Title: The Shadow of Real Quantum Theory

Speakers: Alexander Wilce

Series: Quantum Foundations

Date: March 07, 2024 - 11:00 AM

URL: https://pirsa.org/24030106

Abstract: Local tomography (or tomographic locality) is the principle that the state of a composite systems is determined by the probabilities it assigns to outcomes of experiments performed separately on the component systems. It's well known that complex quantum theory enjoys, and real quantum theory lacks, this feature. This means that a composite of two real quantum systems has additional "global" degrees of freedom. What if we could simply factor these out? In this talk, I'll describe how this can be done, not only for real quantum theory, but for essentially any probabilistic theory. The result is a locally tomographic theory we call the "locally tomographic shadow" of the original. I will also discuss what this shadow theory looks like in the case of real quantum theory. (This is joint work with Howard Barnum and Matthew Graydon).

Zoom link

The Shadow of Real Quantum Theory

Alex Wilce
Susquehanna University

Joint work with Howard Barnum and Matthew Graydon

PI, March 7, 2024

arXiv:2308.16494

Overview

Local Tomography (LT) posits that the state of a composite system AB, is determined by the joint probabilities it assigns to separate, "local" measurements on A and B.

Classical probability theory and *complex* QM satisfy LT, but Real QM ($\mathbb{R}\mathbf{QM}$) does not.

This is clear on dimensional grounds, but let's look a bit deeper.

Let $\mathbf{H}$, $\mathbf{K}$ be (here, f.d.) real Hilbert spaces. Let $\mathcal{L}_s(\mathbf{H})$, $\mathcal{L}_a(\mathbf{H})$ be the spaces of symmetric, resp. anti-symmetric operators on $\mathbf{H}$, and similarly for $\mathbf{K}$. Set

$$\mathcal{L}_{ss} := \mathcal{L}_s(\mathbf{H}) \otimes \mathcal{L}_s(\mathbf{K}) \text{ and } \mathcal{L}_{aa} := \mathcal{L}_a(\mathbf{H}) \otimes \mathcal{L}_a(\mathbf{K}).$$

Then

$$\mathcal{L}_s(\mathbf{H} \otimes \mathbf{K}) = \mathcal{L}_{ss} \oplus \mathcal{L}_{aa}$$

— an *orthogonal* decomposition w.r.t. trace inner product.

So if ρ's a density operator on $\mathbf{H} \otimes \mathbf{K}$,

$$\rho = \rho_{ss} + \rho_{aa}$$

with $\rho_{ss} \in \mathcal{L}_{ss}$ and $\rho_{aa} \in \mathcal{L}_{aa}$.

Given effects $a \in \mathcal{L}_s(\mathbf{H})$ and $b \in \mathcal{L}_s(\mathbf{K})$, $a \otimes b \in \mathcal{L}_{ss}$, so $\text{Tr}((a \otimes b)\rho_{aa}) = 0$. Hence,

$$\text{Tr}((a \otimes b)\rho) = \text{Tr}((a \otimes b)\rho_{ss}).$$

States with the same $\mathcal{L}_{ss}$ component are *locally indistinguishable* in real QM.

Question: Can we just "factor out" the non-local degrees of freedom — here, $\mathcal{L}_{aa}$ — to obtain a LT theory?

Yes! — and not just for $\mathbb{R}$**QM**, but non-LT probabilistic theories (GPTs) very generally.

We call the resulting theory the *locally tomographic shadow* of the original one.

First, we need to say what we mean by a probabilistic theory.

Plan

1. Probabilistic Theories revisited
2. Construction of the LT shadow
3. The shadow of $\mathbb{R}$**QM**
4. Some questions

Probabilistic Models

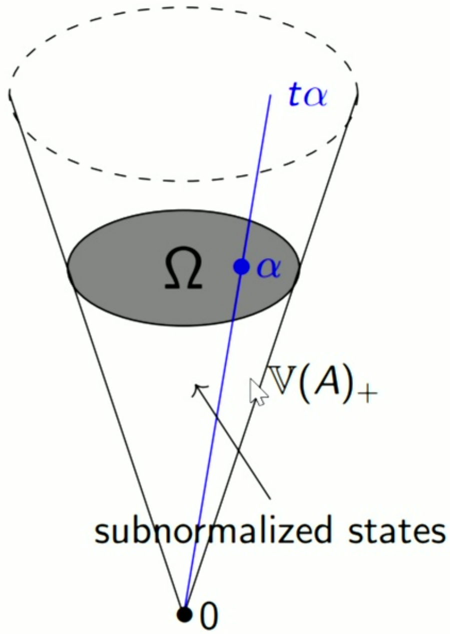
For our purposes, a **probabilistic model** is pair $(\mathbb{V}, u)$ where

- $\mathbb{V}$ is an ordered real vector space, with positive cone $\mathbb{V}_+$;
- u is a strictly positive linear functional on $\mathbb{V}$.

States are elements $\alpha \in \mathbb{V}_+$ with $u(\alpha) = 1$.

Effects ("measurement outcomes") are elements $a \in \mathbb{V}^*$ with $0 \leq a \leq u$: $a(\alpha)$ is the probability of a's of occurring in state α. The functional u is the unit effect of the model.

A rough picture:



Example: For a f.d. quantum system with Hilbert space $\mathbf{H}$, $\mathbb{V} = \mathcal{L}_s(\mathbf{H})$, ordered as usual, and $u = \text{Tr}(\cdot)$. States are density operators. Identifying $\mathbb{V}^*$ with $\mathbb{V}$ (using trace duality), u becomes $\mathbf{1}$ and effects are positive operators between $\mathbf{0}$ and $\mathbf{1}$.

Remark: A more basic framework starts with a collection $\mathcal{M} = \{E, F, ...\}$ of outcome-sets for various experiments.

Let $X = \bigcup \mathcal{M}$: a probability weight on $\mathcal{M}$ is a function $\alpha : X \to \mathbb{R}$ with $\alpha \geq 0$ and $\sum_{x \in E} \alpha(x) = 1$ for all $E \in \mathcal{M}$. Let $\Omega =$ set of all probability weights on $\mathcal{M}$ and $\mathbb{V} = \text{Span}(\Omega) \leq \mathbb{R}^X$, ordered pointwise. Then

$$u : \alpha \mapsto \sum_{x \in E} \alpha(x)$$

is independent of $E \in \mathcal{M}$, and gives a unit functional with $\Omega = u^{-1}(1) \cap \mathbb{V}_+$.

Gleason's Theorem tells us that if $\mathcal{M}$ is the set of orthonormal bases of $\mathbf{H}$, with $\dim(\mathbf{H}) > 2$, this construction gives $\mathbb{V} = \mathcal{L}_s(\mathbf{H})$ and $u = \text{Tr}(\ \cdot\ )$, as above.

Standing assumptions and conventions:

- All models are finite-dimensional;
- We identify $\mathbb{V}$ with $\mathbb{V}^{**}$;
- $L^k(\mathbb{V}_1, ..., \mathbb{V}_k)$ is the space of k-linear forms on $\mathbb{V}_1 \times \cdots \times \mathbb{V}_k$;
- $\otimes$ is always the tensor product of vector spaces/linear maps,

  where

- we take $\mathbb{V} \otimes \mathbb{W} := \mathcal{L}^2(\mathbb{V}^*, \mathbb{W}^*)$ i.e., $(\alpha \otimes \beta)(a, b) := \alpha(a)\beta(b)$

Composites

A non-signaling composite of $(\mathbb{V}_1, u_1)$ and $(\mathbb{V}_2, u_2)$ is a model $(\mathbb{V}, u)$ plus positive bilinear mappings

$$m : \mathbb{V}_1 \times \mathbb{V}_2 \to \mathbb{V}$$

$$\pi : \mathbb{V}_1^* \times \mathbb{V}_2^* \to \mathbb{V}^*$$

such that

(i) $\pi(a, b)m(\alpha, \beta) = a(\alpha)b(\beta)$

(ii) $\pi(u_1, u_2) = u$.

Note m defines a linear mapping

$$m : \mathbb{V}_1 \otimes \mathbb{V}_2 \to \mathbb{V},$$

and π dualizes to give another,

$$\pi^* : \mathbb{V} \to \mathcal{L}^2(\mathbb{V}_1^*, \mathbb{V}_2^*)^* = \mathbb{V}_1 \otimes \mathbb{V}_2.$$

We refer to $\pi(a,b)$ as a product effect. Harmless abuse of notation: $\pi(a,b) = a \otimes b$, so as to embed $\mathbb{V}_1^* \otimes \mathbb{V}_2^*$ into $\mathbb{V}^*$.

Then $\pi^*(\omega)(a,b) = \omega(a \otimes b)$ is essentially ω restricted to product effects. We say the composite $(\mathbb{V}, u)$ is locally tomographic (LT) iff product effects separate states — that is,

$$\omega_1(a \otimes b) = \omega_2(a \otimes b) \forall a, b \ \Rightarrow \ \omega_1 = \omega_2.$$

Equivalently:

$$\pi^* : \mathbb{V} \simeq \mathbb{V}_1 \otimes \mathbb{V}_2.$$

Two extremal cases:

- The minimal tensor product $\mathbb{V}_1 \otimes_{\text{min}} \mathbb{V}_2$ has cone generated by separable states.
- The maximal tensor product $\mathbb{V}_1 \otimes_{\text{max}} \mathbb{V}_2$ has cone generated by tensors positive on product effects.

The definition of a composite just says we have positive linear mappings

$$\mathbb{V}_1 \otimes_{\text{min}} \mathbb{V}_2 \xrightarrow{m} \mathbb{V} \xrightarrow{\pi^*} \mathbb{V}_1 \otimes_{\text{max}} \mathbb{V}_2$$

composing to the identity.

A probabilistic theory is a functor $\mathbb{V} : \mathcal{C} \to \mathbf{Prob}$, where

- $\mathcal{C}$ is a symmetric monoidal category ("actual" physical systems and processes, or mathematical proxies for these)
- $\mathbb{V}(AB)$ is a non-signaling composite of $\mathbb{V}(A)$ and $\mathbb{V}(B)$
- $\mathbb{V}(I) = \mathbb{R}$.

We assume $\mathbb{V}$ is *injective on objects*, which makes $\mathbb{V}(\mathcal{C})$ a subcategory of **Prob**, with a well-defined monoidal structure given (on objects) by

$$\mathbb{V}(A), \mathbb{V}(B) \mapsto \mathbb{V}(AB).$$

Even if $(\mathcal{C}, \mathbb{V})$ is not LT, we can ask what the world it describes "looks like" to local agents.

We need to assume that systems preferred decompositions into local pieces, so replace $\mathcal{C}$ with its strictification $\mathcal{C}^*$:

- objects are finite lists $\vec{A} = (A_1, ..., A_n)$ of objects $A_i \in \mathcal{C}$ $(A_i \neq I)$ standing for

$$\Pi\vec{A} := \Pi_{i=1}^{n} A_i := A_1(A_2(\cdots(A_{n-1}A_n)\cdots)$$

  in $\mathcal{C}$ with the indicated decomposition.
- Morphisms from $\vec{A}$ to $\vec{B}$ are morphisms $\Pi\vec{A} \to \Pi\vec{B}$ in $\mathcal{C}$.

Let $\widetilde{\mathbb{V}}(\vec{A})$ be the space $\bigotimes_i \mathbb{V}(A_i)$, ordered by the cone

$$\widetilde{\mathbb{V}}(\vec{A})_+ := \mathrm{LT}_{\vec{A}}(\mathbb{V}(\Pi A)_+)$$

of local shadows of elements of $\mathbb{V}(\Pi\vec{A})_+$.

With $\widetilde{u}_{\vec{A}} = u_{A_1} \otimes \cdots \otimes u_{A_n}$, $(\widetilde{\mathbb{V}}(\vec{A}), \widetilde{u}_{\vec{A}})$ is a model, the *locally tomographic shadow* of $(\mathbb{V}(A), u_A)$ with respect to the given decomposition.

Notation: Write

$$\widetilde{\mathbb{V}}(\vec{A}) \boxtimes \widetilde{\mathbb{V}}(\vec{B}) \ := \ \widetilde{\mathbb{V}}(\vec{A}\vec{B}).$$

In particular, for $A, B \in \mathcal{C}$,

$$\mathbb{V}(A) \boxtimes \mathbb{V}(B) \ = \ \widetilde{\mathbb{V}}(A, B) \ = \ \mathrm{LT}(\mathbb{V}(AB)).$$

As a vector space, this is just $\mathbb{V}(A) \otimes \mathbb{V}(B)$, but ordered by the cone $\widetilde{\mathbb{V}}(A, B)_+$ generated by local shadows $\widetilde{\omega}$ of states $\omega \in \mathbb{V}(AB)$.

The effect cone $\widetilde{\mathbb{V}}(A_1, ..., A_n)^*_+$ has a nice characterization:

Lemma: $\widetilde{\mathbb{V}}(A_1, ..., A_n)^*_+ \simeq \mathbb{V}^*(\Pi_i A_i)_+ \cap (\bigotimes_i \mathbb{V}^*(A_i))$.

In the bipartite case:

$$(\mathbb{V}(A) \boxtimes \mathbb{V}(B))^*_+ \simeq \mathbb{V}(AB)^*_+ \cap (\mathbb{V}(A)^* \otimes \mathbb{V}(B)^*).$$

A positive linear mapping $\Phi : \mathbb{V}(A) \to \mathbb{V}(B)$ satisfying these conditions is *locally positive* (with respect to the specified decompositions).

The linear mapping ϕ in part (c) is then uniquely determined. We call it the *shadow* of Φ, writing $\phi = \mathrm{LT}(\Phi)$.

Lemma: *If $\Phi : \mathbb{V}(A) \to \mathbb{V}(B)$ is locally positive, then $\phi = LT(\Phi)$ is positive as a mapping $\widetilde{\mathbb{V}}(A_1, ..., A_m) \to \widetilde{\mathbb{V}}(B_1, ..., B_n)$.*

.

Call a morphism $\Pi\vec{A} \xrightarrow{\phi} \Pi\vec{B}$ *local* iff $\mathbb{V}(\phi) : \mathbb{V}(\Pi\vec{A}) \to \mathbb{V}(\Pi\vec{B})$ is locally positive (relative to the preferred factorizations of A and B).

Write $\mathrm{Loc}(\mathcal{C}, \mathbb{V})$ for the monoidal subcategory (it is one) of $\mathcal{C}^*$ having the same objects but only local morphisms.

Lemma: $\widetilde{\mathbb{V}} : Loc(\mathcal{C}, \mathbb{V}) \to$ **Prob** *is a locally tomographic probabilistic theory — the* locally tomographic shadow *of* $\mathbb{V}$.

For simplicity, let $\mathbf{H} = \mathbf{K}$, writing $\mathcal{L}_s$ for $\mathcal{L}_s(\mathbf{H})$. Recall

$$\mathcal{L}_s(\mathbf{H} \otimes \mathbf{H}) = \mathcal{L}_{ss} \oplus \mathcal{L}_{aa},$$

where

$$\mathcal{L}_{ss} = \mathcal{L}_s(\mathbf{H}) \otimes \mathcal{L}_s(\mathbf{H}) \text{ and } \mathcal{L}_{aa} = \mathcal{L}_a(\mathbf{H}) \otimes \mathcal{L}_a(\mathbf{H}).$$

Then LT is just the projection onto $\mathcal{L}_{ss}$ — and *this* is just $\text{Sym} \otimes \text{Sym}$, where $\text{Sym}(a) := \frac{1}{2}(a + a^*)$.

For simplicity, let $\mathbf{H} = \mathbf{K}$, writing $\mathcal{L}_s$ for $\mathcal{L}_s(\mathbf{H})$. Recall

$$\mathcal{L}_s(\mathbf{H} \otimes \mathbf{H}) = \mathcal{L}_{ss} \oplus \mathcal{L}_{aa},$$

where

$$\mathcal{L}_{ss} = \mathcal{L}_s(\mathbf{H}) \otimes \mathcal{L}_s(\mathbf{H}) \text{ and } \mathcal{L}_{aa} = \mathcal{L}_a(\mathbf{H}) \otimes \mathcal{L}_a(\mathbf{H}).$$

Then LT is just the projection onto $\mathcal{L}_{ss}$ — and *this* is just $\mathrm{Sym} \otimes \mathrm{Sym}$, where $\mathrm{Sym}(a) := \frac{1}{2}(a + a^*)$.

But $\mathrm{Sym} \otimes \mathrm{Sym}$ is not a positive mapping!

Example: Let $\{x, y\}$ be an ONB for $\mathbb{R}^2$, and let

$$z = \frac{1}{\sqrt{2}}(x \otimes y + y \otimes x)$$

and set

$$W := (\mathrm{Sym} \otimes \mathrm{Sym})(P_z) = \tfrac{1}{2}(P_x \otimes P_y + P_y \otimes P_x) + S \otimes S$$

where $Sx = \frac{1}{2}y$ and $Sy = \frac{1}{2}x$.

This is not a positive operator. For instance, If $v = x \otimes x - y \otimes y$, then $Wv = -\frac{1}{4}v$.

So $(\mathcal{L}_s \boxtimes \mathcal{L}_s)_+ = \mathrm{LT}(\mathcal{L}_+(\mathbf{H} \otimes \mathbf{H}))$ is strictly larger than $\mathcal{L}_{ss} \cap \mathcal{L}_+$.

A priori we now have four cones:

$$(\mathcal{L}_s \otimes_{\mathrm{min}} \mathcal{L}_s)_+ \leq \mathcal{L}_{ss} \cap \mathcal{L}_+ < (\mathcal{L}_s \boxtimes \mathcal{L}_s)_+ \leq (\mathcal{L}_s \otimes_{\mathrm{max}} \mathcal{L}_s)_+.$$

In fact,

Theorem: All *of these embeddings are strict.*

(The hard one is the last. The proof uses the existence of unextendible product bases.)

LT($\mathbb{R}$**QM**) provides us with an interesting "foil" probabilistic theory, related to but distinct from both complex and real finite-dimensional real quantum theory.

One can also build a LT probabilistic theory by starting with $\mathbb{R}$**QM** and "closing up" under the minimal or maximal tensor product. But these theories are impoverished: the one using $\otimes_{\text{min}}$ has no entangled states. The one using $\otimes_{\text{max}}$ has no entangled effects.

In contrast, LT($\mathbb{R}$**QM**) admits both.

Question 1: Compact Closure

The effect $\epsilon : a \otimes b \mapsto \mathrm{Tr}(ab^t)$ is not local. Hence, LT($\mathbb{R}$**QM**) does not inherit the compact structure of $\mathbb{R}$**QM**. If $\mathcal{C}$ is compact closed, when *is* LT($\mathcal{C}, \mathbb{V}$) compact closed?

Question 2: LT and Complex QM

How does LT interact with the restriction-of-scalars and complexification functors $(-)_{\mathbb{R}} : \mathbb{C}\mathbf{QM} \to \mathbb{R}\mathbf{QM}$, $(-)^{\mathbb{C}} : \mathbb{R}\mathbf{QM} \to \mathbb{C}\mathbf{QM}$?

Question 3. The Shadow of InvQM

In (Barnum, Graydon, AW, Quantum 2020), we constructed a non-LT theory **InvQM**, containing finite-dimensional real and quaternionic QM and also a relative of complex QM. What is LT(**InvQM**)?

Question 4: Non-deterministic shadows

Not all processes in $\mathbb{R}$**QM** are local. Suppose Alice and Bob agree that their joint state is ω. This is consistent with the true global state being any $\mu \in \mathsf{LT}^{-1}_{A,B}(\omega)$.

If μ evolves under a (global) process

$$\phi : \mathbb{V}(AB) \to \mathbb{V}(CD),$$

the result will be *one of* the states in $\phi(\mathsf{LT}^{-1}_{A,B}(\omega))$. But if ϕ isn't local, these needn't lie in a single fibre of $\mathsf{LT}_{C,D}$: parties C and D might observe any of the different states in $\mathsf{LT}_{C,D}(\phi(\mathsf{LT}^{-1}_{A,B}(\omega)))$, giving the impression that ϕ acted indeterministically.

How should one quantify this extra layer of uncertainty?