Title: Models of anyons with symmetry: a bulk-boundary correspondence

Speakers: Fiona Burnell

Collection: Higher Categorical Tools for Quantum Phases of Matter

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Abstract: I will describe models with on-site symmetry that permutes anyons with non-trivial mutual statistics, and show that the action of this symmetry on the boundary can effectively be that of a non-invertible symmetry such as Kramers-Wannier duality. I will sketch some implications of this for anomalies in non-invertible symmetries. Finally, I will introduce a construction (based on idempotent completion) that allows us to realize all possible anyon permuting symmetries of a given topological order in an on-site way.

Models of anyons with symmetry: a bulk-boundary correspondence

Fiona J Burnell
(University of Minnesota, Twin Cities)

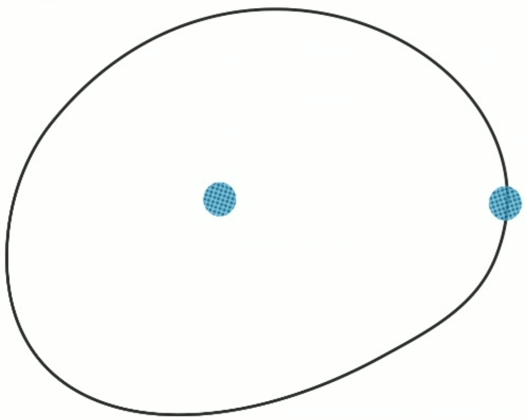
Collaborators: Kevin Walker
Michael Levin, Lukasz Fidkowski, Chris Heinrich

Themes of the talk

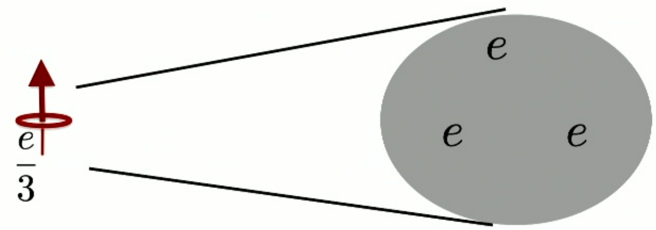
- Symmetries of anyons in 2 dimensions
- Generalized, non-invertible symmetries of 1-dimensional systems
- How are these related? And how might we find all of them, and learn what they do?

Symmetries and anyons

Exotic self-statistics: anyons
(Halperin; Arovas, Schrieffer, Wilczek; …)



Fractional electric charge



Symmetries and anyons

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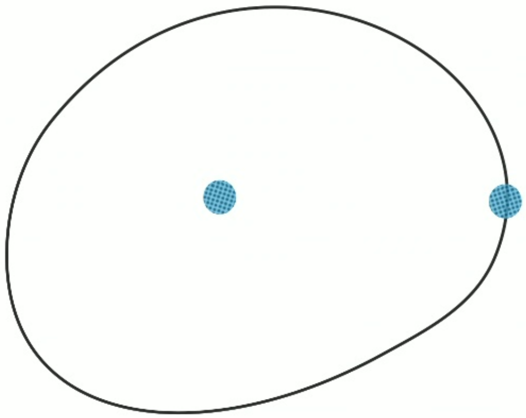
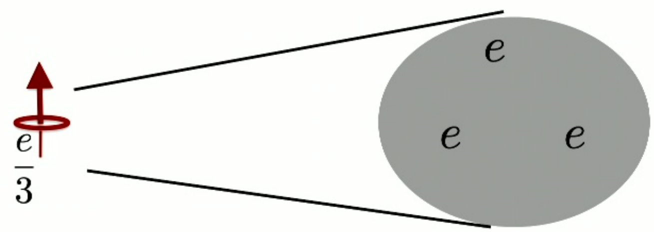


Double exchange: Fermions (+1)

Bosons (+1)

FQHE: $|\Psi\rangle \Rightarrow e^{i2\pi/3}|\Psi\rangle$

Fractional electric charge



Symmetries and anyons: Symmetry fractionalization

Wen (Projective symmetry group); Essin & Hermele; Lu & Vishwanath; Levin & Stern; …

(See review by Xie Chen)

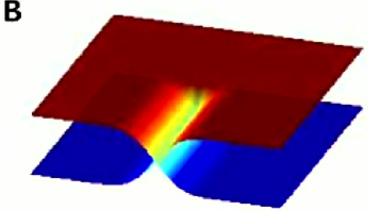
- Anyons are emergent — no need to follow the "usual" rules for quantization of (symmetry) charges
- In nature when we look for systems with emergent particles, this is more the rule than the exception

quantum spin liquids

Anyon-permuting symmetries

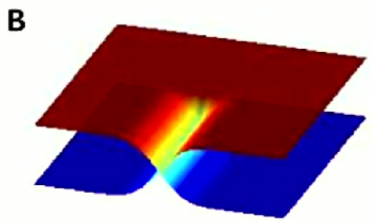
Bombin '10; Kitaev & Kong '11; Barkeshli & Qi '12, …

- Charge fractionalization is not the only interesting symmetry of anyons
- Some symmetries can act on the anyon types themselves, permuting them.
- Why care? Symmetry defects can realize non-abelian defects! (e.g. defects realized by google group)



Barkeshli & Qi

Anyon-permuting symmetries: some examples



Barkeshli & Qi

- 2 layers with the same topological order: layer exchange (lattice defects in fractional states in bands with Chern number 2 or higher — c.f. Moire materials)

$$e \leftrightarrow \bar{e}$$

$$m \leftrightarrow \overline{m}$$

- $\mathbb{Z}_N$ Topological order: charge conjugation symmetry

$$\begin{pmatrix} S & e & m & \epsilon \\ e & 1 & -1 & -1 \\ m & -1 & 1 & -1 \\ \epsilon & -1 & -1 & 1 \end{pmatrix}$$

- Toric code: Two anyons (e and m) that are both bosons can be exchanged

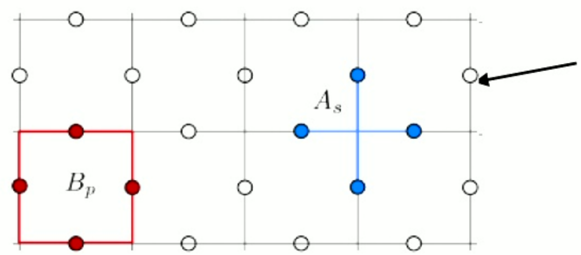
More examples: Barkeshli, Bonderson, Cheng, & Wang '19

Anyon permuting symmetry in the Toric code:

Bravyi & Kitaev '98; Kitaev '03; Levin-Wen '04; …

- Recall the Toric code…

Hilbert space: 2 states per edge



$\sigma_i^z \in \{-1, 1\}$

$$B_P = \prod_{i \in \partial p} \sigma_i^x \qquad A_v = \prod_{i \in * v} \sigma_i^z$$

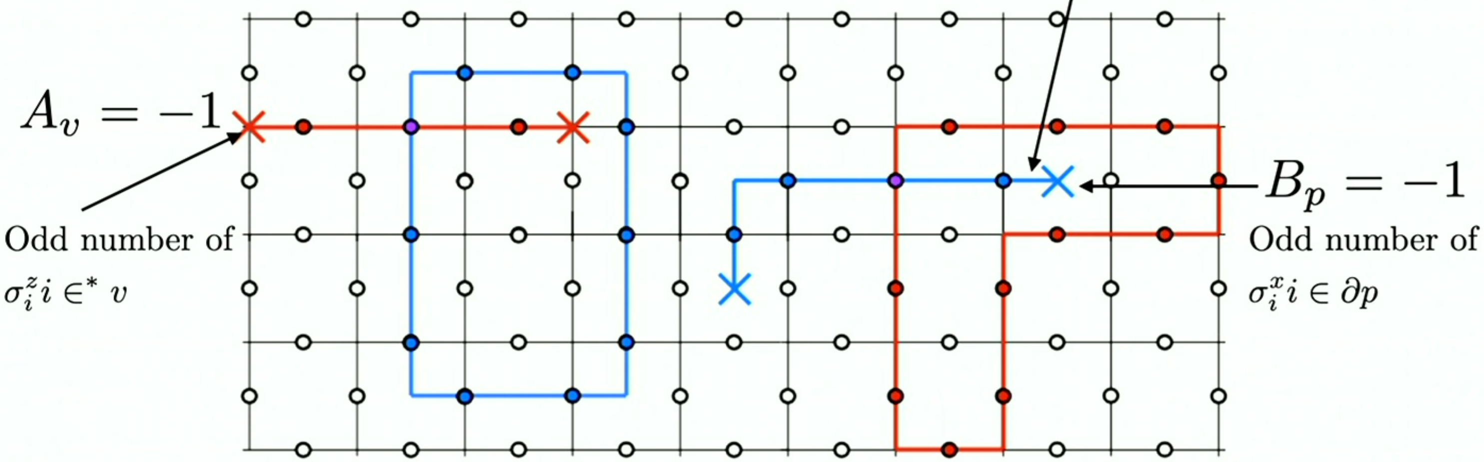
$$H = -\sum_p B_p - \sum_v A_v$$

Hamiltonian: commuting projectors

Toric code topological order

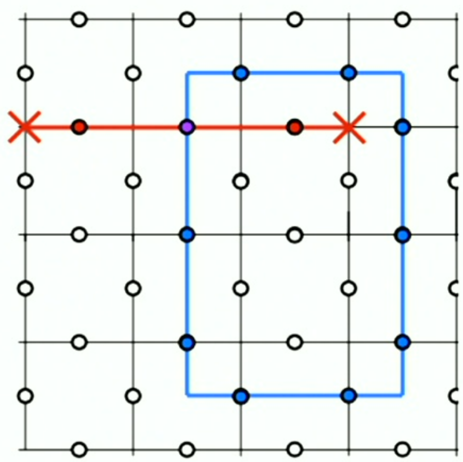
$$B_P = \prod_{i \in \partial p} \sigma_i^x \qquad A_v = \prod_{i \in ^* v} \sigma_i^z$$

Operator: $S_v = \prod_i \sigma_i^x$ **Operator:** $S_p = \prod_i \sigma_i^z$



Topological order and symmetry

$$S_p^{-1}S_v^{-1}S_pS_v = -1$$



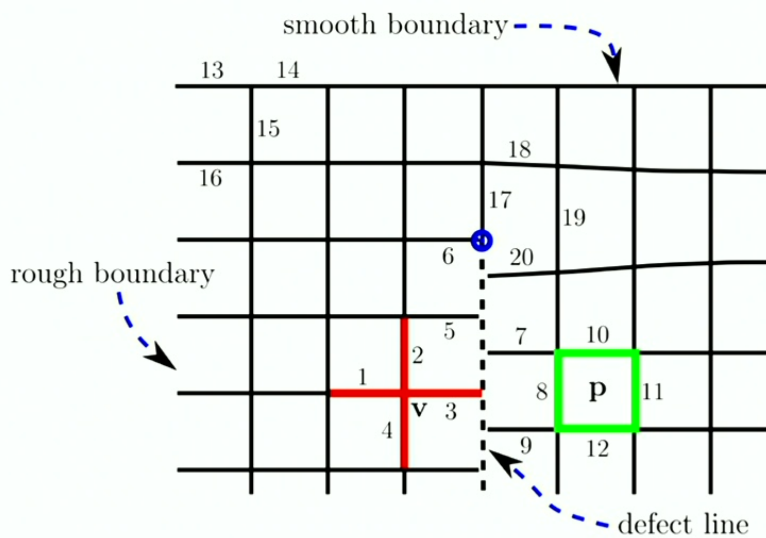
- e and m are bosons
- e and m are “mutual semions” (string operators anti-commute)
- e x m = fermion
- *Topologically, e and m are the same! There is a symmetry of the TQFT that interchanges them*

Anyon-permuting defects are non-abelian

Bombin '10; Kitaev & Kong '11

- The point: any defect that permutes e & m must be able to eat up a fermion, and thus a pair must have at least a 2-dimensional Hilbert space

- Example: (topologically the same as in the google experiment)



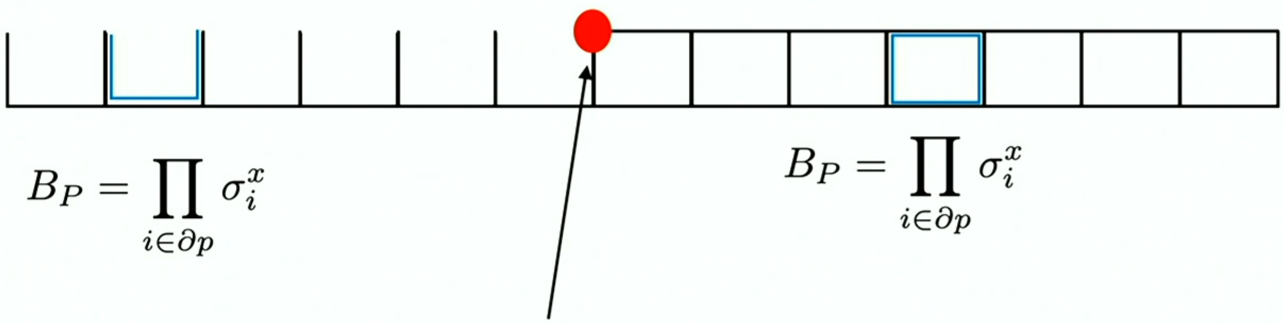
Bravyi & Kitaev '98

Anyon permuting symmetry and Toric code boundaries

- Gapped boundaries: condense an anyon to obtain the vacuum (Levin '13)
- Since e and m are bosons, both can condense

Rough ("e condensing") = boundary to Higgs phase

Smooth ("m condensing") = boundary to confined phase



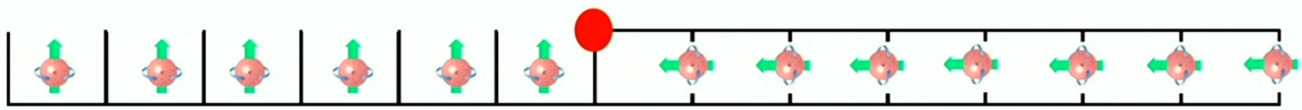
$$B_P = \prod_{i \in \partial p} \sigma_i^x$$

$$B_P = \prod_{i \in \partial p} \sigma_i^x$$

Interface: non-abelian bound state (Majorana)

- 2 gapped boundaries, related by anyon-permuting symmetry

Toric code boundaries and Kramers-Wannier duality



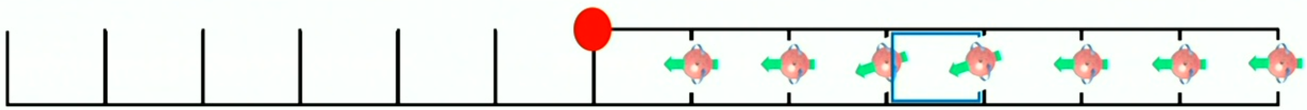
- Two views of the 1D Ising model:
  - σ^z on the direct lattice
  - σ^x on the dual lattice

- Both represent the same bulk of 1D Ising model, but the transformation is not quite unitary (finicky boundary condition issues)

Toric code boundaries and spin models

$$H = -J \sum S_i^z S_{i+1}^z - h \sum S_i^x$$

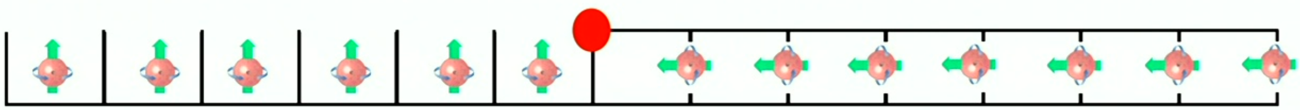
Where are the spins?
Option 2: spins on vertical edges



$$B_P = \prod_{i \in \partial p} \sigma_i^x = S^z S^z$$

- "Ferromagnetic" phase of the Ising model (dual domain walls are fluctuating)
- Symmetry action = product of vertex terms
- Kramers-Wannier dual of the plaquette-centered representation

Non-abelian defects and Kramers-Wannier duality



- A defect between these also behaves like a non-abelian anyon

- Applications: 1D superconducting wires, … etc?

Kitaev; Fu & Kane; Lutchyn, Sau, & Das Sarma; Oreg, Refael, & Von Oppen; Fendley; Clarke, Alicea, & Shtengel

Summary: e-m swap symmetry in the Toric code

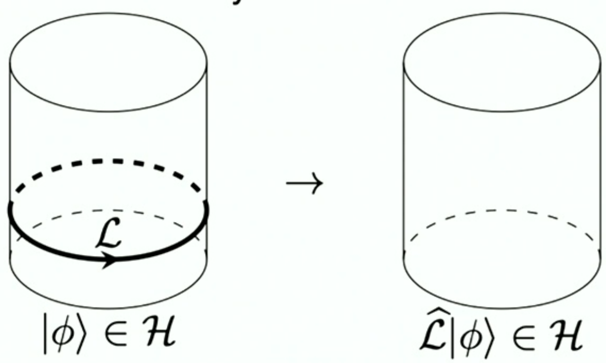
- There is a symmetry at the level of the topological order (though not at the level of Kitaev's Hamiltonian)
- This symmetry also relates the two gapped boundaries (e condensing and m condensing)
- When spins are assigned to the boundaries, these two are also related by Kramers-Wannier duality, suggesting a link between the boundary duality and the bulk anyon-permuting symmetry

Non-invertible symmetry: who cares about Kramers-Wannier anyway?

- Idea: the notion of symmetry can be generalized in interesting and physically meaningful ways, that do not require a strict group-like multiplication rule

Fendley, Aasen, Mong, Wen, Ji, Kong, Choi, Cordova, Hsin, Lam, Shao, Bhardwaj, Bottini, Schafer-Nameki, Tiwari, Gaiotto, Johnson-Freyd…

Bulk: topological line operators act on the boundary

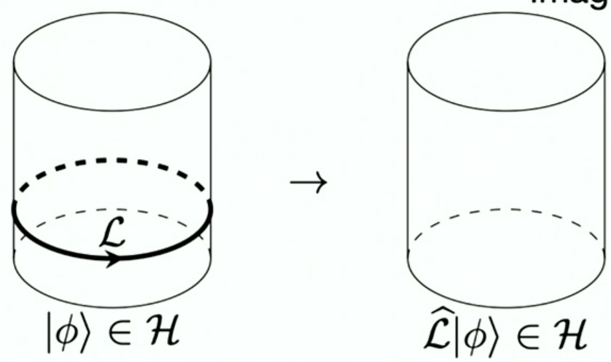


Boundary Hilbert space (States in the "system")

Image: Lin & Shao '23

Bulk topological line operators as non-invertible symmetries

Image: Lin & Shao '23



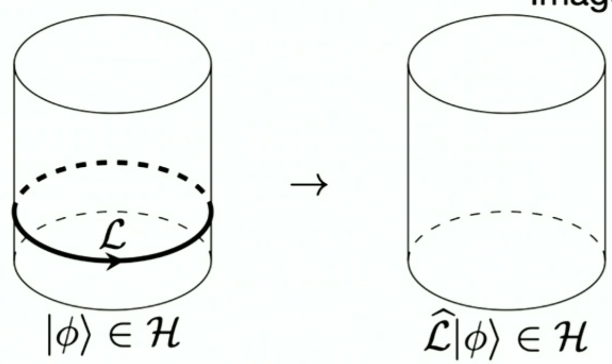
- Example: Ising CFT (Ji & Wen '19, Lin & Shao '23, and many others)

$$\mathcal{L} = \sigma$$

Non-invertible: $\sigma \times \sigma = 1 + \psi$ ← Think of this line as the usual Ising symmetry operator

Bulk topological line operators as non-invertible symmetries

Image: Lin & Shao '23



- If the line leaves the boundary state invariant, it is "symmetric" (e.g. Ising critical point is self-dual)
- True even if line operators do not have inverses
- May be useful for finding new structures in quantum field theories (conformal bootstrap in higher dimension, crossing symmetry of S matrices, etc.)

Some morals

- The Toric code topological order has an anyon-permuting symmetry, which is not, however, a symmetry of the lattice model itself.

- The Toric code also has two gapped boundaries, related by this same symmetry, associated with condensing e or m

- By associating a spin model with the boundary, we can think of these as related to Kramers-Wannier duality, which becomes a symmetry of the Ising model at its critical point.

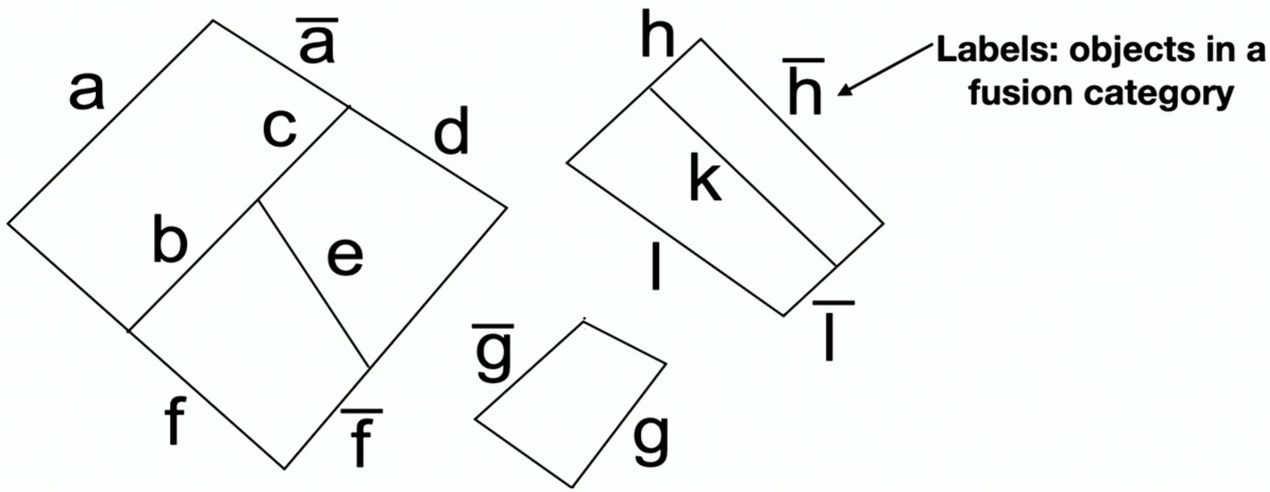
Goals of this talk:

- Bulk-boundary correspondence between anyon-permuting symmetries and dualities of the boundary theory (see also Lychtman et al '21, Wen & Potter '23)
- Consequences for anomalies
- Exploring the space of dualities using "completed category" approach

General topological orders: string nets

Kitaev '03; Levin & Wen '05; Kitaev & Kong '12; Lan & Wen '14; Lin, Levin, & **FJB** '20

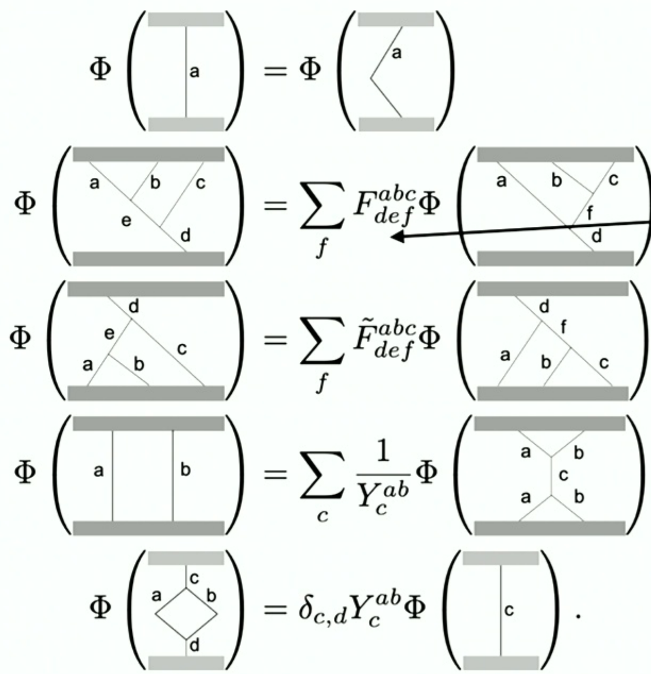
- This construction can be generalized to realize any "Drinfeld center"
- General features: 0 correlation length, gapped boundaries



General topological orders: string nets

- Ground state wave functions: local moves

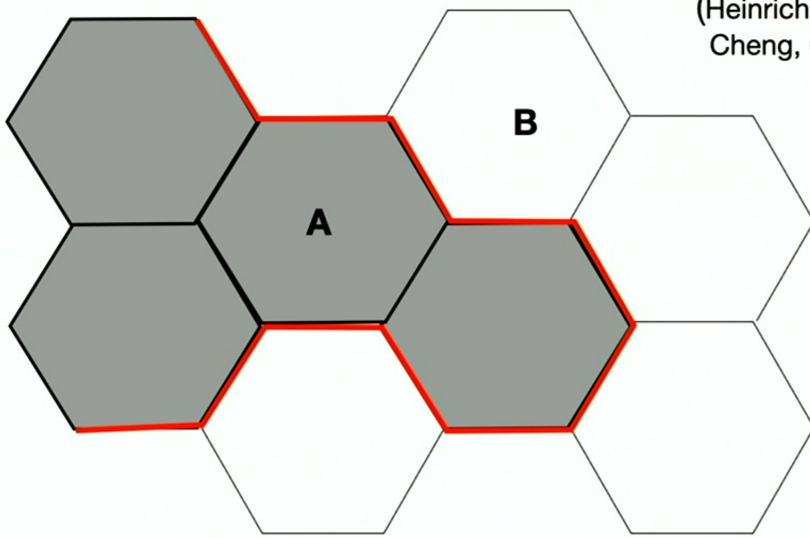
Images: Lin, Levin, **FJB** ‘20



Data: F’s, Y’s, etc.

Enhancing the symmetry: shaded string nets

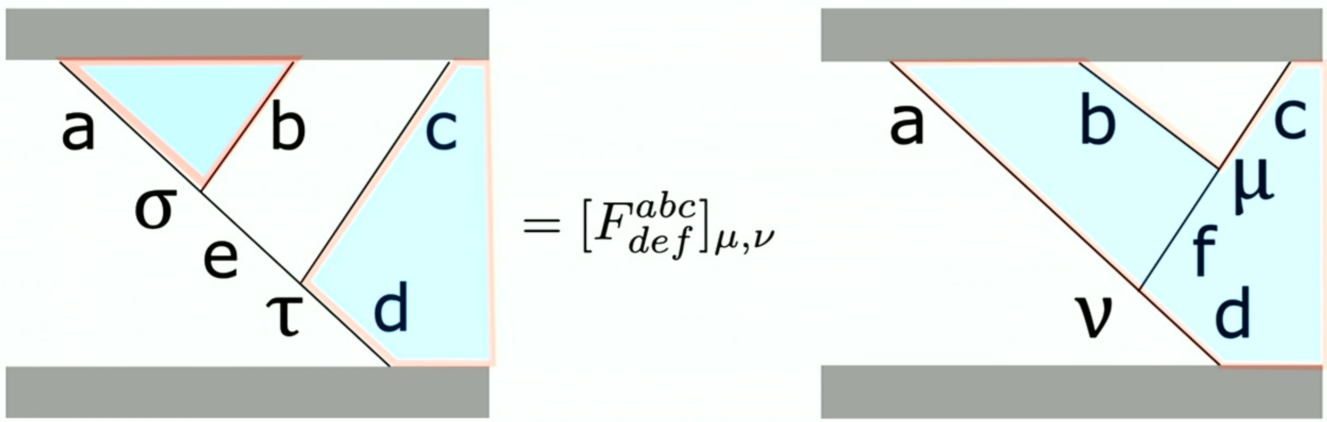
(Heinrich, FJB, Fidkowski, & Levin '16; Cheng, Gu, Jiang, & Qi '16; FJB and Walker, TBD)



- String nets are not unique: can have A, B, … that realize the same topological order. (Morita equivalence)
- Given a (Morita) equivalence between two string nets A and B, introduce a shading with 2 colors
- Within A (B), edges are as for string net A (B)
- New labels describe A/B & B/A interfaces

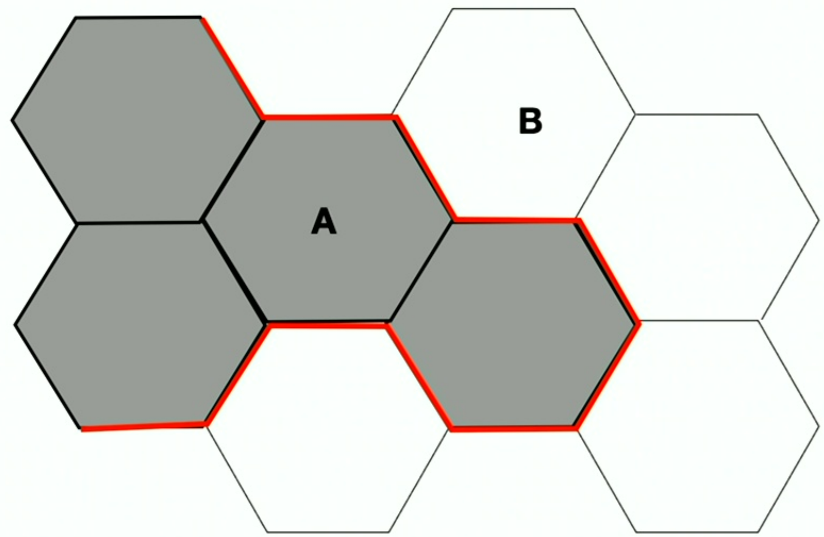
Are these string nets?

- Morita equivalence ensures that rules for how red edges fluctuate are as required for a string net Hamiltonian



- Sufficient to ensure that the bulk describes a topological symmetry action

The shaded Ising string net



$A = \mathrm{Vec}(G) = \mathbb{Z}_2$

A edges: $1, \psi$

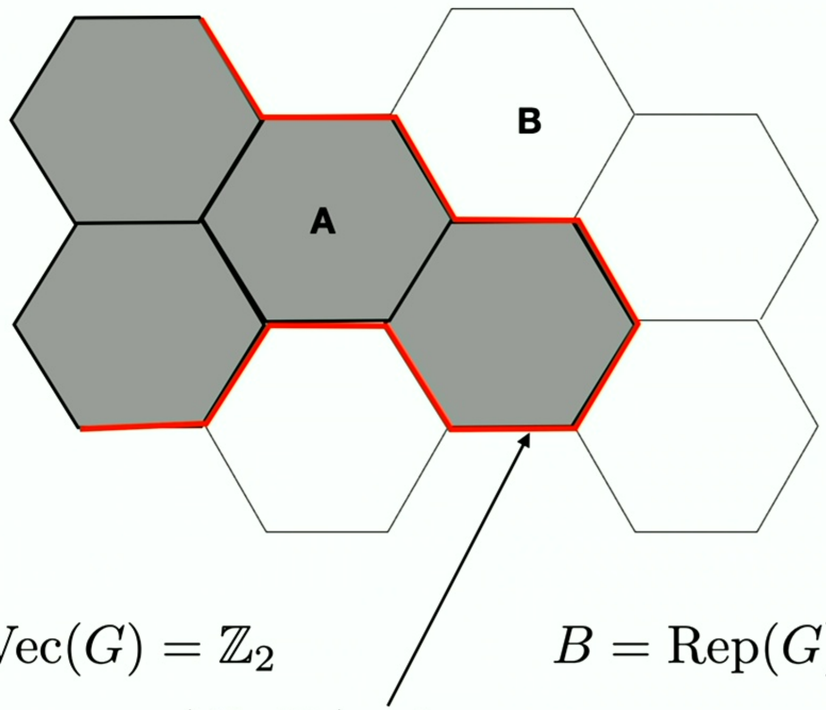
$\psi \times \psi = 1$

$B = \mathrm{Rep}(G) = \mathbb{Z}_2$

B edges: $1, \tilde{\psi}$

$\tilde{\psi} \times \tilde{\psi} = 1$

The shaded Ising string net



$A = \mathrm{Vec}(G) = \mathbb{Z}_2$

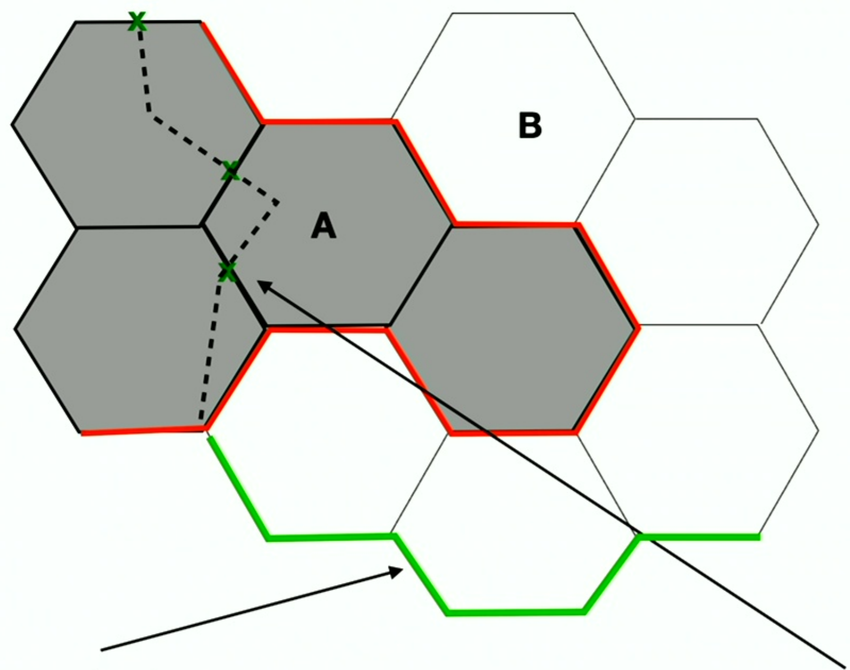
$B = \mathrm{Rep}(G) = \mathbb{Z}_2$

AB, BA edges: σ

$\sigma \times \sigma = 1 + \psi$

Heinrich, FJB, Fidkowski, & Levin '16;
Cheng, Gu, Jiang, & Qi '16

Shaded Ising string net: anyons



Operator: $S_v = \prod_i \sigma_i^x$ **Operator:** $S_p = \prod_i \sigma_i^z$

Heinrich, FJB, Fidkowski, & Levin ’16;
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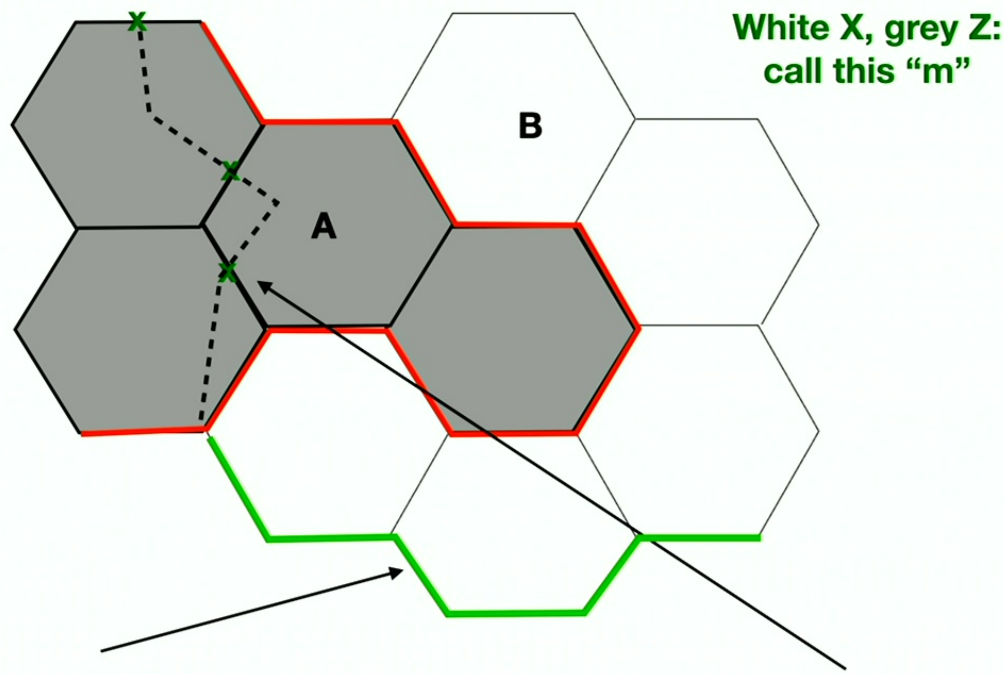

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Shaded Ising string net: Symmetries

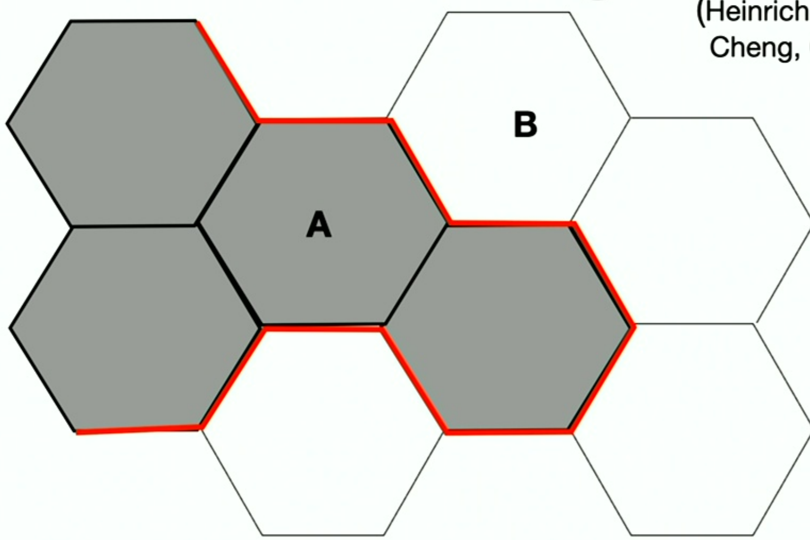
- $A \simeq B$, so there is a global $\mathbb{Z}_2$ symmetry that interchanges grey and white regions.

- This on-site symmetry interchanges e and m

- Compare: symmetry swap + local moves

Enhancing the symmetry: shaded string nets

(Heinrich, FJB, Fidkowski, & Levin '16; Cheng, Gu, Jiang, & Qi '16; FJB and Walker, TBD)



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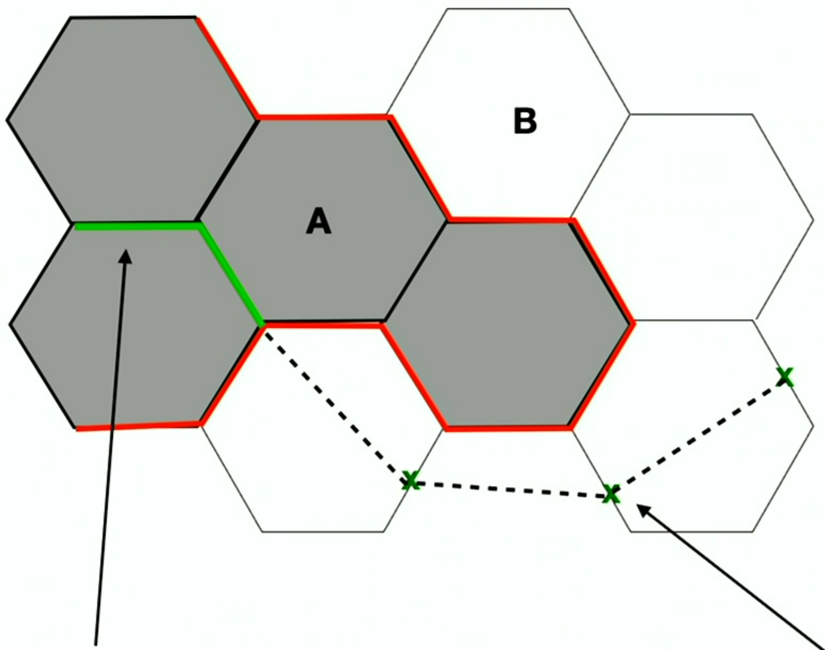
Heinrich, FJB, Fidkowski, & Levin ’16;
Cheng, Gu, Jiang, & Qi ’16

Shaded Ising string net: anyons



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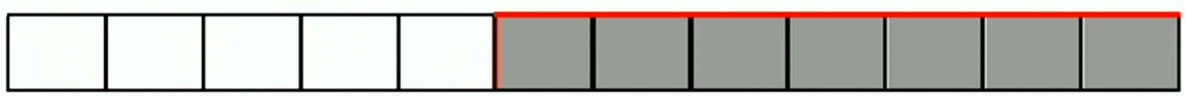
- This on-site symmetry interchanges e and m
- Compare: symmetry swap + local moves

Shaded Ising string net: gapped boundaries?

- The rough white boundary (condense white e/ grey m)



- The smooth white boundary (condense white m/ grey e)

Heinrich, FJB, Fidkowski, & Levin ’16;
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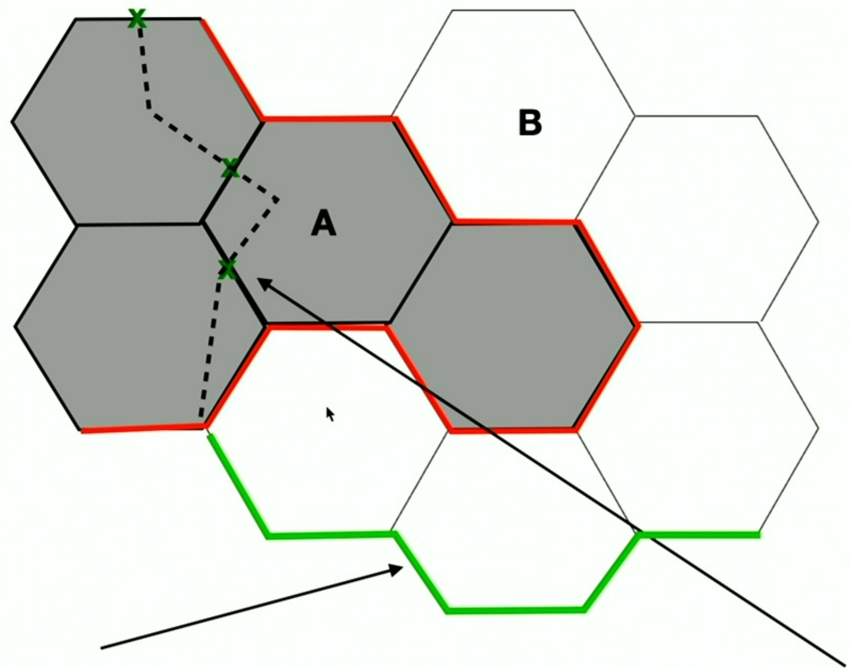
Shaded Ising string net: anyons



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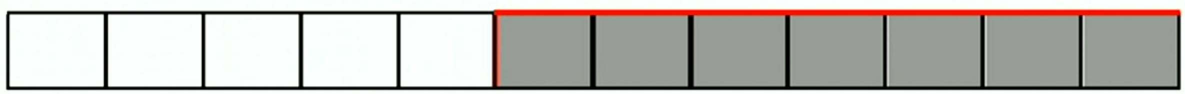
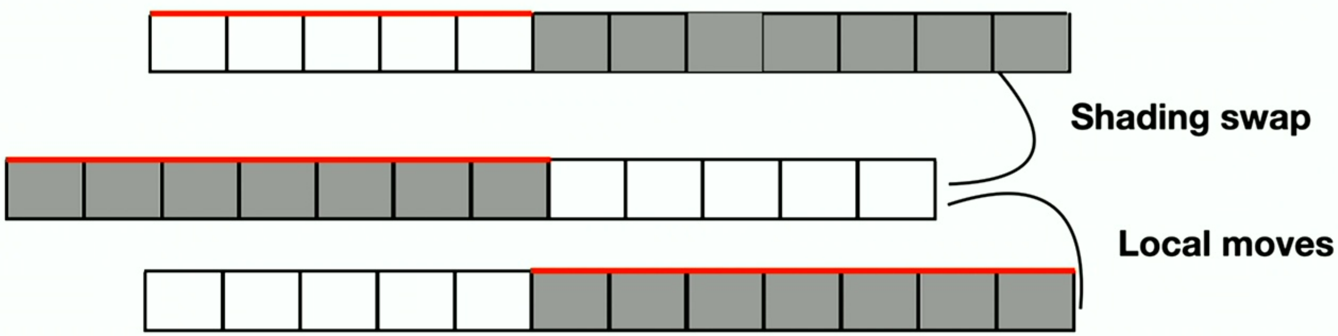


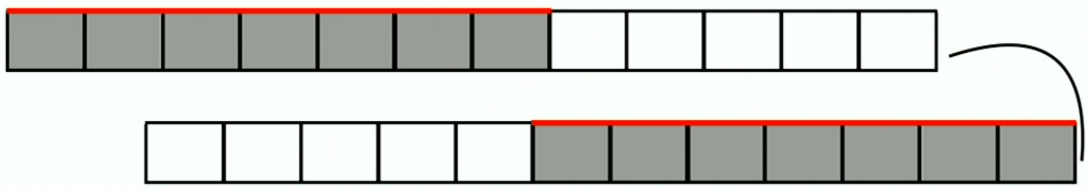
- The smooth white boundary (condense white m/ grey e)

Shaded Ising string net: Symmetries

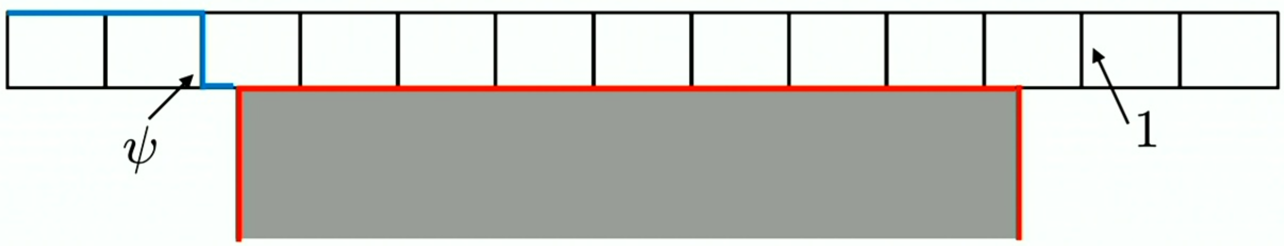
- $A \simeq B$, so there is a global $\mathbb{Z}_2$ symmetry that interchanges grey and white regions.
- This on-site symmetry interchanges e and m
- … And it interchanges each boundary color with its Kramers-Wannier dual



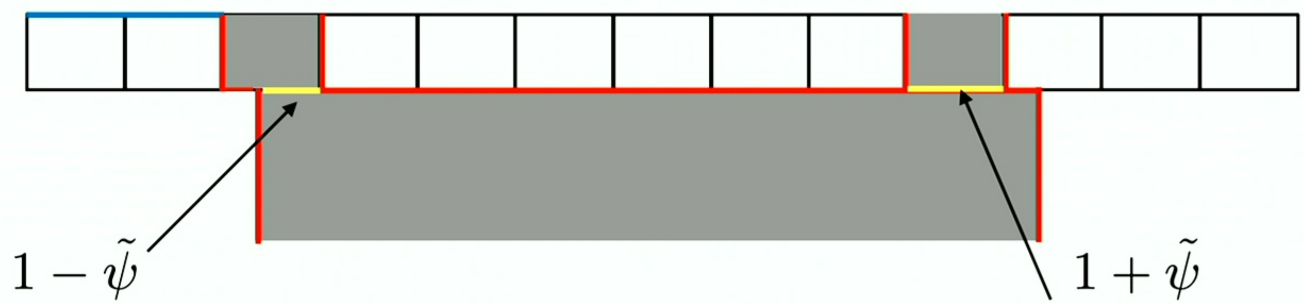
Shaded Ising string net boundaries and spin models



Let's examine the local moves:

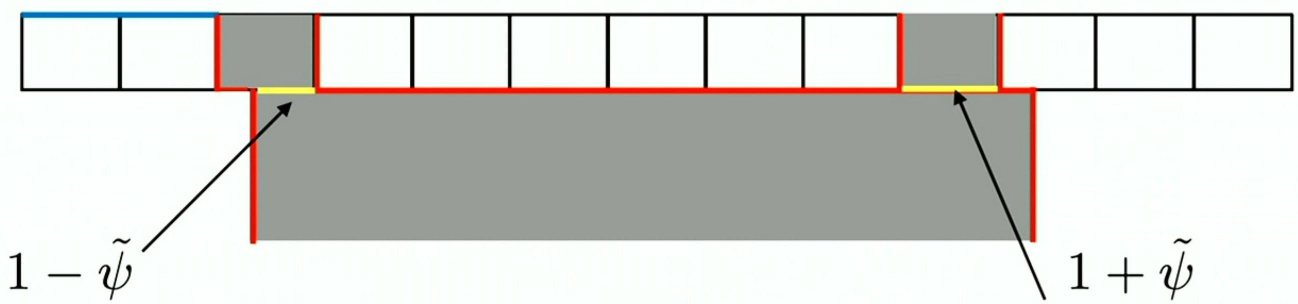


Shaded Ising string net boundaries and spin models



- Information about "occupancy" of edges perpendicular to boundary becomes information about relative phase of 2 possible edges parallel to boundary

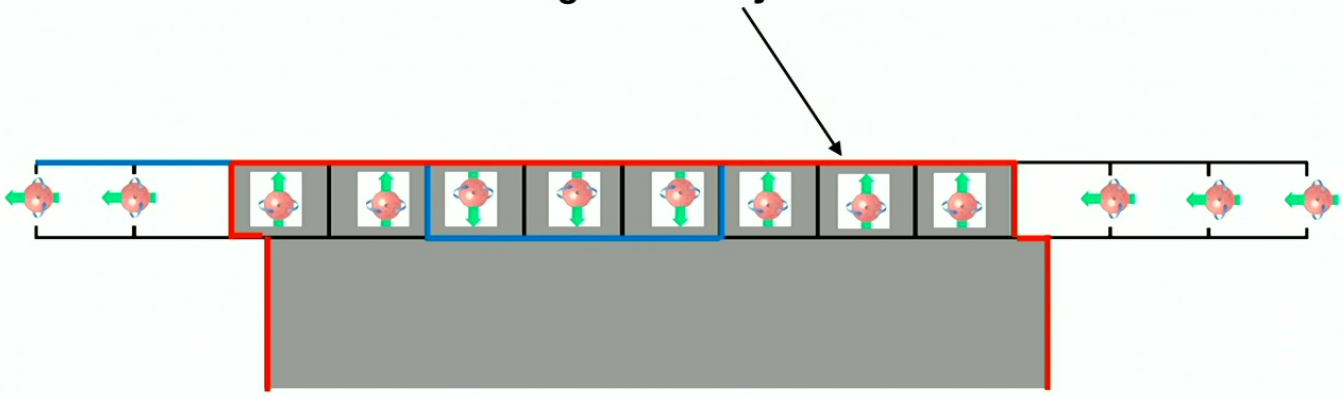
Shaded Ising string net boundaries and spin models



- Information about "occupancy" of edges perpendicular to boundary becomes information about relative phase of 2 possible edges parallel to boundary

Shaded Ising string net boundaries and spin models

This is now a rough boundary…



- Interchanges information about “edge” spins and “plaquette” spins
- I.e. interchanges the spin model with its Kramers-Wannier dual

Shaded Ising string net: morals

- On-site shading swap symmetry that permutes e and m anyons
- On boundary: swap shadings, and compare to configuration with the original shading using local moves carries out Kramers-Wannier duality
- Duality (topological) defects have the same topological data as boundaries of shaded regions (which end at Majorana zero modes)

Shaded Ising string net: What does this teach us?

Can there be symmetric shaded Ising string net boundaries?

- Gapped boundaries: condense bosons
- No boson is invariant under the symmetry!
- Thus boundaries either break symmetry, or must be gapless.

Shaded string nets: general picture

- Anytime there are 2 ways to realize the same string net, we have a shading
- If these 2 ways “look the same” locally, the shading induces a symmetry
- At the gapped boundary, this symmetry carries out a duality between different phases of the same spin model. (Self-dual point is always gapless…)
- Two boundary conditions (smooth grey, smooth white) imply gapped boundaries break the symmetry. But topological symmetry operators are (probably) not affected by this…

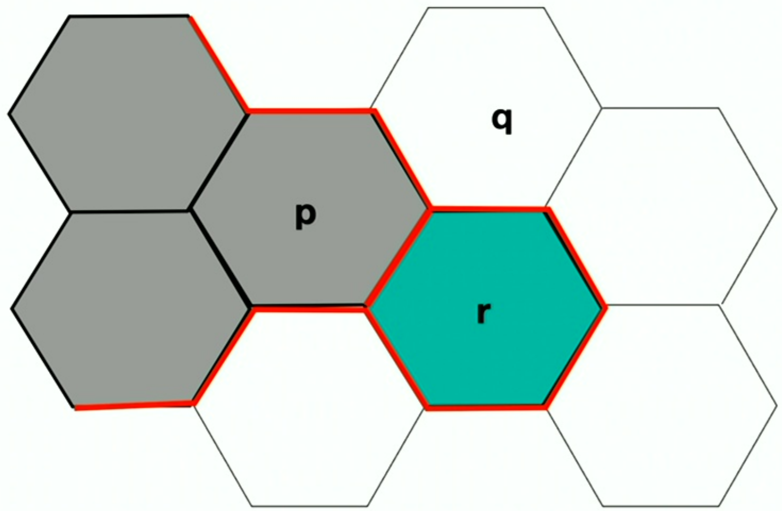
Questions of interest

- Given a model with global G symmetry, what are all of its (potentially non-invertible) symmetries?
- Given a topological order, what are all of its potential anyon-permuting symmetries?
- What are the associated symmetry defects like? (Can we extract their fusion rules etc.?)

Answer: Shade the string net by "completing" the fusion category

Completed category

Input: fusion category $\mathcal{C}$ (Data for a string net)

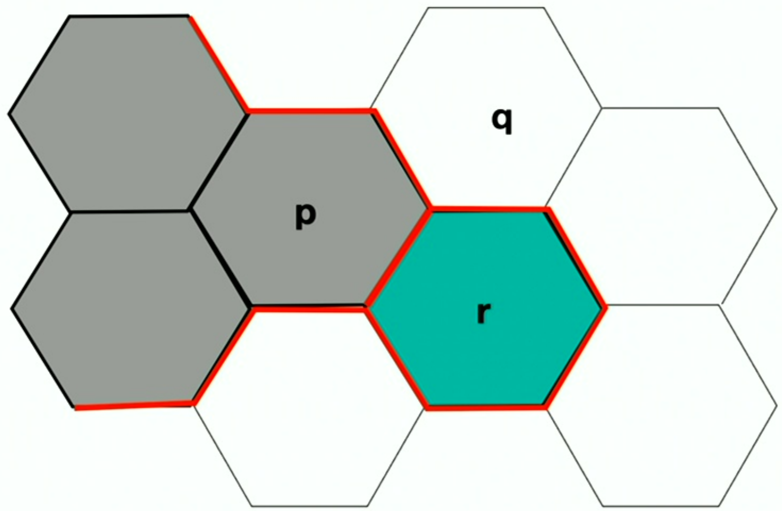


Regions $p, q, r, \ldots$ associated with *Algebra objects* of $\mathcal{C}$

(= gapped boundaries of the string net)

Completed category

Input: fusion category $\mathcal{C}$ $(= \ \mathrm{Vec}\ (G))$



Regions $p, q, r, \ldots$ associated with *Algebra objects* of $\mathcal{C}$

$\mathcal{C} = \mathrm{Vec}(G)$: Specified by H, ω (Ostrik)

(Always at least 2 colors, related by group Fourier transform)

Shaded completed string net: what is it?

- Topological order: as for string net from any one of the colors.
- 1-1 correspondence between bulk regions and possible gapped boundaries of a string net with this topological order
- Contains edges associated with all non-trivial Morita equivalences (dualities). Can systematically obtain the resulting topological data to understand defects.

Example: completion of $\mathbb{Z}_2 \times \mathbb{Z}_2$

(See also: Li, Oshikawa, & Zheng '23; Albert et al '21)

$$H = 1,\ \mathbb{Z}_2^{(A)},\ \mathbb{Z}_2^{(B)},\ \mathbb{Z}_2^{(AB)}\ (\omega = \text{trivial})$$

$$H = \mathbb{Z}_2 \times \mathbb{Z}_2\ (\omega = \text{trivial}, \text{non-trivial})$$

- These correspond to known boundary phases of matter with the same symmetry:

$$H = \mathbb{Z}_2 \times \mathbb{Z}_2\ (\omega = \text{trivial}, \text{non-trivial})$$

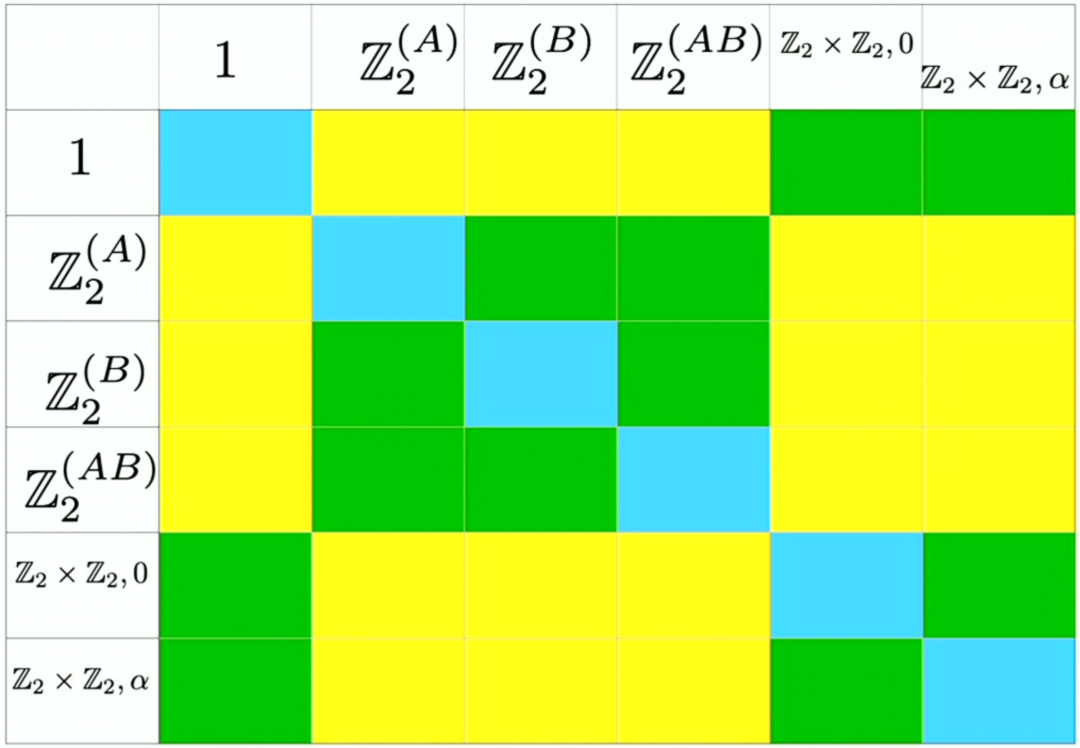
Two different symmetric phases

$H = 1$ Fully ordered phase (SSB)

$H = \mathbb{Z}_2^{(A)},\ \mathbb{Z}_2^{(B)}$ Partially ordered: SSB of one copy of $\mathbb{Z}_2$

$H = \mathbb{Z}_2^{(AB)}$ Partially ordered: symmetry reduced to diagonal subgroup

(See also: Li, Oshikawa, & Zheng '23; Albert et al '21)

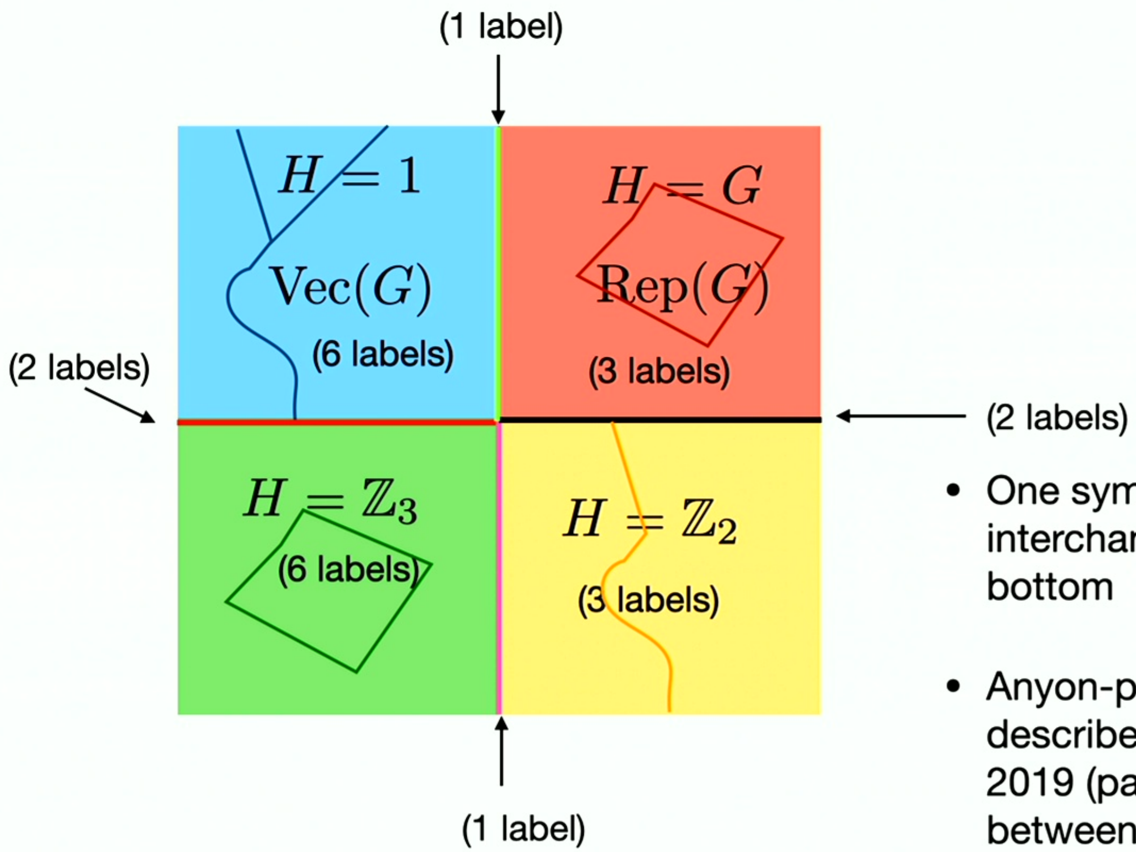


4 labels (like $\mathbb{Z}_2 \times \mathbb{Z}_2$ string net)

2 labels (partial E-M duality)

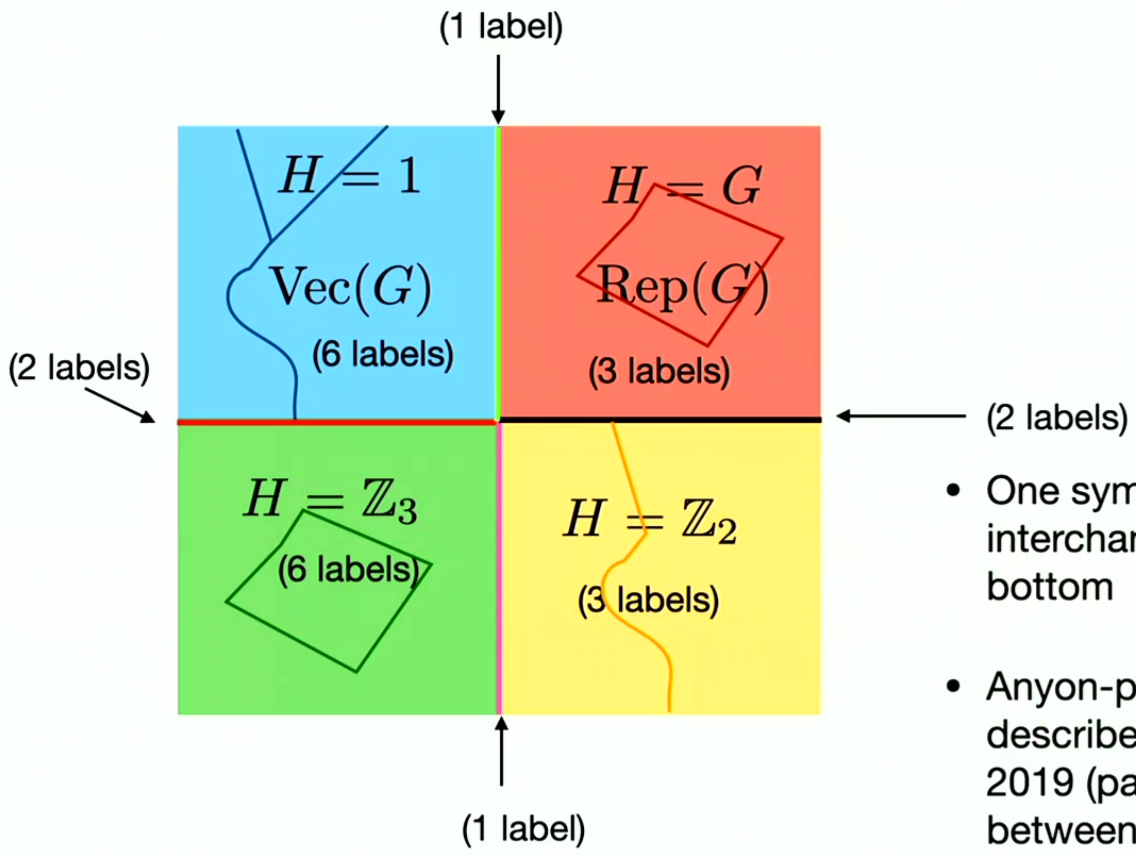
1 label (full E-M duality)

Completion of S3



- One symmetry, interchanging top and bottom
- Anyon-permuting symmetry described in Barkeshli et al 2019 (partial E-M duality between 2 anyons)

Completion of S3



- One symmetry, interchanging top and bottom
- Anyon-permuting symmetry described in Barkeshli et al 2019 (partial E-M duality between 2 anyons)

What do we learn?

- Completion: generates many new dualities, including some that are symmetries of the shaded string net
- Symmetries are realized in an on-site manner (anyon permuting symmetries in bulk and boundary dualities)
- Systematic procedure for constructing topological lines separating different regions, including potential non-invertible symmetries

Summary

- Introduced lattice models with anyon permuting symmetry, and discussed the relationship between this symmetry and duality/ non-invertible symmetry
- "Anomalies" of non-invertible symmetry are related to obstruction to gapping edge of the SET in a symmetric way
- Categories associated with simple discrete symmetries can be "completed", revealing their potential non-invertible symmetry structure.