Title: Once extended unitary topological field theories

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Collection: Higher Categorical Tools for Quantum Phases of Matter

Date: March 22, 2024 - 9:15 AM

URL: https://pirsa.org/24030093

Abstract: Motivated by the dagger category of Hilbert spaces, we explore the mathematical axiomatization of unitary topological field theory (TFT) using dagger categories. Recent advancements have elucidated the structure of higher dagger categories, paving the way for a precise definition of extended unitary TFTs. I will present an explicit formulation of once extended TFTs utilizing various versions of dagger bicategories. This framework enables a complete classification of unitary extended two-dimensional TFTs with arbitrary symmetry groups in terms of their unitary 2-representations. This is joint work in progress with Lukas Muller.

Once extended unitary TFT

Joint with Lukas Müller

Luuk Stehouwer

Perimeter Institute: Higher Categorical Tools for Quantum Phases of Matter

March 22, 2024

DALHOUSIE UNIVERSITY

Outline

1. Slides: unitary Atiyah TFT
   - Positive pairings on categories
   - Subtleties for fermions
   - Formulation of unitarity with dagger categories
2. Blackboard: unitary extended TFT
   - What could be a dagger 2-category?
   - $O(2)$-equivariance and spin-statistics
   - Classification of 2d unitary TFTs.

Unitarity in QFT

	Hermitian	Unitary
State space	Hermitian vector space	Hilbert space
Observables	$*$-algebra	C^*-algebra
TFT	$\mathbb{Z}_2$-equivariant functor	$\dagger$-functor
Extended TFT	equivariant functors	higher $\dagger$-functors

Unitarity in Atiyah TFT: Attempt 0

Consider symmetric monoidal functors

$$\mathrm{Bord}_{d,d-1} \to \mathrm{Hilb}\,.$$

Problem: $\mathrm{Hilb}^{fd} \to \mathrm{Vect}^{fd}$ is an equivalence of categories.

Idea: remember structure of $\dagger : \mathrm{Hilb} \to \mathrm{Hilb}^{\mathrm{op}}$

Hermitian pairings

Let $\mathcal{C}$ be a symmetric monoidal category with $\mathbb{Z}/2$-action

$$c \mapsto \overline{c} \qquad \overline{\overline{c}} \cong c.$$

Definition (Jeff Egger)

A *Hermitian form* on $x \in \mathcal{C}$ is a nondegenerate pairing $\overline{x} \otimes x \to 1$ such that

$$\begin{tikzcd}
\overline{\overline{x} \otimes x} \arrow[r, "\overline{h}"] \arrow[d] & \overline{1} \arrow[r] & 1 \\
\overline{\overline{x}} \otimes \overline{x} \arrow[r] & x \otimes \overline{x} \arrow[r, "\sigma_{x,\overline{x}}"] & \overline{x} \otimes x \arrow[u, "h"']
\end{tikzcd}$$

commutes.

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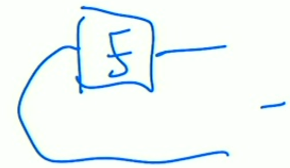
Hermitian adjoints



If (x_1, h_1) and (x_2, h_2) are objects with Hermitian pairings and $f : x_1 \to x_2$ a morphism, let $f^T : \overline{x}_2 \to \overline{x}_1$ denote the dual morphism with respect to the pairing.

The Hermitian adjoint $f^\dagger : x_2 \to x_1$ is

$$x_2 \cong \overline{\overline{x_2}} \xrightarrow{\overline{f}^T} \overline{\overline{x_1}} \cong x_1.$$

Dagger categories

Let Herm $\mathcal{C}$ be the category of pairs (x, h) with $\mathrm{Hom}_{\mathrm{Herm}\,\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}$.

Then this is a $\dagger$-category, i.e.

$$\dagger : \mathcal{C} \to \mathcal{C}^{\mathrm{op}} \quad \dagger(x) = x \quad \dagger^2 = \mathrm{id}_{\mathcal{C}}\,.$$

Example: let $\mathcal{C} = \mathrm{Vect}^{fd}$. Then $\mathrm{Herm}(\mathrm{Vect}^{fd})$ are finite-dimensional Hermitian vector spaces.

Idea: make the bordism category in a $\dagger$-category.

$\mathbb{Z}/2$-equivariant TFTs

For unitary QFT, the partition function Z satisfies $Z(\overline{X}) = \overline{Z(X)}$, where $\overline{X}$ is the orientation reversal.
Idea: categorify
Orientation reversal gives a $\mathbb{Z}/2$-action $Y \mapsto \overline{Y}$ on $\mathrm{Bord}_{d,d-1}$.

Definition (Freed-Hopkins)

A *Hermitian (or reflection) TFT* is a $\mathbb{Z}/2$-equivariant functor

$$\mathrm{Bord}_{d,d-1} \to \mathrm{Vect}\,.$$

Hermitian TFTs preserve Hermitian pairings.

Theorem (S.)

Hermitian TFTs are equivalent to †-functors

$$\mathrm{Herm}(\mathrm{Bord}_{d,d-1}) \to \mathrm{Herm}(\mathrm{Vect}).$$

Including fermions and spin

Let $\mathrm{Bord}_{d,d-1}^{\mathrm{Spin}}$ be the spin bordism category.

Definition

A *fermionic TFT* is a symmetric monoidal functor

$$\mathrm{Bord}_{d,d-1}^{\mathrm{Spin}} \to \mathrm{sVect}\,.$$

Given a spin structure $P \to Y$, define $\overline{P} := P \times_{\mathrm{Spin}} \mathrm{Pin}^{+} \setminus P$.

Which Hermitian forms on $\mathrm{Bord}_{d,d-1}^{\mathrm{Spin}}$ should be positive definite?

$0+1$d fermionic TF

Consider a 1-dimensional fermionic TFT

$$Z : \mathrm{Bord}_{1,0}^{\mathrm{Spin}} \to \mathrm{sVect},$$

determined by the state space $Z(\mathrm{pt}) = \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ graded by $(-1)^F$ equipped with a representation $Z(c)$ of $\mathrm{Spin}(1)$. Suppose Z is a Hermitian TFT.
The two Hermitian pairings

$$\overline{\mathcal{H}} \times \mathcal{H} \to \mathbb{C}$$

differ by $Z(c)$.
$\implies$ if both of these are Hilbert spaces, then Z does not depend on the spin structure.
Conclusion: have to be careful which Hermitian forms we allow on $\mathrm{Bord}_{d,d-1}^{\mathrm{Spin}}$.

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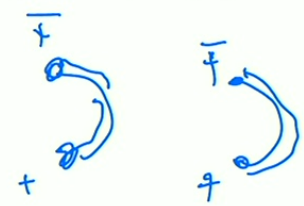

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Unitary TFT

I denote by P my preferred Hermitian pairings on $\mathrm{Bord}^{\mathrm{Spin}}_{d,d-1}$.
Let $\mathrm{Bord}^{\mathrm{Spin},P}_{d,d-1} \subseteq \mathrm{Herm}\,\mathrm{Bord}^{\mathrm{Spin}}_{d,d-1}$ denote the corresponding full subcategory.

Definition

A *unitary fermionic TFT* is a symmetric monoidal $\dagger$-functor

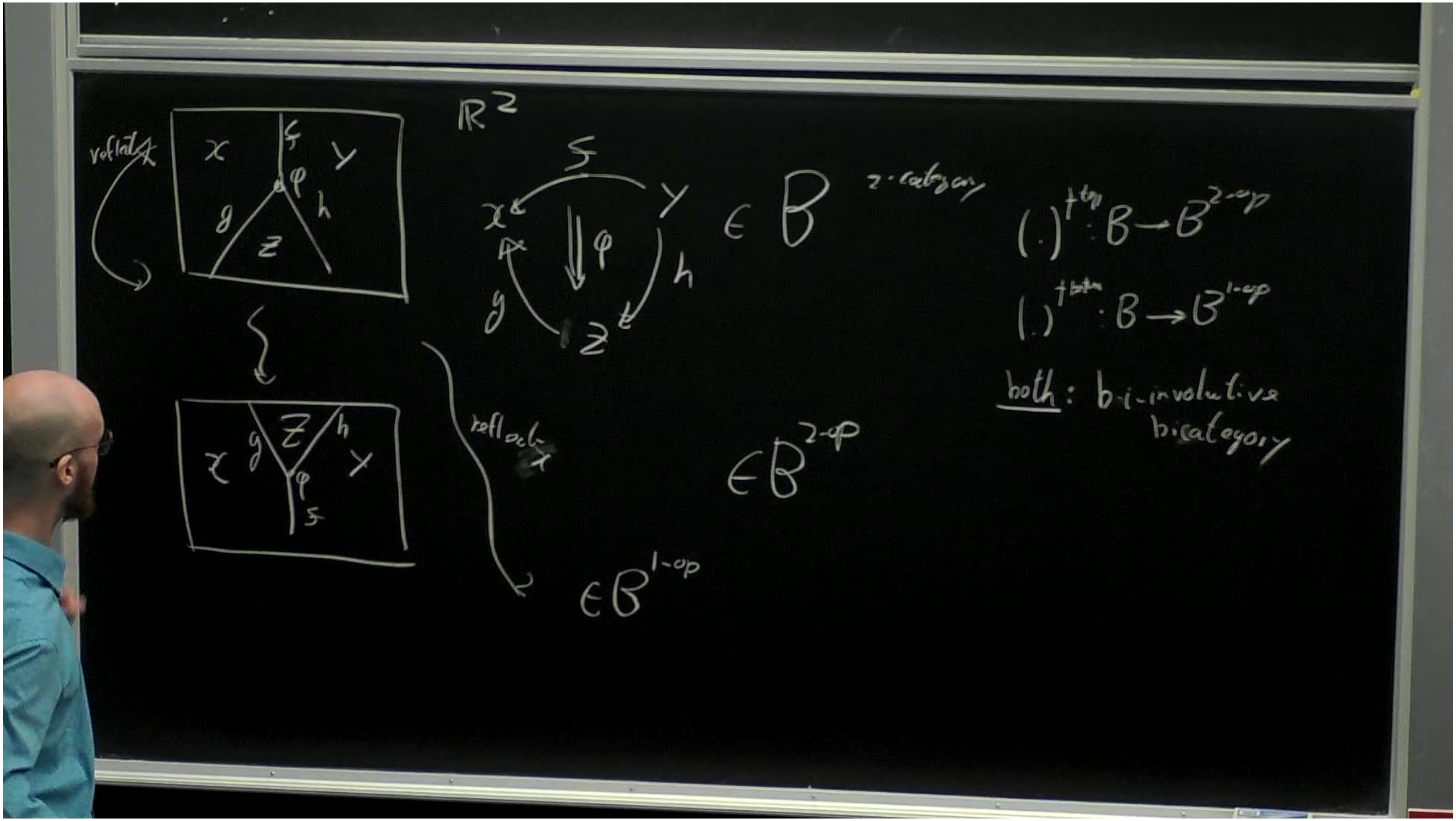
$$\mathrm{Bord}^{\mathrm{Spin},P}_{d,d-1} \to \mathrm{sHilb}\,.$$

Remark

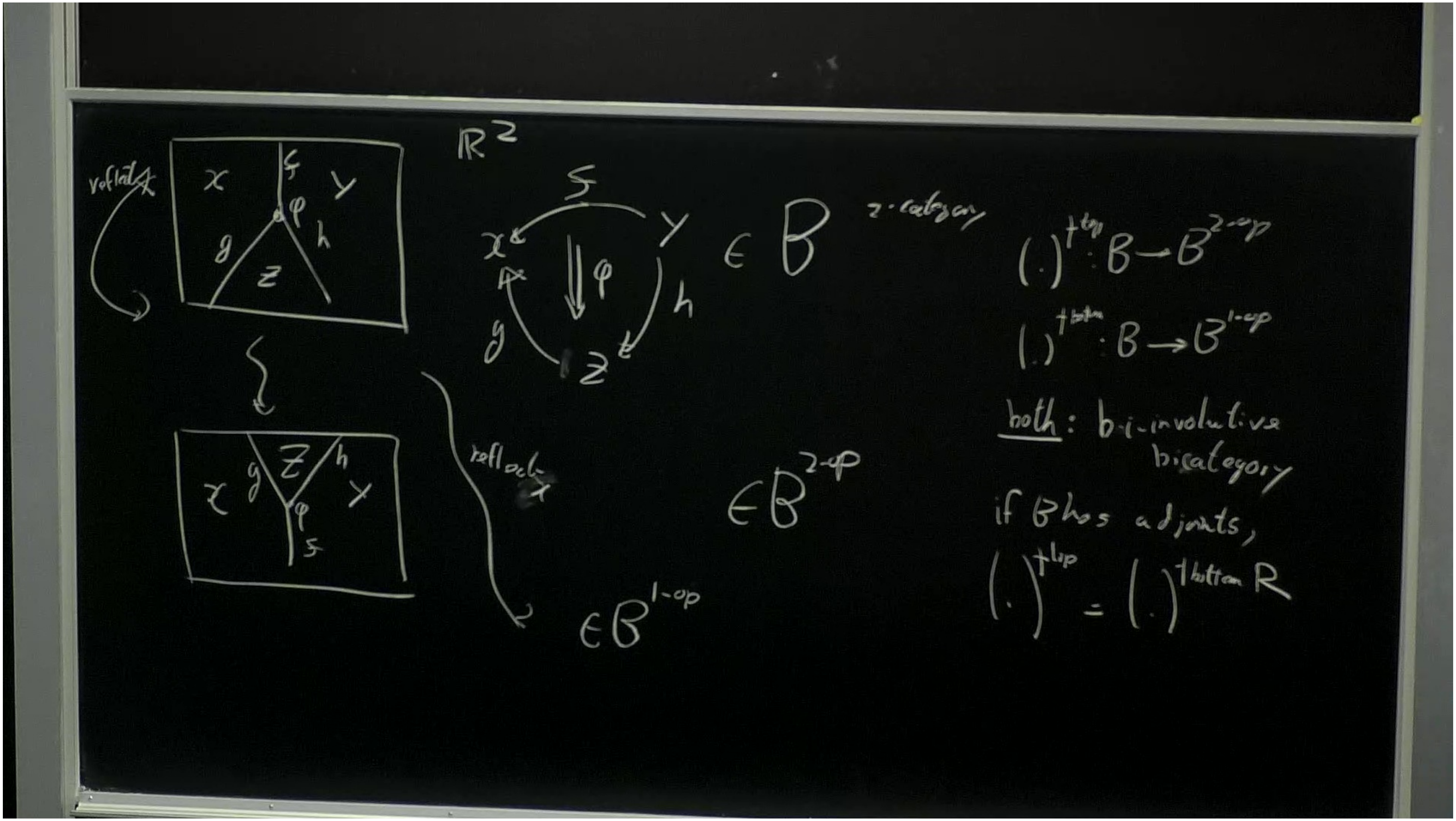
The same as Freed-Hopkins' reflection positive TFTs.

Summary

1. $\dagger$-categories are strict anti-involutions $\dagger : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$
2. Symmetric monoidal categories admit a canonical weak anti-involution $(.)^* : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$.
3. One can twist this anti-involution by a $\mathbb{Z}/2$-action $\overline{(.)} : \mathcal{C} \to \mathcal{C}$
4. To strictify the new weak anti-involution $\overline{(.)}^* : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ we need to choose Hermitian pairings $x \cong \overline{x}^*$.



reflect
$\mathbb{R}^2$
$\in \mathcal{B}$
2-category
$(\cdot)^{\dagger_{top}} : \mathcal{B} \to \mathcal{B}^{2\text{-op}}$
$(\cdot)^{\dagger_{bottom}} : \mathcal{B} \to \mathcal{B}^{1\text{-op}}$
both: bi-involutive bicategory
reflect
$\in \mathcal{B}^{2\text{-op}}$
$\in \mathcal{B}^{1\text{-op}}$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reflect
$\in \mathcal{B}^{2\text{-op}}$
if $\mathcal{B}$ has adjoints,
$(\cdot)^{\dagger_{top}} = (\cdot)^{\dagger_{bottom}\, R}$
$\in \mathcal{B}^{1\text{-op}}$

Previously $\mathcal{C} \xrightarrow{(\cdot)^\dagger} \mathcal{C}^{op}$

$\mathcal{B} \xrightarrow[(\cdot)^{RR}]{(\cdot)^*} \mathcal{B}^{1-op}$

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \mathrm{Serre}"] & \mathcal{B} \end{tikzcd}$$

$\mathcal{B}_2 = s\mathrm{lat}_{\mathbb{C}}$ $s\mathrm{Vect}_{\mathbb{C}}$-enriched

$\circlearrowleft$

$\overset{}{\underset{\cdot}{—}}$ $(-1)^F_\circ$

$\mathbb{Z}/_2 \times B\mathbb{Z}/_2 = \Omega_{\le 2}\, O \leftarrow O \leftarrow O_2$

$\mathcal{B}_1 = \mathrm{Bord}^{\mathrm{Spin}}_{d,d-1,d-2}$

$\circlearrowleft$

$\underset{\cdot}{—}$ $(-1)^F$

$B_1 \longrightarrow B_2$ $\mathbb{Z}/2$ - equiv : hermitian
$B\mathbb{Z}/2$ - equiv : spin-statistics] $O(2)$-equiv

(proven for nonextended)
spin-st theorem : $O(1)$-$\dagger$ $\Longrightarrow$ $O(2)$-equivariance

$(f \circ g)^\dagger = (-1)^{|f||g|} g^\dagger f^\dagger$

$O(1)$ herm pairing on $\mathcal{C} \in \mathrm{sCat}_{\mathbb{C}}$:

$$\mathcal{C} \overset{d}{\simeq} \overline{\mathcal{C}}^{op}$$ weak anti-involution

Guess: $O(2)$-pairings that are positive:
1. d is a $\dagger$

$SO(2)$-pairings

$$N_{\mathcal{C}} \circ (-1)^F_{\mathcal{C}} \cong \mathrm{id}_{\mathcal{C}} \Longleftrightarrow$$

super CY-structure $t(f \circ g) = t(g \circ f)$

$$t: \mathrm{Hom}(x,y) \times \mathrm{Hom}(y,x) \longrightarrow \mathbb{C}$$

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Guess: $O(2)$-pairings that are positive:

1. d is a $\dagger$
2. $\mathrm{Hom}(x,y) \simeq \mathrm{Hom}(y,x)^* \overset{\dagger}{\simeq} \overline{\mathrm{Hom}(x,y)}^*$ sHerm enriched

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$$\langle v,w\rangle = (-1)^{|v||w|}\overline{\langle w,v\rangle}$$

$$\langle v,v\rangle = (-1)^{|v|}\overline{\langle v,v\rangle}$$

$$\overline{\psi_1\psi_2} = \overline{\psi_1}\,\overline{\psi_2}$$

$O(1)$ - herm

Example $(A,*)$ super $*$-alg $\Rightarrow$ A-Mod weak anti-involution

$$M \longmapsto \mathrm{Hom}_A(M,A)$$

for which hermitian pairings are A-valued inner products on M

$\lambda(ab) = \lambda(ba)$

Frob str $\Longrightarrow$ A-Mod CY

$\lambda(a^*) = \overline{\lambda(a)}$

subex $A = \mathbb{C}l_{1,*}$

$\Rightarrow$ Arf theory $Z(X) = (-1)^{\mathrm{Arf}(X)}$

$\langle v, w \rangle = (-1)^{|v||w|} \overline{\langle w, v \rangle}$

$\langle v, v \rangle = (-1)^{|v|} \overline{\langle v, v \rangle}$

$\overline{\psi_1 \psi_2} = \overline{\psi_1}\, \overline{\psi_2}$

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Ex $A = \mathbb{C}$

A-Mod $=$ s Vect $\quad d^2 \overset{\gamma}{\simeq} id_B \Longrightarrow$ trivial theory s.t. $Z(S^1)$ neg dof

1d. unitary : $\mathrm{sHilb} \ni \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ + symmetry

$$1 \to \mathbb{Z}_2^F \to G_f \to G_b \to 1$$

$$R : G_f \to GL_{\mathbb{R}}(\mathcal{H})$$

$R(g)\, i = i\, R(g)$ if g preserves time $R(g)$ unitary

$R(g)\, i = -i\, R(g)$ — reverses — $R(g)$ anti-unitary

$$R((-1)^F) = (-1)^F$$

Final theorem in progress:

has s_1

a 2d unitary TFT with symmetry G_f is

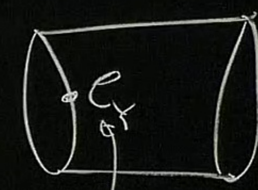
$\mathcal{C}$ super 2 Hilbert

$\rho(g) : \mathcal{C} \longrightarrow \overline{\mathcal{C}}^{\theta(g)}$ † functors   preserves $C\ell$

$\rho(g)\rho(h) \simeq \rho(gh)$ $\rho((-1)^F) \cong (-1)^F$

$\rho(\gamma) : \rho(g_1) \Longrightarrow \rho(g_2)$

Rem Y^{d-2}



admit unitary 2-rep of G_F