ that condenses on a gapped edge i.e.,

$$\lim_{|x-y|\to\infty} \langle e^{i m_I \phi^I(x)} e^{-i m_I \phi^I(y)} \rangle \neq 0, \quad \forall \vec{m} \in \mathcal{M}$$

- Backscattering terms

$$\mathcal{L}_{\text{b.s.}} = \sum_{\{\Lambda_i\}} U_{\Lambda_i}(x) \cos\left(\Lambda_i^I K_{I,J} \phi^J - f_i(x)\right),$$

Null vectors $\{\Lambda_\text{i}\}$: $\Lambda_i^T K\, \Lambda_j = 0, \forall i, j$

Haldane PRL (1995)
Levin PRX (2013)

Symmetry Constraints on Gappability

- With global symmetry: condensable bosons must be *charge neutral*

- Consider $U(1)_C$ first and assume symmetric gapped boundary
  - Charged operators have short range correlation functions
  - Charge neutral: $m^T K^{-1} t = 0, \quad \forall\, m \in \mathcal{M}$
- Suppose Hall-conductance is fractional: $\sigma_{xy} = t^T K^{-1} t \neq 0 \mod 1$
- Then, $t \notin M$ (recall $m^T K^{-1} m = 0\ \forall m \in M$)

Symmetry Constraints on Gappability

- Similar arguments for $U(1)_r$:
  - No SSB (equiv. $\langle V^\dagger_{\vec{m}_i,n_i=0} V_{\vec{m}_i,n_i=0}\rangle \neq 0$)) → $m^T K^{-1} s = 0, \quad \forall\, m \in \mathcal{M}$

- Can show that $s^T K^{-1} s \neq 0 \mod 1$ & $\mathcal{S} = t^T K^{-1} s \neq 0 \mod 1.$

  are sufficient conditions for gapless edge states on the disc.

- Lagrangian algebra condition only detects fractional part (relies on braiding statistics)

Flux Threading Argument

- Assume gapped symmetric edge & show contradiction:

- Generic symmetric local term

$$\delta\mathcal{L}_{\rm edge}(x=\theta R)=\sum_{\{\vec{l}_i\}} V_{K\vec{l}_0}(\theta_0=\theta)$$

$$\int\prod_{i=1}^{N_v}\left[\frac{\mathrm{d}\theta_i}{2\pi}V_{K\vec{l}_i}(\theta+\theta_i)\right]T_{\{\vec{l}_i\}}(\{\theta_i\})$$

$$\sum_i l_i^T t=0$$

$$\sum_i l_i^T s=0$$

- Adiabatic flux insertion of Φ:

$$\tilde{T}_{\{\vec{l}_i\}}(\{\theta_i\})=e^{\,\mathrm{i}\,A_\theta\sum_{i=1}^{N_v}\theta_i l_i^T t}T_{\{\vec{l}_i\}}(\{\theta_i\})$$

- Undone by large gauge transformation

$$U_0=e^{\mathrm{i}\int\mathrm{d}x t_I\phi_I(x)}$$

$$U_0H(\Phi=2\pi)U_0^{-1}=H(\Phi=0)\,.$$

$$|\Phi=0\rangle=e^{\,\mathrm{i}\,\alpha}U_0\,|\Phi=2\pi\rangle$$

Flux Threading Argument

- Assume gapped symmetric edge & show contradiction:
- Generic symmetric local term

$$\delta\mathcal{L}_{\rm edge}(x=\theta R) = \sum_{\{\vec{l}_i\}} V_{K\vec{l}_0}(\theta_0=\theta)$$
$$\int \prod_{i=1}^{N_v} \left[\frac{\mathrm{d}\theta_i}{2\pi} V_{K\vec{l}_i}(\theta+\theta_i)\right] T_{\{\vec{l}_i\}}(\{\theta_i\})$$

$$\sum_i l_i^T t = 0$$
$$\sum_i l_i^T s = 0$$

- Adiabatic flux insertion of Φ:

$$\tilde{T}_{\{\vec{l}_i\}}(\{\theta_i\}) = e^{\,\mathrm{i}\,A_\theta \sum_{i=1}^{N_v} \theta_i l_i^T t} T_{\{\vec{l}_i\}}(\{\theta_i\})$$

- Undone by large gauge transformation

$$U_0 = e^{\,\mathrm{i}\int \mathrm{d}x t_I \phi_I(x)}$$
$$U_0 H(\Phi = 2\pi) U_0^{-1} = H(\Phi=0)\,.$$
$$|\Phi=0\rangle = e^{\,\mathrm{i}\,\alpha} U_0\,|\Phi=2\pi\rangle$$

BUT!

$$U_0\Big(\int \rho_c(x)\mathrm{d}x\Big)U_0^{-1} = \int \rho_c(x)\mathrm{d}x - t^T K^{-1} t$$

$\sigma_{xy} = t^T K^{-1} t \neq 0$ $\Rightarrow$ stable gapless edge.

Flux Threading Argument

- Assume gapped symmetric edge & show contradiction:

- Generic symmetric local term

$$\delta\mathcal{L}_{\mathrm{edge}}(x=\theta R) = \sum_{\{\vec{l}_i\}} V_{K\vec{l}_0}(\theta_0=\theta)$$
$$\int \prod_{i=1}^{N_v} \left[\frac{\mathrm{d}\theta_i}{2\pi} V_{K\vec{l}_i}(\theta+\theta_i)\right] T_{\{\vec{l}_i\}}(\{\theta_i\})$$

$$\sum_i l_i^T t = 0$$

$$\sum_i l_i^T s = 0$$

- Adiabatic flux insertion of Φ:

$$\tilde{T}_{\{\vec{l}_i\}}(\{\theta_i\}) = e^{\mathrm{i}\,A_\theta \sum_{i=1}^{N_v} \theta_i l_i^T t} T_{\{\vec{l}_i\}}(\{\theta_i\})$$

- Can also thread flux Φ_r and show $s\,K^{-1}s \neq 0$ $\Rightarrow$ stable gapless edge.

Symmetric Gapless Edge States for Abelian TOs

$$\mathcal{L}_{\text{bulk}} = -\frac{\epsilon^{\mu\nu\lambda}}{4\pi} K_{I,J} a^I_\mu \partial_\nu a^J_\lambda + \frac{\epsilon^{\mu\nu\lambda}}{2\pi} t_I A_\mu \partial_\nu a^I_\lambda + \frac{\epsilon^{i\mu\nu}}{2\pi} s_I \omega_i \partial_\mu a^I_\nu,$$

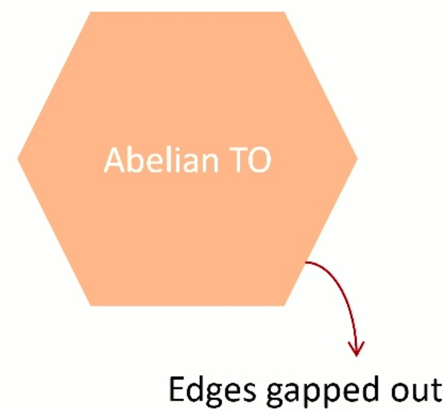
Symmetric Gapped edge ⟺

$$\sigma_{xy} = t^T K^{-1} t \neq 0$$
$$\mathcal{S} = s^T K^{-1} t \neq 0$$
$$\ell_s = s K^{-1} s \neq 0$$

Discrete Rotation Symmetry: Corner Modes

- Consider the Abelian TO on regular n-sided polygon
- If $c = 0, \sigma_{xy} = 0$: counter-propagating edges symmetrically gapped

$$H_{b.s.} = \sum_{\{\Lambda_i\}} U_{\Lambda_i} \int_0^{2\pi} \mathrm{d}\theta \cos\left[\Lambda_i^T K \phi(\theta) - f_i(\theta)\right],$$
$$f_i(\theta + 2\pi/n) = f_i(\theta) + \frac{2\pi}{n}\Lambda_i^T s$$

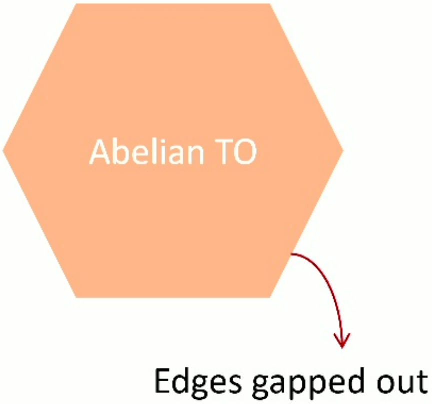


Discrete Rotation Symmetry: Corner Modes

- Consider the Abelian TO on regular n-sided polygon
- If $c = 0, \sigma_{xy} = 0$: counter-propagating edges symmetrically gapped

$$H_{b.s.} = \sum_{\{\Lambda_i\}} U_{\Lambda_i} \int_0^{2\pi} \mathrm{d}\theta \cos\left[\Lambda_i^T K \phi(\theta) - f_i(\theta)\right],$$
$$f_i(\theta + 2\pi/n) = f_i(\theta) + \frac{2\pi}{n} \Lambda_i^T s$$

- Vertex operator at corners $\theta = \theta_0$: $\mathcal{D}_{\mathrm{C}_n}(\theta_0) \sim e^{\mathrm{i}\, s_I \phi^I(\theta_0)/n}$.



Corner Modes on Platonic Solids

Place Abelian TO on 2d surface of 3d platonic solid

Discrete Rotation Symmetry: Corner Modes

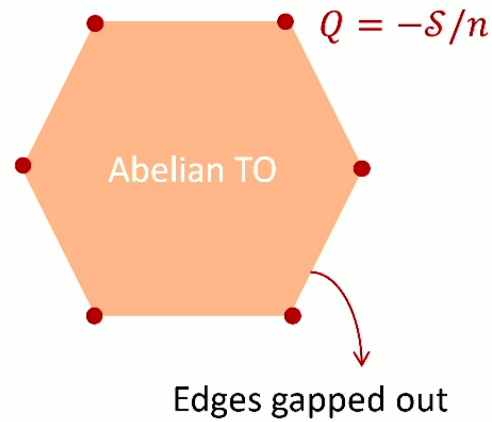
- Consider the Abelian TO on regular n-sided polygon

- If $c = 0, \sigma_{xy} = 0$: counter-propagating edges symmetrically gapped

$$H_{b.s.} = \sum_{\{\Lambda_i\}} U_{\Lambda_i} \int_0^{2\pi} \mathrm{d}\theta \cos\left[\Lambda_i^T K \phi(\theta) - f_i(\theta)\right],$$
$$f_i(\theta + 2\pi/n) = f_i(\theta) + \tfrac{2\pi}{n} \Lambda_i^T s$$

- Vertex operator at corners $\theta = \theta_0$: $\mathcal{D}_{C_n}(\theta_0) \sim e^{\mathrm{i}\, s_I \phi^I(\theta_0)/n}$.

- Corner localized charge: $Q_{C_n} = -\frac{1}{n} t^T K^{-1} s = -\frac{\mathcal{S}}{n} \mod 1$



Corner Modes on Platonic Solids

Place Abelian TO on 2d surface of 3d platonic solid

Faces: regular n-gons, m faces share a vertex

$F, V = \frac{nF}{m}, E = \frac{nF}{2}$ satisfy $V - E + F = 2$ ($\cong S^2$)

$V = \frac{4n}{2(n+m)-mn}$, expect: $Q_{(n,m)} = \frac{2\mathcal{S}}{V} = \mathcal{S}\,\frac{2(n+m)-mn}{2n}$

Corner Modes on Platonic Solids

Place Abelian TO on 2d surface of 3d platonic solid

Faces: regular n-gons, m faces share a vertex

$$F, V = \frac{nF}{m}, E = \frac{nF}{2} \text{ satisfy } V - E + F = 2 \ (\cong S^2)$$

Works even for $c, \sigma_{xy} \neq 0$

$$V = \frac{4n}{2(n+m)-mn}, \text{ expect: } \quad Q_{(n,m)} = \frac{2\mathcal{S}}{V} = \mathcal{S}\,\frac{2(n+m)-mn}{2n}$$

Field Theory Calculation

$$\mathcal{L}_{\text{edge}} = -\frac{K_{I,J}}{4\pi}\partial_x\phi^I\partial_t\phi^J + \frac{t_I\epsilon^{\mu\nu}}{2\pi}A_\mu\partial_\nu\phi^I + \frac{s_I}{2\pi}\omega_x\partial_t\phi^I + \cdots$$

Sew patches together into smooth bulk: anyons tunnel freely

$$\mathcal{H}^0_{\text{gap}} = -\sum_{\{e_i\}}\sum_{j=1}^{N} U_{e_i}\cos\left[e_i^T K(\phi_{j+1,R} - \phi_{j,L})\right]$$

$$\vec{\phi} = (\phi_1, \dots, \phi_{N_K})^T$$

$$\{\vec{\phi}_{j,L/R} | 1 \le j \le N\}$$

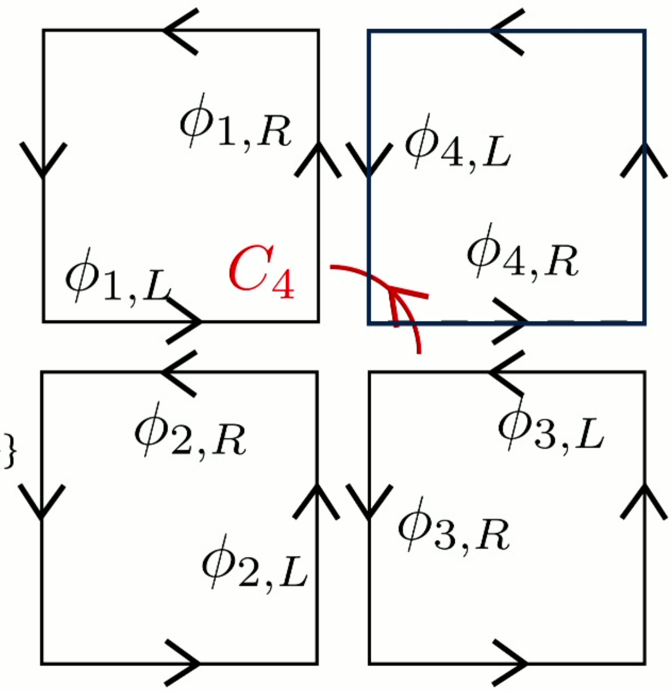
$$\{(e_i)_I = \delta_{i,I} | 1 \le i \le N_K\}$$

Chiral bosons pinned: $\langle\phi_{j+1,R}\rangle = \langle\phi_{j,L}\rangle$

Under C_N around vertex:

$$C_N\phi_{j,L/R}C_N^{-1} = \phi_{j+1,L/R} - \frac{(n-2)\pi}{n}K^{-1}s$$

$$\langle\phi_{j+1,L/R}\rangle = \langle\phi_{j,L/R}\rangle + \frac{(n-2)\pi}{n}K^{-1}s$$



$$N = \frac{2\pi}{(n-2)\pi/n} = \frac{2n}{n-2}$$

Field Theory Calculation

To properly sew edges in this case, require

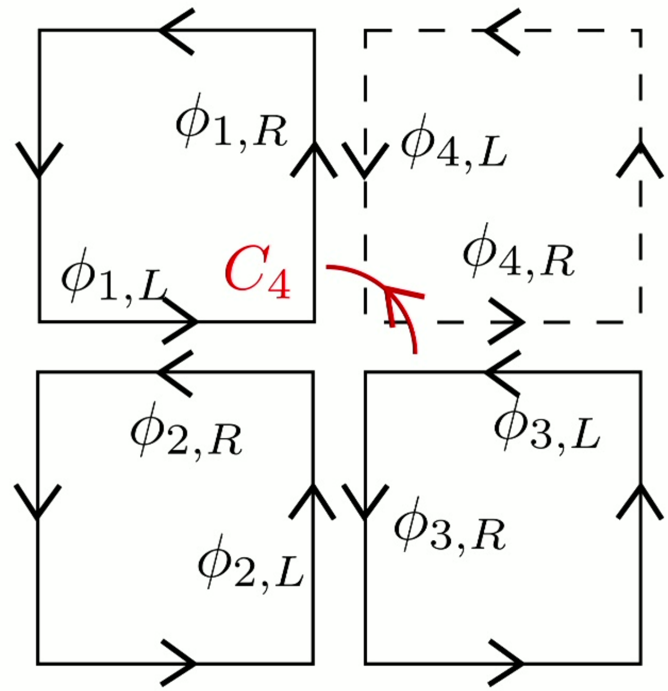
$$\langle\phi_{1,R}\rangle = \langle\phi_{N,L}\rangle = \langle\phi_{m,L}\rangle + (N-m)\frac{(n-2)\pi}{n}$$

Modified gapping term:

$$\mathcal{H}_{\text{gap}} = -\sum_{\{e_i\}} U_{e_i}\Big\{\sum_{j=1}^{m-1}\cos\left[e_i^T K(\phi_{j+1,R}-\phi_{j,L})\right] + \cos\left[e_i^T K(\phi_{1,R}-\phi_{m,L}-(N-m)\tfrac{(n-2)\pi}{n})\right]\Big\}$$

Find corner charge:

$$Q_{n,m} = \sum_{j=1}^{m}\frac{t^T}{2\pi}\langle\phi_{j,R}-\phi_{j,L}\rangle$$
$$\equiv \sum_{j=1}^{m}\frac{t^T}{2\pi}\langle\phi_{j+1,R}-\phi_{j+1,L}\rangle$$
$$= \frac{(N-m)(n-2)}{2n}t^T K^{-1}s \mod 1$$
$$= \frac{2(m+n)-mn}{2n}t^T K^{-1}s \mod 1$$



$$Q_{(n,m)} = \frac{2\mathcal{S}}{V} = \mathcal{S}\,\frac{2(n+m)-mn}{2n}$$

Calculating the Shift

Many-Body Real Space Invariants:

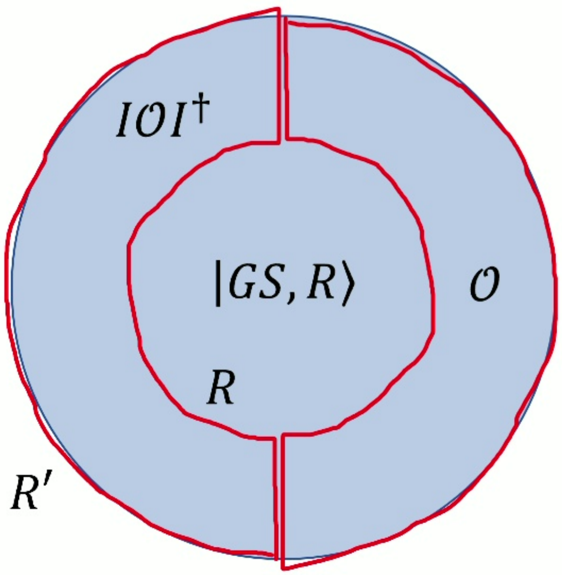
- Ryu et al; Manjunath et al: calculate Shift from *partial rotation* operators

- Topological indices global in *k*-space → must exist local invariants in position space

- RSI's as quantum numbers: define symmetry operators s.t. spectrum independent of boundary conditions

Many-Body RSIs

- To ensure many-body RSIs are invariants, they should be quantum numbers of appropriate symmetry operators
- To ensure locality, use open boundary conditions with cutoff $|\mathbf{r}| < R$
  - But must not depend on boundary truncation, or else they aren't universal

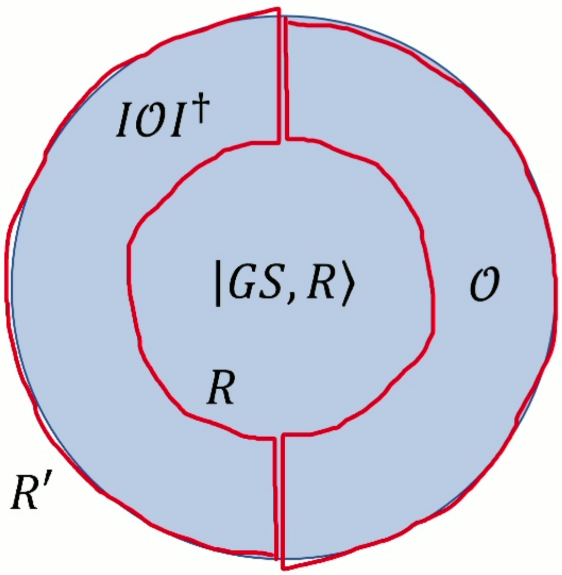


$$|GS, R'\rangle \propto \mathcal{O}\mathcal{I}\mathcal{O}\mathcal{I}^\dagger \, |GS, R\rangle$$

$$e^{\frac{i\pi}{2}\hat{N}} \mathcal{I}|GS, R\rangle = e^{\frac{i\pi}{2}\Delta}|GS, R\rangle$$

Many-Body RSIs

- To ensure many-body RSIs are invariants, they should be quantum numbers of appropriate symmetry operators
- To ensure locality, use open boundary conditions with cutoff $|\mathbf{r}| < R$
  - But must not depend on boundary truncation, or else they aren't universal



$$|GS, R'\rangle \propto \mathcal{O}I\mathcal{O}I^\dagger\,|GS, R\rangle$$

The quantum numbers of $|GS, R\rangle, |GS, R'\rangle$ are different. But eigenvalues of

$$e^{\frac{i\pi}{2}\hat{N}} I|GS, R\rangle = e^{\frac{i\pi}{2}\Delta}|GS, R\rangle$$

Many-Body RSIs

- To ensure many-body RSIs are invariants, they should be quantum numbers of appropriate symmetry operators
- To ensure locality, use open boundary conditions with cutoff $|\mathbf{r}| < R$
  - But must not depend on boundary truncation, or else they aren't universal

Jonah Herzog-Arbeitman
(Princeton)

$$e^{\frac{i\pi}{N}\hat{N}} C_N |GS\rangle = e^{\frac{i\pi}{2}\Delta_1} |GS\rangle$$

$$C_N^2 |GS\rangle = e^{\frac{i2\pi}{N/2}\Delta_2} |GS\rangle$$

$\Delta_1, \Delta_2 \in \mathbb{Z}_{2N} \times \mathbb{Z}_{N/2}$ (N even)

$\Delta_1, \Delta_2 \in \mathbb{Z}_{N} \times \mathbb{Z}_{N}$ (N odd)

Outlook

- Generalisation to non-Abelian TOs
- Non-Abelian corner states from anyon permuting C_N symmetry
- Boundary manifestations of non-trivial polarization?
  - Polarization ill-defined in the presence of non-vanishing Chern number
  - Interface between states with identical C but distinct P can host localized modes