**Title:** Factorisation Quantum Groups

**Speakers:** Alexei Latyntsev

**Collection/Series:** Mathematical Physics

**Subject:** Mathematical physics

**Date:** March 07, 2024 - 11:00 AM

**URL:** https://pirsa.org/24030101

**Abstract:**

This talk explains the theory of "factorisation" or "vertex algebra" analogues of the theory of quantum groups, (2312.07274).

We will first recap the theory of quantum groups and their connections to wider mathematics, due to Drinfeld, Etingof, Kazhdan, Jimbo, Reshetikhin, Turaev, and many others. We will then explain how to give a natural definition of "factorisation" version of these objects, give the basic structure theory about their categories of representations, and derive an explicit model for them. This relates to previous work of Etingof-Kazhdan and Frenkel-Reshetikhin.

We will sketch some physics heuristics for the above theory; briefly, our structures should appear on the category of line operators in 4d theories due to Costello, Witten and Yamazaki, e.g. on the representations Rep U_q(g^) of affine quantum groups.

We will then give some examples. The last part discusses the conjectures that naturally suggest themselves, for instance by analogy to the connections referred to above.

Prerequisites on chiral and vertex algebras will be given at a talk earlier in the week. (1-2pm Tuesday, Bob Room)

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Zoom link

# FACTORISATION QUANTUM GROUPS 2312.07274

$\mathcal{T}$ - 3d TQFT $\rightsquigarrow$ $\mathcal{T}(S^1)$ br. mon

BC $\rightsquigarrow$ $\mathcal{T}(S^1) \to \mathrm{Vect}$

$\mathcal{T}(S^1) \to A\text{-Mod}$, such $A$ is called a quantum group.

If $\mathcal{V}$ - hol bdy to $\mathcal{T}$,

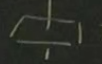
$$V\text{-Mod} \simeq \mathcal{V}(pt) \simeq \mathcal{T}(S^1) = A\text{-Mod}$$

"Kazhdan-Lusztig".

Thm (RT) $\mathcal{T}(S^1)$ is a modular tensor cat, and any such determines $\mathcal{T}$. If $\mathcal{C}$ br mon., any object $V \in \mathcal{C}$ gives an int $\mathcal{T}_V(K)$ of knots $K \subseteq S^3$.

$T = CS(G, k)$, $A = U_q(\mathfrak{g})$

<u>This talk</u>: If $T$ is 4d holl/top, what structure

has $T(S^2)$  $\mathbb{1}_\Xi$ ?

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0. QUANTUM GROUPS

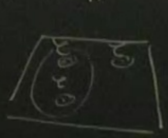
<u>Defn</u> $\mathcal{C}$ is braided monoidal if equivalently:

(1) (Lurie) It extends to a functor

$$U \subseteq \mathbb{R}^2 \rightsquigarrow \mathcal{C}(U) \in \mathrm{Cat}$$

$\mathcal{C}(U) \xrightarrow{\sim} \mathcal{C}(V)$ if $U \hookrightarrow V$ homotopy equiv, $\mathcal{C}(U_1 \sqcup \cdots \sqcup U_n) \simeq \bigotimes \mathcal{C}(U_i)$.

(2) $\mathcal{C}$ is a $\mathbb{E}_2$-algebra in $\mathrm{Cat}$



(3) (If $\mathcal{C} = D(\Xi)$) we have

$$\otimes : \Xi \otimes \Xi \to \Xi \qquad \mathbb{1}_\Xi \in \Xi$$

and a BRAIDING

$$\beta : e_1 \otimes e_2 \xrightarrow{\sim} e_2 \otimes e_1 \quad \text{satisfying hexagon.}$$

$T(S^1) \to A\text{-Mod}$, such $A$ is called a quantum group.

any such determines $T$. If $\mathcal{C}$ be mon., any object $V \in \mathcal{C}$ gives an inv't $T_V(K)$ of knots $K \subseteq S^3$.

Call alg $A$ a QUANTUM GROUP if.

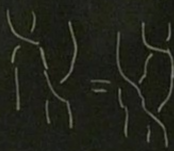
<u>Thm</u> $A$-Mod is braided (i.e. $\mathrm{Rep}A \to \mathrm{Vect}$ monoidal)

iff $A$ is a bialgebra and has an $R$-matrix

$R \in A \otimes A$

$A \otimes A \otimes A$ $\quad (\mathrm{id} \otimes \Delta) R = R_{12} R_{13} \quad (\Delta \otimes \mathrm{id}) R = R_{12} R_{23}$

with $(\mathrm{swap})\Delta = R \Delta R^{-1}$.



<u>Proof</u> Any coproduct $\Delta$ on $A$ induces $\otimes$ on $A$-Mod.

Conversely, any $\otimes$ makes $A \curvearrowright A \otimes A$

$$\Delta : A \xrightarrow{\mathrm{id} \otimes 1 \otimes 1} A \otimes (A \otimes A) \xrightarrow{\mathrm{act}} A \otimes A$$

Likewise, $\beta = R \cdot (\mathrm{swap})$ is a braiding, and given $\beta$,

$R = \beta(1)$ is an $R$-matrix

$\square$

## 0. QUANTUM GROUPS

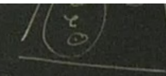
(2) $\mathcal{C}$ is a $\mathbb{E}_2$-algebra in Cat

(3) (If $\mathcal{C} = D(\mathcal{E})$) we have

$$\otimes : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E} \qquad \mathbb{1}_{\mathcal{E}} \in \mathcal{E} \qquad \&$$

and a BRAIDING

$$\beta : e_1 \otimes e_2 \xrightarrow{\sim} e_2 \otimes e_1 \quad \text{satisfying hexagon.}$$



## 1. FACTORISATION Q. GPS

<u>Defn</u> $\mathcal{C}$ is br. mon iff

(4) $\mathcal{C}$ extends to a constr. sheaf of cats on

$$\mathrm{Ran}\,\mathbb{R}^2 = \{ \text{finite } S \subseteq \mathbb{R}^2 \}$$

with a decomposition comm. alg str.

$\mathrm{Ran}\,\mathbb{R}^2$ is a <u>comm. decomp. space</u>:

$$\begin{tikzcd} & (\mathrm{Ran}\,\mathbb{R}^2 \times \mathrm{Ran}\,\mathbb{R}^2)_0 \arrow[dl, "j"'] \arrow[dr, "\cup"] & \\ \mathrm{Ran}\,\mathbb{R}^2 \times \mathrm{Ran}\,\mathbb{R}^2 & & \mathrm{Ran}\,\mathbb{R}^2 \end{tikzcd}$$

giving $\otimes^{ch} = \cup_* j^*$ sym mon on $\mathrm{Shv}(\mathrm{Ran}\,\mathbb{R}^2)$

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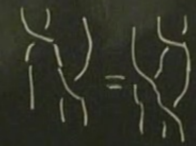
Can replace $\mathbb{R}^2$ by any $X$

<u>Thm</u> (BD) Translation-equivariant D-Modules on $\mathrm{Ran}\,\mathbb{C}$ with decom. comm alg str are vertex algebras.

$A\otimes A\otimes A$ $\quad (\mathrm{id}\otimes\Delta)R = R_{12}R_{13} \quad (\Delta\otimes\mathrm{id})R = R_{12}R_{23}$

with $(\mathrm{swap})\Delta = R\Delta R^{-1}$.



$R = \beta(1)$ is an $R$-matrix $\square$

Space $Y$ has a DECOMPOSITION STR if:

<u>Lem</u> TFAE:

(1) $Y$ is an ($\mathbb{E}_1$-)algebra object in $\mathrm{Spaces}^{\mathrm{corr}}$

$$Y_1 \qquad Y_1 \leftarrow C \rightarrow Y_2$$

(2) ($Y$ d) there is product + unit

$$Y\times Y \leftarrow C \rightarrow Y \qquad pt \leftarrow E \rightarrow Y$$

Examples: groups $Y = G$, moduli stacks $Y = \mathcal{M}_A$,

diagonal $Y\times Y \xleftarrow{\Delta} Y \rightarrow Y$, associative Ran space:

$$\mathrm{Ran}_{\mathbb{E}_1} X = \mathrm{colim}(X \to X^2 \rightrightarrows X^3 \rightrightarrows \cdots)$$

<u>ordered</u> finite subsets of $X$.

$\mathrm{Ran}\,\mathbb{R} = \{\text{finite subsets} \ldots\}$

with a decomposition comm. alg str.

$\mathrm{Ran}\,\mathbb{R}^2$ is a comm. decomp. space:

$$(\mathrm{Ran}\,\mathbb{R}^2 \times \mathrm{Ran}\,\mathbb{R}^2)_0 \quad i \swarrow \quad \searrow u$$

**Thm (BD)** Translation-equivariant D-Modules on $\mathrm{Ran}\,\mathbb{C}$ with decom. comm alg str $\Longleftrightarrow$ vertex algebras.

Likewise, if $\mathcal{C} \to Y$ is a sheaf of $\mathrm{QCoh}_Y$-cats

e.g. $\mathcal{C} = \mathrm{QCoh}_Y$, $\mathrm{DMod}_Y$, $A\text{-Mod}(\mathrm{QCoh}_Y)$, ...

Prop TFAE:

(1) $(Y, \mathcal{C})$ is $\mathbb{E}_1$-alg in cat $\mathrm{Spaces}^{\mathrm{corr}}_{\mathrm{cat}}$ $(Y_1, \mathcal{C}_1)$

$$\varphi : q^* \mathcal{C}_1 \to p^* \mathcal{C}_2.$$

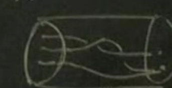
(2) (classical case) $(Y, \mathcal{E})$ has

$$\otimes_{\mathcal{E}} : q^*(\mathcal{E} \boxtimes \mathcal{E}) \to p^* \mathcal{E} \quad 1_{\mathcal{E}} \in \Gamma(Y, \mathcal{E})$$

BRAIDINGS

$$\mathrm{Ran}_{\mathbb{E}_2} X = \mathrm{colim}\big(X \to \underset{\circlearrowleft\, B_2}{X^2} \rightrightarrows \underset{\circlearrowleft\, B_3}{X^3} \rightrightarrows \cdots\big)$$

braided finite subsets of $X$



If $\mathcal{C}$ group stack, braided-commutative.

(2) (Y d) there is product + unit

$$\begin{tikzcd} & C \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y\times Y & & Y \end{tikzcd} \qquad \begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ pt & & Y \end{tikzcd}$$

PROP: TFAE:

(1) $Y$ is a (strong) $\mathbb{E}_2$-alg in $\mathrm{Span}^{corr}$

(2) ($Y$ d) in addition to product, unit,

$$\begin{tikzcd} & C \arrow[ddl, bend right, "(\mathrm{swap})\cdot q"'] \arrow[ddr, bend left, "p"] \arrow[d, "\tilde\sigma"] & \\ & C \arrow[dl, "q"] \arrow[dr, "p"'] & \\ Y\times Y & & Y \end{tikzcd}$$

satisfying hexagon

Also:

(1) $(Y,\tau)$ is a strong $\mathbb{E}_2$-alg in $\mathrm{Spans}^{corr}/\mathrm{Cat}$

(2) (d) in addition, have

$$\begin{tikzcd} q^*(\tau\boxtimes\tau) \arrow[r, "\otimes_\tau"] \arrow[d, equal] \arrow[dr, phantom, "s\Downarrow\beta"] & p^*\tau \arrow[d, equal] \\ \tilde\sigma^* q^*(\tau\boxtimes\tau) \arrow[r, "\tilde\sigma^*\otimes_\tau"'] & \tilde\sigma^* p^*\tau \end{tikzcd}$$

satisfying hexagon.

$\varphi : q^*\mathcal{C}_1 \to p^*\mathcal{C}_2.$

(2) (classical case) $(Y,\xi)$ has

$$\otimes_\xi : q^*(\xi \boxtimes \xi) \to p^*\xi \qquad 1_\xi \in \Gamma(Y,\xi)$$

If $\mathcal{C}$ groupstack, braided-commutative.

Explicitly, over

$$\begin{tikzcd} & c \arrow[dl] \arrow[dr] & \\ (y_1, y_2) & & y \end{tikzcd} \qquad \begin{tikzcd} & (g_1,g_2) \arrow[dl] \arrow[dr] & \\ (g_1, g_2) & & g_1 * g_2 \end{tikzcd} \qquad \begin{tikzcd} & (z,w) \arrow[dl] \arrow[dr] & \\ (z, w) & & (z,w) \end{tikzcd}$$

have functors

$$\otimes_c : \mathcal{C}_{y_1} \otimes \mathcal{C}_{y_2} \longrightarrow \mathcal{C}_y \qquad 1_\mathcal{C} \in \mathcal{C}_1$$

and braiding

$$\beta_c : A_1 \otimes_c B_2 \xrightarrow{\sim} B_2 \otimes_{\sigma(c)} A_1$$
$$A_1 \in \mathcal{C}_{y_1} \quad B_2 \in \mathcal{C}_{y_2}$$

Examples

| | decomp $\mathbb{E}_1$-cat | decom $\mathbb{E}_2$-cat |
|---|---|---|
| $Y = pt$ | Mon. | br. mon. |
| $(Y, \Delta)$ | a sheaf of fibrewise monoidal cats $\mathcal{C}_z \otimes \mathcal{C}_z \to \mathcal{C}_z$ | fibrewise br. mon. |
| $Y = G$ | multiplicative | + respects br. comm. |
| $Y = \mathrm{Ran}\,\Sigma$ | fact mon cat | br. fact mon cat. |

satisfying hexagon

$\tilde{\sigma}^* q^*(\tau \otimes \tau) \xrightarrow[\tilde{\sigma}^* \otimes_\tau]{} \tilde{\sigma} p^* \tau$ satisfying hexagon.

$\Delta$, viewed as a decomp. $\mathbb{E}_n$-str, is compatible with any other decomp. str. on $\Psi$. So

$$(\mathrm{Ran}\, \Sigma, \ \mathrm{ch}, \ \Delta)$$

can be viewed as $\mathrm{Ran}(\Sigma \times \mathbb{R}^n)$

Rmk. BD Grassmannian

Fix background $\mathcal{E}$, $\otimes = \otimes'$.

$A$-alg is a DECOMP QUANTUM GRP

Thm $A$-Mod has a decomp braiding iff it is a decomp bialg

$$\Delta : A \to A \otimes' A \quad (A \otimes A \to A)$$

and $R \in A \otimes' A$ satisfying

( —— )

and ( — )

$(Y, Y_2)$ $\quad Y$

$(z,w)$ $\quad (z,w)$ $\quad (z,w)$

have functors

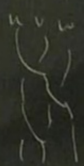
$$\otimes_c : \mathcal{C}_{Y_1} \otimes \mathcal{C}_{Y_2} \longrightarrow \mathcal{C}_Y \qquad 1_{\mathcal{C}} \in \mathcal{C}_1$$

| | | |
|---|---|---|
| $(Y, \Delta)$ | a sheaf of fibrewise monoidal cats $\mathcal{C}_z \otimes \mathcal{C}_z \to \mathcal{C}_z$ | fibrewise br. mo. |
| $Y = G$ | multiplicative | + respects br. comm. |
| | fact mon cat | br. fact mon cat. |

$(—)$ implies SYBE:

$$R_{12}(u,v)\, R_{13}(u,w)\, R_{23}(v,w)$$
$$= R_{23}(v,w)\, R_{13}(u,w)\, R_{12}(u,v),$$

$u, v, w \in Y$




Take background $\mathcal{C} = H\text{-Mod}(\mathcal{D}\mathrm{Mod}^{\wedge}{}_{\mathrm{Ran}\,\mathbb{C}})$

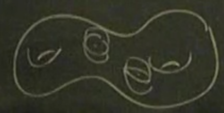
($H$ - almost cocomm. bialg). $V \in H\text{-Mod}$.

Thm $V$-Mod fact braided iff

$Y : V \otimes V \to V((z-w)) \qquad \Delta : V \to V \otimes V$

and $R : V \otimes V \to \mathbb{C}((z))$ satisfying $(—)$.

Ranal. BD Grassmannian



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( —— )

and ( —— )

EXAMPLES + QUESTIONS.

1. (DJ) $U_\hbar(sl_2)$ $U_\hbar(g)$

Thm (EK) there is a functor

$$\begin{array}{ccc} \{\text{Lie bialgebras}\} & \to & \{\text{quantum groups}\} \\ g & \longmapsto & U_\hbar(g) \\ \text{Zhu} \uparrow & & \uparrow \text{Zhu} \\ \{\text{vertex Lie bialg}\} & \overset{?}{\dashrightarrow} & \{\text{vertex q groups}\} \\ & & V_{\hbar}(g) \end{array}$$

2. Algebras of BPS states $\mathcal{M}$ = Moduli of CY3 ($K_{CP2}$

$E_2$ Alg in $E_1$ cong $(H(M), \cup)$-Mod

| | |
|---|---|
| $H(M,\varphi)^\vee$, CoHA$^\vee$, Joyce VA | $U_q(n)$ |
| $H^\cdot(M)$, $\oplus_i$, hol. Joyce VA | $U_q(t)$ |
| BPS Lie Alg (Davison) | $n$ |

$\mathrm{Rep}\, U_q(g) \simeq$ double of $U_q(n)$-Mod in $\mathrm{Rep}\, U_q(t)$)

$(Y, Y_2)$ $\quad Y$

$(z, w) \quad (z,w) \quad (z,w)$

have functors

$$\otimes_c : \mathcal{C}_{Y_1} \otimes \mathcal{C}_{Y_2} \longrightarrow \mathcal{C}_Y \qquad 1_{\mathcal{C}} \in \mathcal{C}_1$$

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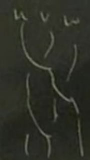


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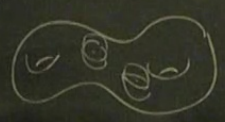

$\Delta : A \to A \otimes A$

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$V_{\hbar,k}(g)$

2. Algebras of BPS states $\mathcal{M}$ = Moduli of CY3 ($K_{C^2/\mathbb{Z}}$

$E_1$ Alg $\mathbb{E}_1$ coalg $(H^\cdot(M), \cup)$-Mod

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$H^\cdot(M)$, $\oplus$, hol. Joyce VA | $U_q(t)$

BPS Lie Alg (Davison)

$\operatorname{Rep} U_q(g) \simeq$ double of $U_q(n)$-Mod (Rep $U_q(t)$)


u, v, w ∈ V
Y : V ⊗ V → V((z − w))
Δ : V → V ⊗ V
and R : V ⊗ V → ℂ((z)) satisfying (—).
3. 𝒞 - braided fact cat / Ran Σ
V ∈ Γ(Ran Σ, 𝒞),
do we get invt τ_V(K)
Examples
decomp cat
decom E₂-cat
mon.
br. mon.
a sheaf of
fibrewise
monoidal cats
br. mon.
multiplicative
+ respects br. comm.
fact mon cat
br. fact mon cat.