Title: Categorical Aspects of Symmetry in Fermionic Systems

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Collection: Higher Categorical Tools for Quantum Phases of Matter

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Abstract: In this talk we explore the symmetry in 1+1d fermionic systems from a category theory perspective. We argue that It requires additional structure than a tensor category over sVect, and rather captured by a fusion category equipped with a braided central functor from a specific braided fusion category, dependent on the chiral central charge. The same data defines a 1-morphism in the 4-category of braided tensor categories, which lead us to a 4d-3d-2d picture via cobordism hypothesis.

This talk is based on an ongoing work with K. Inamura.

Categorical Aspects of Symmetry in Fermionic Systems

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c.f. K. Inamura arXiv:2206.13159

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Higher Categorical Tools for Quantum Phases of Matter

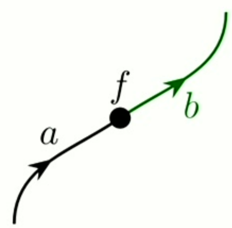
Intro and Summary

Generalized Symmetry

- Generalized Symmetry $\overset{\text{def}}{\Leftrightarrow}$ **topological defects**
- Can/should be described by **(higher)-category**
- "Quantum matter" motivations:
  - Classification of phases
  - Lattice models (e.g. Anyonic chain/ tensor networks)

Symmetry, Defects, Category

- Generalized symmetry ↔ topological defects
- 1+1d systems: line defects ↔ objects in linear category

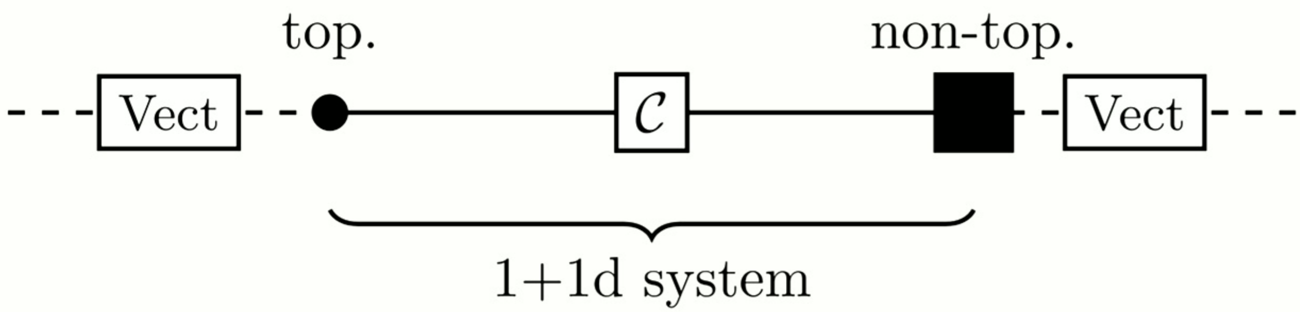


$$a, b \in \mathcal{C}$$
$$f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$$

- Finite 0-form symmetry in 1+1d: (spherical) **Fusion category**.
- (We assume semi-simplicity (from unitarity).)

SymTFT in Bosonic Systems

- $\mathcal{C}$: fusion cat → $\mathrm{TV}(\mathcal{C})$ **2+1d TQFT**
- SymTFT: 3d-2d picture/ "sandwich"



- Topological boundary ↔ $\mathcal{M} \in \mathrm{Mod}(\mathcal{C})$ ↔ Topological manipulations (gaugings) in 1+1d system

Symmetry in Fermionic Systems

- **Fermionic systems?**
- $\mathrm{Vect} \rightsquigarrow \mathrm{sVect}$, i.e. **superfusion category?**
- Not enough if we want to talk about
  the fermion parity $(-1)^F$ and its gaugings (aka. **bosonization**).
- Roughly, it should include specified objects:
  transparent fermion π and fermion parity $(-1)^F$ + **spin-statistics**.
- $(-1)^F$ can be **anomalous**, depending on the chiral central charge
  $c_- = c_L - c_R$.

Upshot

- $c_- = 0$ case: a **$Z(\mathrm{Vect}_{\mathbb{Z}_2})$-central algebra** (module tensor category, or relative fusion category)

$$\left(\begin{array}{ccc} & & Z(\underline{\mathcal{C}}) \\ & \overset{(F,\sigma)}{\nearrow} & \downarrow \text{Forget} \\ Z(\mathrm{Vect}_{\mathbb{Z}_2}) & \xrightarrow{F} & \underline{\mathcal{C}} \end{array} \right) \in \mathbf{BrFus}(Z(\mathrm{Vect}_{\mathbb{Z}_2}), \mathrm{Vect})$$

- $\underline{\mathcal{C}}$: fusion category, (F, σ): braided tensor functor.
- $\mathbf{BrFus}$: **4-category of braided fusion categories** (Haugseng 2017; Johnson-Freyd and Scheimbauer 2017; Brochier, Jordan, and Snyder 2021)
  → **4d-3d-2d picture** (Huston et al. 2023)

Outline

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BrFus (Brochier, Jordan, and Snyder 2021)

Drinfeld Center

- $\mathcal{C}$: fusion cat $\rightsquigarrow$ $Z(\mathcal{C})$: **braided** fusion cat
- $(X \in \mathcal{C}, \sigma_{X,-}) \in Z(\mathcal{C})$
  - $\sigma_{X,-} : X \otimes - \to - \otimes X$, natural isomorphism.
- $(X, \sigma_{X,-}) \otimes (Y, \sigma_{Y,-}) = (X \otimes Y, \sigma_{X,-} \circ \sigma_{Y,-})$
  - Braiding: $\sigma_{X,Y}$
- Forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$
- $\mathcal{B} \to Z(\mathcal{C})$: braided tensor functor
  $\Leftrightarrow$ a pair : $F : \mathcal{B} \to \mathcal{C}$ (tensor)
  and $\sigma_{b,-} : F(b) \otimes - \to - \otimes F(b)$ for all $b \in \mathcal{B}$.

$\mathbf{BrFus}$

- Rigid symmetric monoidal 4-category $\mathbf{BrFus}$
- $\mathcal{B} \in \mathrm{Obj}(\mathbf{BrFus})$: braided fusion category
- **$\mathcal{B}$-central algebra** (aka fusion category relative to $\mathcal{B}$)

$$\left(\begin{array}{ccc} & & Z(\underline{\mathcal{C}}) \\ & \overset{(F,\sigma)}{\nearrow} & \Big\downarrow \mathrm{Forget} \\ \mathcal{B} & \xrightarrow{\quad F \quad} & \underline{\mathcal{C}} \end{array} \right) \in \mathbf{BrFus}(\mathcal{B}, \mathrm{Vect})$$

- $\sigma_{b,c} : F(b) \otimes c \xrightarrow{\sim} c \otimes F(b)$

Balanced Tensor Product

- $(\mathcal{C}_1, F_1, \sigma_1)$: $\mathcal{B}^{\mathrm{bop}}$-central algebra
  $(\mathcal{C}_2, F_2, \sigma_2)$: $\mathcal{B}$-central algebra $\Longrightarrow \mathcal{C}_1 \boxtimes_{\mathcal{B}} \mathcal{C}_2$: fusion cat
- $(c_1 \otimes F_1(b)) \underset{\mathcal{B}}{\boxtimes} c_2 \cong c_1 \underset{\mathcal{B}}{\boxtimes} (F_2(b) \otimes c_2)$
- monoidal structure: $(c_1 \underset{\mathcal{B}}{\boxtimes} c_2) \otimes (c_1' \underset{\mathcal{B}}{\boxtimes} c_2') := (c_1 \otimes c_1') \underset{\mathcal{B}}{\boxtimes} (c_2 \otimes c_2')$
- consistency, F-symbol $\Leftarrow$ central structures

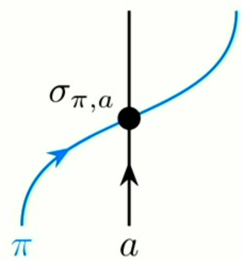
k-morphisms of $\mathbf{BrFus}$

- $\mathbf{BrFus}(\mathcal{B}_1, \mathcal{B}_2) = \{(\mathcal{B}_1 \boxtimes \mathcal{B}_2^{\mathrm{bop}})$-central algebra$\}$
  - **composition**:
    $$\mathbf{BrFus}(\mathcal{B}_1, \mathcal{B}_2) \times \mathbf{BrFus}(\mathcal{B}_2, \mathcal{B}_3) \xrightarrow{\boxtimes_{\mathcal{B}_2}} \mathbf{BrFus}(\mathcal{B}_1, \mathcal{B}_3).$$
  - **identity**: $\mathcal{B} \boxtimes \mathcal{B}^{\mathrm{bop}} \xrightarrow{((\mathrm{id},\mathrm{id}),(\sigma_{\mathcal{B}},\sigma_{\mathcal{B}}^{-1}))} Z(\mathcal{B}) \to \mathcal{B}$
- 2-morphisms $2\mathrm{Hom}_{\mathbf{BrFus}(\mathcal{B}_1,\mathcal{B}_2)}(\mathcal{C}, \mathcal{D})$: "Centered" $\mathcal{C} - \mathcal{D}$ bimodules.
  - vertical/horizontal compositions: balanced tensor products
- 3-morphisms: Functors.
- 4-morhpisms: Natural isomorphisms.

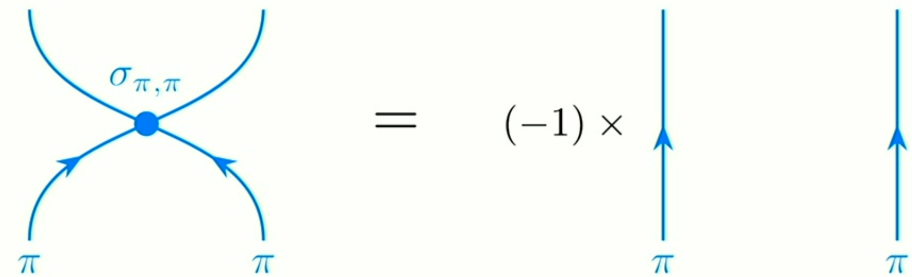
π and $(-1)^F$

Transparent Fermion

- Fermionic system $\ni$ heavy fermionic excitation; decouples from IR phys.
- π: worldline of such a excitation.



transparency



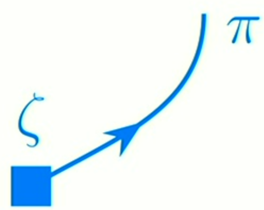
Fermionic statistics

- transparency: $\sigma_{\pi,a} : \pi \otimes a \xrightarrow{\sim} a \otimes \pi$
- fermionic half-braiding: $\sigma_{\pi,\pi} = -1$.

π-supercategory (Brundan and Ellis 2017)

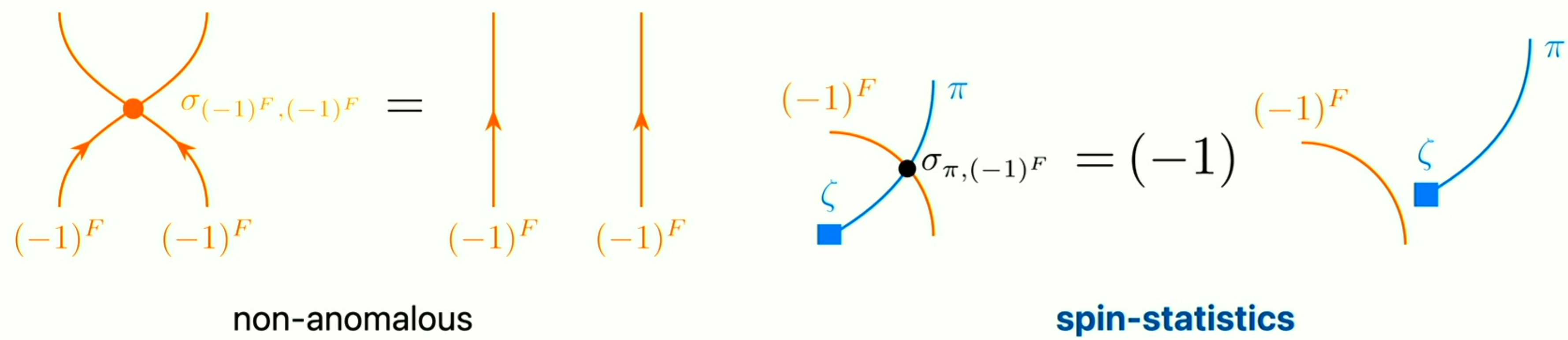
- Symmetry including $\pi \Rightarrow \underline{\mathrm{sVect}}$-central algebra
- $\underline{\mathrm{sVect}}$: underling category of sVect
- Decoupling of π from IR physics: **odd** isometry $\zeta : 1 \xrightarrow{\sim} \pi$
- $\Rightarrow \mathcal{C} := \mathrm{sVect} \underset{\underline{\mathrm{sVect}}}{\boxtimes} \underline{\mathcal{C}}$

  : monoidal Π-supercategory (Brundan and Ellis 2017)



(non-anomalous) Fermion Parity (1)

- Fermion parity $(-1)^F$ ← spin structure



- $\sigma_{(-1)^F,a} : (-1)^F \otimes a \to a \otimes (-1)^F$
- $\sigma_{(-1)^F,(-1)^F} = 1$, $\quad \sigma_{\pi,(-1)^F}$: nontrivial

(non-anomalous) Fermion Parity (2)

- braiding among $\{\pi, (-1)^F\} \to Z(\mathrm{Vect}_{\mathbb{Z}_2})$
- Symmetry is specified by **$Z(\mathrm{Vect}_{\mathbb{Z}_2})$-central algebra**

$$
\begin{array}{ccc}
 & & Z(\underline{\mathcal{C}}) \\
 & \overset{(F,\sigma)}{\nearrow} & \big\downarrow \mathrm{Forget} \\
Z(\mathrm{Vect}_{\mathbb{Z}_2}) & \xrightarrow{\quad F \quad} & \underline{\mathcal{C}} \\
\cup & & \cup \\
\{1, f, e, m\} & \longmapsto & \{1, \pi, (-1)^F, \pi(-1)^F\}
\end{array}
$$

- sVect-enrichment: $\mathcal{C} = \mathrm{sVect}_{\mathcal{Z}_2} \boxtimes_{Z(\mathrm{Vect}_{\mathbb{Z}_2})} \underline{\mathcal{C}}$

4d-3d-2d picture

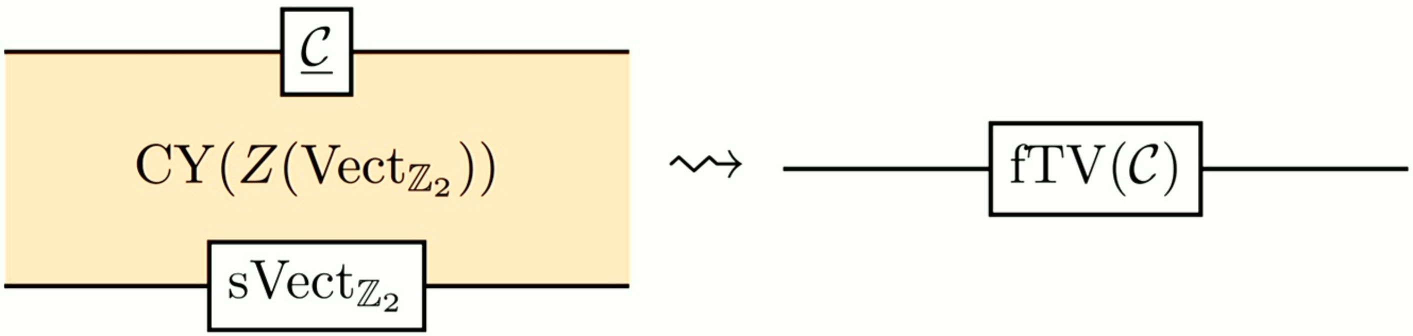
Crane-Yetter from $\mathbf{BrFus}$ (Huston et al. 2023)

- $\mathbf{BrFus}$: rigid sym. mon. 4-category
- **Cobordism hypotesis** : $\mathcal{B} \rightsquigarrow$ **Crane-Yetter** TQFT $\mathrm{CY}(\mathcal{B})$
  - $\underline{\mathcal{C}} \in \mathbf{BrFus}(\mathcal{B}, \mathrm{Vect}) \rightsquigarrow$ boundary condition of $\mathrm{CY}(\mathcal{B})$.
  - $\mathcal{M} \in 2\mathrm{Hom}_{\mathbf{BrFus}(\mathcal{B},\mathrm{Vect})}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$

    $\rightsquigarrow$ bounday interface

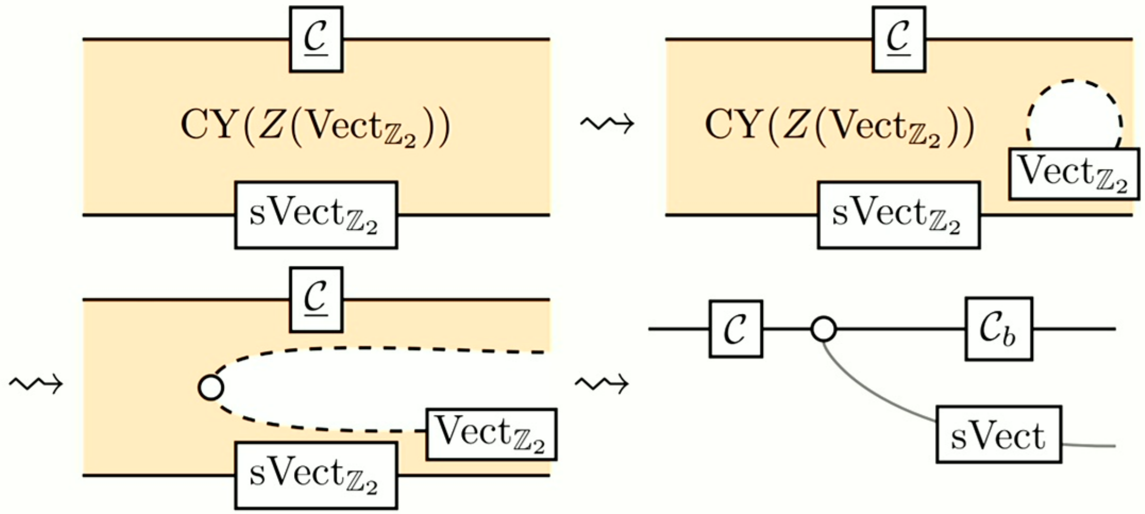
Crane-Yetter to fermionic Turaev-Viro

- superfution category: $\mathcal{C} = \mathrm{sVect}_{\mathbb{Z}_2} \boxtimes_{Z(\mathrm{Vect}_{\mathbb{Z}_2})} \underline{\mathcal{C}}$
- fermionic Turaev-Viro of $\mathcal{C}$ (Bhardwaj, Gaiotto, and Kapustin 2017) by sandwiching:



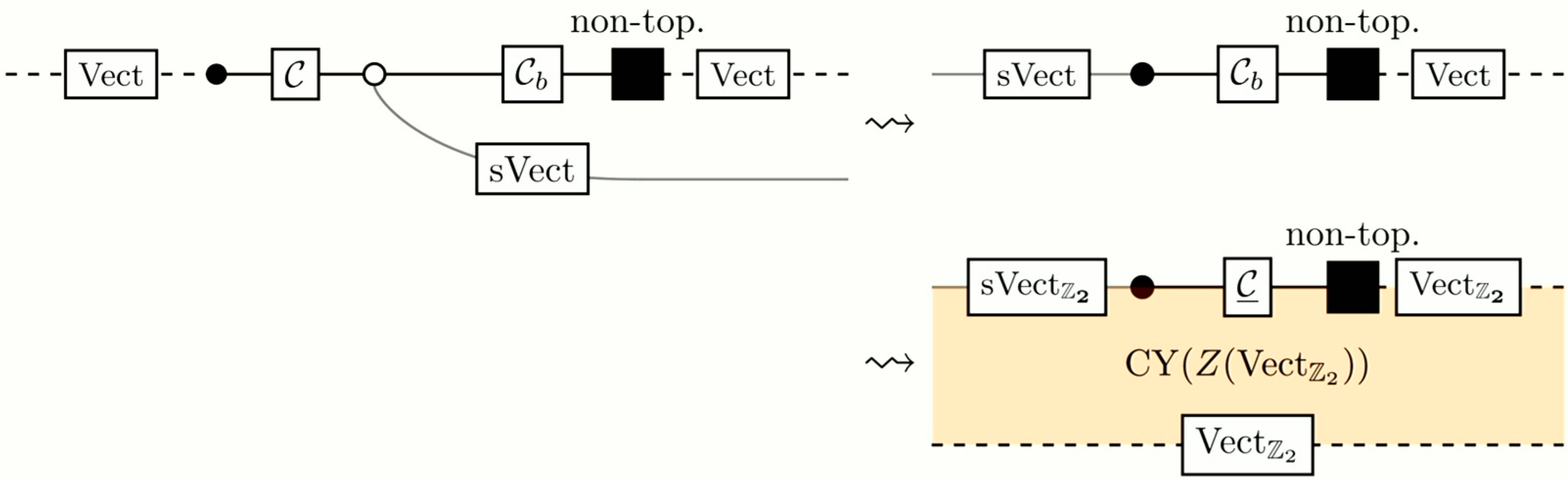
Bosonic Shadow

- $\mathrm{Vect}_{\mathbb{Z}_2} : Z(\mathrm{Vect}_{\mathbb{Z}_2}) \xrightarrow{\simeq} \mathrm{Vect}$ in $\mathbf{BrFus}$



- Bosonic shadow $\mathcal{C}_b = \mathrm{Vect}_{\mathbb{Z}_2} \boxtimes_{Z(\mathrm{Vect}_{\mathbb{Z}_2})} \underline{\mathcal{C}}$
- Morita equivalence $\mathcal{C} \sim \mathcal{C}_b \boxtimes \mathrm{sVect}$

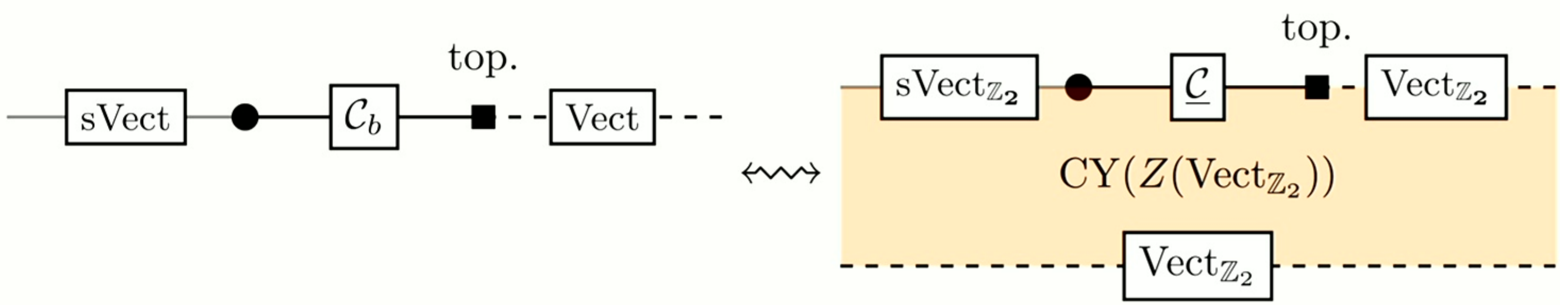
SymTFT



- Topological manipulation ↔ fermionic condensation in $\mathcal{C}_b$
  $\leftrightarrow \mathcal{M} \in 2\mathrm{Hom}(\mathrm{sVect}_{\mathbb{Z}_2}, \underline{\mathcal{C}})$

Gapped phases (Inamura 2023)

- 1+1d gapped phases with $\mathcal{C}$ action
  (possibly SSB'ed but preserving $(-1)^F$)
  $\leftrightarrow$ bosonic condensation in $\mathcal{C}_b$
  $\leftrightarrow \mathcal{N} \in 2\mathrm{Hom}(\underline{\mathcal{C}}, \mathrm{Vect}_{\mathbb{Z}_2})$.



$c_- \neq 0$

- $Z(\mathrm{Vect}_{\mathbb{Z}_2}) \rightsquigarrow \mathrm{Rep}(\mathrm{Spin}(2c_-))$.
  - $\mathrm{Rep}(\mathrm{Spin}(0)) := Z(\mathrm{Vect}_{\mathbb{Z}_2})$
  - $\mathrm{Rep}(\mathrm{Spin}(1)) := \mathbf{Ising}$
  - $\mathrm{Rep}(\mathrm{Spin}(-2c_1)) := \mathrm{Rep}(\mathrm{Spin}(2c_1))^{\mathrm{bop}}$
- $\mathrm{Rep}(\mathrm{Spin}(2c_-))$

$$\left(\begin{array}{ccc} & & Z(\underline{\mathcal{C}}) \\ & \nearrow^{(F,\sigma)} & \big\downarrow \text{Forget} \\ \mathrm{Rep}(\mathrm{Spin}(2c_-)_1) & \xrightarrow{F} & \underline{\mathcal{C}} \end{array}\right) \in \mathbf{BrFus}(\mathrm{Rep}(\mathrm{Spin}(2c_-)_1), \mathrm{Vect})$$

Summary

- Fermionic symmetry:

$$\left(\begin{array}{ccc} & & Z(\underline{\mathcal{C}}) \\ & \overset{(F,\sigma)}{\nearrow} & \big\downarrow \text{Forget} \\ Z(\mathrm{Vect}_{\mathbb{Z}_2}) & \xrightarrow{F} & \underline{\mathcal{C}} \end{array}\right) \in \mathbf{BrFus}(Z(\mathrm{Vect}_{\mathbb{Z}_2}), \mathrm{Vect})$$

- Bosonic shadow $\mathcal{C}_b = \mathrm{Vect}_{\mathbb{Z}_2} \boxtimes_{Z(\mathrm{Vect}_{\mathbb{Z}_2})} \underline{\mathcal{C}}$
  - $\mathcal{C} \overset{\text{Morita}}{\sim} \mathcal{C}_b \boxtimes \mathrm{sVect}$
- Cob. hyp. → 4d-3d-2d picture
- $c_- \neq 0 \Rightarrow Z(\mathrm{Vect}_{\mathbb{Z}_2}) \rightsquigarrow \mathrm{Rep}(\mathrm{Spin}(2c_-))$.

Prospect

- Fermionic anyonic chain
- Higher dimensions
- Parafermions