Title: Quantum homotopy groups

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Collection: Higher Categorical Tools for Quantum Phases of Matter

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Abstract: An *open-closed tqft* is a tqft with a choice of boundary condition. Example: the sigma model for a sufficiently finite space, with its Neumann boundary. Slogan: every open-closed tqft is (sigma model, Neumann boundary) for some "quantum space". In this talk, I will construct homotopy groups for every such "quantum space" (and recover usual homotopy groups). More precisely, these "groups" are Hopf in some category. Given a "quantum fibre bundle" (a relative open-closed tqft), I will construct a Puppe long exact sequence. Retracts in 3-categories and a higher Beck-Chevalley condition will make appearances. This project is joint work in progress with David Reutter.

Quantum Homotopy Groups

Higher Categorical Tools for Quantum Phases of Matter

Perimeter Institute, 21 March 2024

Theo Johnson-Freyd, Perimeter & Dalhousie

Based on joint work in progress with David Reutter

these slides: http://categorified.net/QuantumHomotopy.pdf

Sponsor message: TQFT Spring School, 20-25 May, St. John's NL
http://categorified.net/TQFT2024/

Classical homotopy groups:

Any space $X \rightsquigarrow$ • groupoid $\pi_{\leq 1} X$.

encodes the action of π_1 on π_k.

• $\pi_2 X, \pi_3 X, \ldots : \pi_{\leq 1} X \to \mathrm{AbGp}, \quad x \mapsto \pi_k(X, x)$

if X is sufficiently finite

$n+1$D sigma model w/ nD Neumann boundary

The sigma model "knows" the homotopy gps:

$\mathcal{H}\Big($  $\leftarrow S^k \times D^{n-k} \Big) = \mathbb{C}[\pi_k(X,x)]$ as a Hopf alg:

Dirichlet b.c. for $x \in X$

Neumann b.c. elsewhere

$m = \mathrm{Pants}^{k+1} \times D^{n-k}$

$\Delta = S^k \times \mathrm{cochaps}^{n-k+1}$

Goal: • Think of every $n+1$D TQFT w/ b.c. as a "sigma model w/ Neumann b.c" for some "quantum space"

• extract homotopy gps of this "quantum space".

The quantum fundamental gpoid

framing is OK!

Suppose Q is a (at least) once-extended open-close $n+1$D TQFT

The 1-category $Q(D^{n-1})$ is symmetric monoidal if $n \geq 3$.

 $=$: $Q(D^{n-1}) \times Q(D^{n-1}) \xrightarrow{\otimes} Q(D^{n-1})$

braiding $=$ 

Tannakian Philosophy: Every symmetric monoidal category $(\mathcal{C}, \otimes)$ should be thought of as $\mathrm{Rep}(\mathcal{G})$ for some gpoid $\mathcal{G} =$ "Spec $\mathcal{C}$".

$\{\text{Points of } \mathcal{G}\} = \{\text{fibre functors } \mathcal{C} \longrightarrow \mathrm{Vec}\}$.

Think of these as choices  or maybe  ← "dir b.c."

← "neum b.c."

$\mathcal{C} \longrightarrow \mathrm{Vec}$ $\qquad$ $\mathcal{C} \longrightarrow \mathrm{Vec}$

Motivating calculation: If Q is a sigma model with Neumann b.c., then $Q(D^{n-2}) = \mathrm{Rep}(\pi_{\leq 1} \text{ target space})$.

So in an arbitrary open-closed TQFT, define $\pi_{\leq 1} Q := \mathrm{Spec}\ Q(D^{n-2})$.

The quantum higher homotopy gps

Classically, $\pi_k X$ is an $\boxed{\text{abelian gp}}$ $\boxed{\text{w/ an action by } \pi_{\leq 1} X}$.

Quantize: $\boxed{\text{Commutative and cocommutative Hopf alg}}$ $\boxed{\text{internal to } \mathrm{Rep}(\pi_{\leq 1} X)}$

So I'm after Hopf algebra objects internal to $Q(D^{n-1})$.

Strategy: Take a solid n-manifold $(M^n, \partial M^{n-1})$. Take a "bite" out of the boundary. Apply Q. This gives an object of $Q(D^{n-1})$.

 : $\mathrm{Vec} = Q(\varnothing) \longrightarrow Q(D^{n-1})$

Theorem: $(S^k \times D^{n-k}) \setminus$ bite is a Hopf alg in D^{n-1}.

Multiplication $= (S^k \times \text{chaps}^{n+1-k}) \setminus$ bite

chaps := solid pants.

Comultiplication $= (\text{pants}^{k+1} \times D^{n-k}) \setminus$ bite

e.g.  $= \text{chaps}^2$.

"CPT thm": antipode $= (\text{reflect one coord}) \times (\text{reflect one coord})$, then untwist framings

The bite makes $S^k \times D^{n-k} \simeq S^k$ into a based sphere. $\text{pants}^{k+1} = \pi_k$ composition.

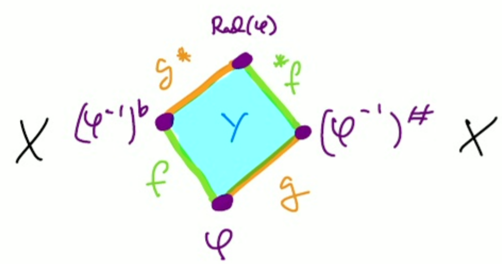
Constructing Hopf algebras

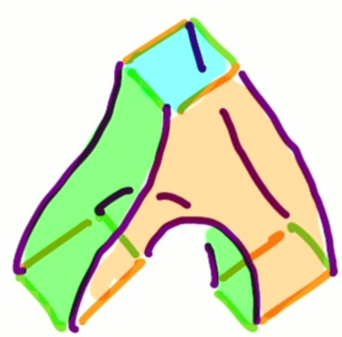
(see also Reutter's 2017 Perimeter talk)

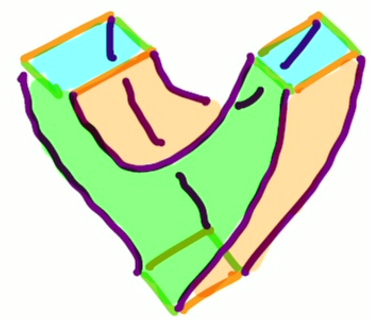
The theorem takes place in a (once-extended) bordism category. So I am allowed to prove it in a more-extended bordism category.

Corollary: Suppose $\mathcal{C}$ is an $(\infty,3)$-category w/ all adjoints. Given a (1-)retract $Y \underset{g}{\overset{f}{\rightleftarrows}} X$, $fg \overset{\varphi}{\Rrightarrow} \mathrm{id}_X$, can get a Hopf alg $H(Y,f,g,\varphi)$ in the braided monoidal $(\infty,1)$-category $\mathrm{End}^{(2)}_{\mathcal{C}}(X) = \mathrm{End}_{\mathrm{End}_{\mathcal{C}}(X)}(\mathrm{id}_X)$.

Our pf is categorical/algebraic. Here is the topological interpretation:

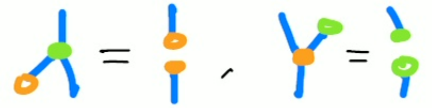
$H :=$ 

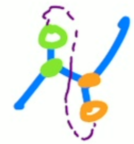

multiplication


comultiplication

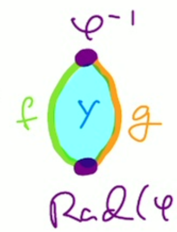
Interpretation: X = vac, Y = some QFT, f = Neumann b.c., g = Dirichlet b.c., φ = "Neum ∧ Dir"

Twisted Frobenius – Hopf algebras

A bialgebra  is Frobenius-Hopf if it is equipped with an integral  and a cointegral , meaning , such that the composition  is an antiautomorphism. Our Hopf algebra looks Frobenius, but it isn't quite: the framings are wrong. Rather, it is **twisted Frobenius Hopf:** the integral and cointegral are (co)valued in some nontrivial invertible object $I =$

Facts: (1) Twisted Frobenius Hopf $\Rightarrow$ Hopf:  is the antipode.

(2) If $\mathcal{B}$ is rigid braided monoidal and Karoubi complete, the converse holds: every Hopf alg in $\mathcal{B}$ admits a unique twisted Frobenius Hopf str.

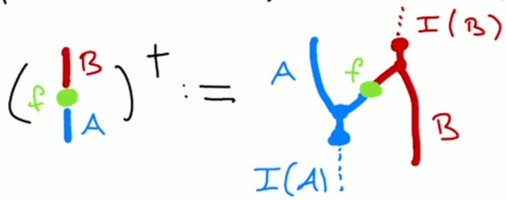
(3) In my example, the invertible is $I(Y, f, g, \varphi) =$  $\in End^{(2)}(X)$.

w/o "Rad", this would be $id^{(2)}(X)$.
"Rad" = "Radford" twists up the framings.

(Twisted) Frobenius-Hopf-Beck-Chevalley squares

A commuting square of adjunctible functors
$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "c"'] \arrow[dr, phantom, "\cong"] & \bullet \arrow[d, "b"] \\ \bullet \arrow[r, "d"'] & \bullet \end{tikzcd}$$
is **Beck-Chevalley** if the vertical adjoint
$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \\ \bullet \arrow[u, "c^*"] \arrow[r, "d"'] & \bullet \arrow[u, "b^*"'] \end{tikzcd}$$
strongly commutes, where $(-)^*$ denotes the categorical adjoint.

A homomorphism between Frobenius(-Hopf) algebras doesn't have a categorical adjoint, but it does have a **linear adjoint** defined by the Frobenius pairings. In the twisted case, the adjoint is off by some invertible objects:



up to a suppressed power of the antipode

Definition: A square
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$
is (twisted) FHBC if there is an iso $I(A)I(D) \simeq I(B)I(C)$ s.t.

$$\begin{tikzcd} A\, I(A)^{-1} I(C) \arrow[r] & B\, I(A)^{-1} I(C) \arrow[d, phantom, "\wr\wr"] \\ & B\, I(B) I(D)^{-1} \\ C \arrow[uu] \arrow[r] \arrow[uur, phantom, "\circ"] & D \arrow[u] \end{tikzcd}$$

N.B. Iso is unique $\Leftarrow$ $tr(s^2)$ is invertible.
Such H is called regular.

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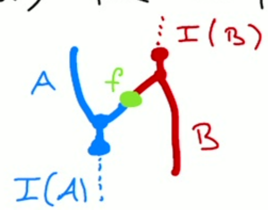
$$\left(f \, {}^{B}_{A} \right)^{\dagger} :=$$


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$$\begin{tikzcd} A\, I(A)^{-1} I(C) \arrow[r] & B\, I(A)^{-1} I(C) \arrow[d, phantom, "\cong" rotate=90] \\ C \arrow[u] \arrow[r] \arrow[ur, phantom, "\circ"] & D \arrow[r] & B\, I(B)\, I(D)^{-1} \end{tikzcd}$$

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Such H is called *regular*.

(Twisted) Frobenius-Hopf exact sequences

Proposition: If $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence of finite gps which is exact at B, then the square of group algebras

$$\begin{array}{ccc} \mathbb{K}A & \xrightarrow{\mathbb{K}f} & \mathbb{K}B \\ \downarrow & & \downarrow \mathbb{K}g \\ \mathbb{K} & \longrightarrow & \mathbb{K}C \end{array}$$

is FHBC. The converse (FHBC $\Rightarrow$ middle-exact) holds if $|\mathrm{Ker}(g)| \neq 0$ in $\mathbb{K}$ (e.g. if $\mathbb{K}B$ is regular)

Definition: A sequence $\cdots \to A \to B \to C \to \cdots$ of Hopf algebras in a braided monoidal category should have

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & \circ & \searrow & \\ A & & & & C \\ & \searrow & & \nearrow & \\ & & \mathbb{1} & & \end{array}$$

at each entry.

A sequence of (twisted Frobenius) Hopf algebras is **Frob-Hopf exact** if these squares are FHBC.

Example: If $A \xrightarrow{\circ} B \xrightarrow{\circ} C$ is FH exact at B and A and C are regular, then $B = \mathbb{1}$ is the trivial Hopf algebra.

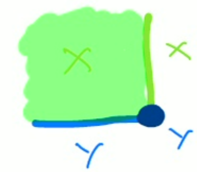
FH exactness is not very strong in the irregular case.

The quantum Puppe sequence

Given a fibre bundle $\begin{matrix} F \to Y \\ \downarrow \\ X \end{matrix}$, get a LES of $\pi_{\le 1} Y$-equivariant homotopy gps $\dots \to \pi_k F \to \pi_k Y \to \pi_k X \to \pi_{k-1} F \to \dots$

Quantum encoding: $X \rightsquigarrow$ sigma model w/ Neum b.c.

$\begin{matrix} Y \\ \downarrow \\ X \end{matrix} \rightsquigarrow$ another b.c., a corner



This is a relative open-closed TQFT.

Main Theorem: Every relative open-closed TQFT produces a FHLES of quantum homotopy gps.

Example: Calculate for 

The Hopf algebras end up measuring fusion rings of observables in bulk + boundary. The differential measures the Hopf link. 

Corollary: Bulk is invertible $\iff$ boundary observables have nondegen. "higher S-matrix".

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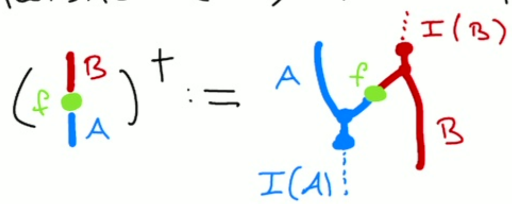

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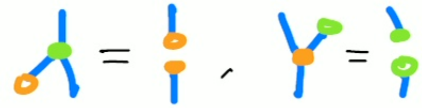
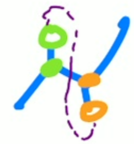
A bialgebra  is Frobenius-Hopf if it is equipped with an integral  and a cointegral , meaning , such that the composition  is an antiautomorphism. Our Hopf algebra looks Frobenius, but it isn't quite: the framings are wrong. Rather, it is **twisted Frobenius Hopf:** the integral and cointegral are (co)valued in some nontrivial invertible object $I =$

Facts: (1) Twisted Frobenius Hopf $\Rightarrow$ Hopf:  is the antipode.

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