Title: Motion Groupoids

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Collection: Higher Categorical Tools for Quantum Phases of Matter

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Abstract: The braiding statistics of point particles in 2-dimensional topological phases are given by representations of the braid groups. One approach to the study of generalised particles in topological phases, loop particles in 3-dimensions for example, is to generalise (some of) the several different realisations of the braid group.

In this talk I will construct for each manifold M its motion groupoid Mot_M, whose object class is the power set of M. I will discuss several different, but equivalent, quotients on motions leading to the motion groupoid. In particular that the quotient used in the construction Mot_M can be formulated entirely in terms of a level preserving isotopy relation on the trajectories of objects under flows -- worldlines (e.g. monotonic `tangles').

I will also give a construction of a mapping class groupoid MCG_M associated to a manifold M with the same object class. For each manifold M I will construct a functor $F \colon Mot_M \to MCG_M$, and prove that this is an isomorphism if π_0 and π_1 of the appropriate space of self-homeomorphisms of M is trivial. In particular there is an isomorphism in the physically important case $M=[0,1]^n$ with fixed boundary, for any $n\in\mathbb{N}$.

I will discuss several examples throughout.

Motion groupoids

arXiv:2103.10377, with Paul Martin, João Faria Martins

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Brief overview

(I) Construction of the motion groupoid $\mathrm{Mot}_{\underline{M}}$ of a pair $\underline{M} = (M, A)$.
Morphisms are equivalence classes of continuous flows of ambient space M which fix A, acting on $\mathcal{P}M$. Recover classical definition of the motion group associated to a manifold M and a submanifold $N \in \mathcal{P}M$, by looking at the morphism group at N. Obtain groups isomorphic to braid groups, loop braid groups.

(II) Construction of mapping class groupoid $\mathrm{MCG}_{\underline{M}}$.
Morphisms are now equivalence classes of homeomorphisms of M, fixing A. The object set is again $\mathcal{P}M$. Again obtain groups isomorphic to braid groups, loop braid groups.

(III) Construction of functor $\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$.
We prove that this is an isomorphism when π_0 and π_1 of space of homeomorphisms of M fixing A are trivial. E.g. $\underline{M} = ([0,1]^n, \partial[0,1]^n)$.

Motivation

AIM: To construct algebraic structures useful for modelling generalised particle motion in topological phases.

- Very general ambient space, particle types allowed.
- Study object sets in a unified way, questions about skeletons etc.
- Allows access to higher categorical structures e.g. monoidal.
- Facilitates passage between motions and generalised tangles/ defect TQFT
- Morphisms which do not start and end in the same configuration allowed.
- Expect interesting new algebraic structures

Space of self-homeomorphisms of a manifold M

Let $\underline{M} = (M, A)$ be a pair of a manifold and a subset. Let $\mathrm{H}_{\underline{M}} \subset \mathbf{Top}(M, M)$ is the set of homeomorphisms of M which fix A pointwise with the compact-open topology. Notice this also has a group structure.

Lemma

(Hatcher) Let X be a compact space and Y a metric topological space with metric d. Then

(i) the function

$$d'(f, g) \; := \; \sup_{x \in X} d(f(x), g(x))$$

is a metric on $\mathbf{Top}(X, Y)$; and

(ii) the compact open topology on $\mathbf{Top}(X, Y)$ is the same as the one defined by the metric d'.

GROUPOIDS OF SELF HOMEOMORPHISMS

Let $\underline{M} = (M, A)$ be a pair of a manifold and a subset.

Lemma
There is a (left) group action

$$\sigma\colon \mathrm{H}_{\underline{M}} \times \mathcal{P}M \to \mathcal{P}M$$
$$(\mathfrak{f}, N) \mapsto \mathfrak{f}(N).$$

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Groupoids of self homeomorphisms

Proposition
Thus there is an action groupoid $\mathrm{Homeo}_{\underline{M}}$ obtained from σ. Explicitly the object set is $\mathcal{P}M$ and the morphisms in $\mathrm{Homeo}_{\underline{M}}(N, N')$ are triples $(\mathfrak{f}, N, \mathfrak{f}(N))$ where

- $\mathfrak{f}: M \to M$ is a homeomorphism,
- $\mathfrak{f}(N) = N'$,
- $\mathfrak{f}$ fixes A pointwise.

We will denote triples $(\mathfrak{f}, N, \mathfrak{f}(N)) \in \mathrm{Homeo}_{\underline{M}}(N, N')$ as $\mathfrak{f}: N \rightsquigarrow N'$.
Identity: $\mathrm{id}_M: N \rightsquigarrow N$ Inverse: $\mathfrak{f}: N \rightsquigarrow N' \mapsto \mathfrak{f}^{-1}: N' \rightsquigarrow N$.

We will also sometimes identify $\mathrm{Homeo}_{\underline{M}}(N, N')$ with the projection to the first element of the triple. Then can equip morphism sets with a topology and $\mathrm{H}_{\underline{M}} = \mathrm{Homeo}_{\underline{M}}(\varnothing, \varnothing) = \mathrm{Homeo}_{\underline{M}}(M, M)$ and every $\mathrm{Homeo}_{\underline{M}}(N, N') \subseteq \mathrm{H}_{\underline{M}}$. Notice each self-homeomorphism $\mathfrak{f}$ of M will belong to many such $\mathrm{Homeo}_{\underline{M}}(N, N')$.

Flows

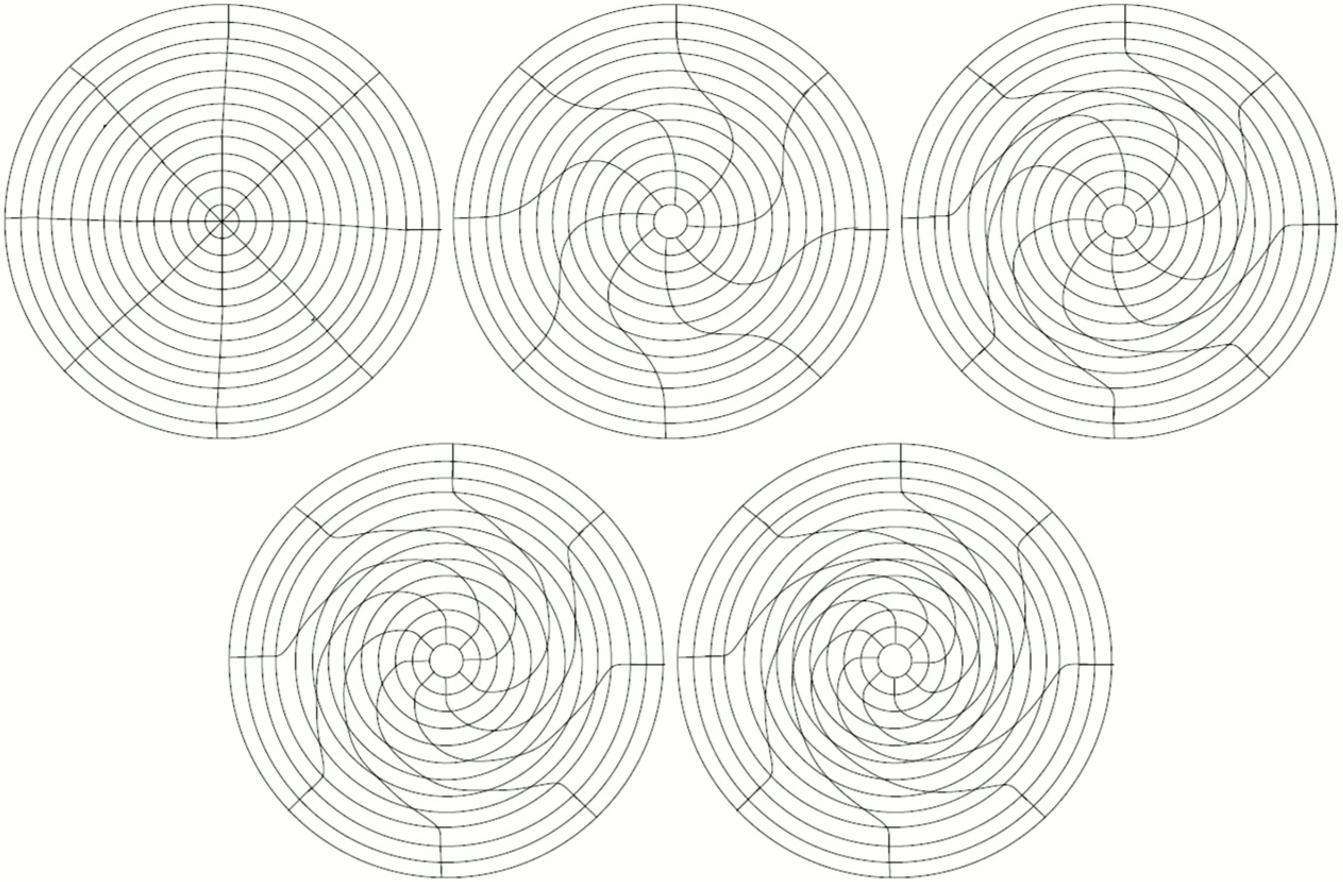
Definition
Fix a manifold, subset pair $\underline{M} = (M, A)$. A flow in $\underline{M}$ is a map $f \in \mathsf{Top}(\mathbb{I}, \mathrm{H}_{\underline{M}})$ with $f_0 = \mathrm{id}_M$. Define,

$$\mathrm{Flow}_{\underline{M}} = \{f \in \mathsf{Top}(\mathbb{I}, \mathrm{H}_{\underline{M}}) \mid f_0 = \mathrm{id}_M\}.$$

Example
For any manifold M the path $f_t = \mathrm{id}_M$ for all t, is a flow. We will denote this flow Id_M.

Example
For $M = S^1$ (the unit circle) we may parameterise by $\theta \in \mathbb{R}/2\pi$ in the usual way. Consider the functions $\tau_\phi : S^1 \to S^1$ ($\phi \in \mathbb{R}$) given by $\theta \mapsto \theta + \phi$, and note that these are homeomorphisms. Then consider the path $f_t = \tau_{t\pi}$ ('half-twist'). This is a flow.

EXAMPLE $M = D^2$



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Obtaining new flows from old

Lemma
Let M be a manifold. For any flow f in $\underline{M} = (M, A)$, then $(f^{-1})_t = f_t^{-1}$ is a flow.

NOTE: Proof uses that $\mathrm{H}_{\underline{M}}$ is a topological group when M is locally compact and locally connected (Arens). This means the product map and inverse map are continuous.

Lemma
Let M be a manifold. There exists a set map

$$\bar{\ } : \mathrm{Flow}_{\underline{M}} \to \mathrm{Flow}_{\underline{M}}$$
$$f \mapsto \bar{f}$$

with

$$\bar{f}_t = f_{(1-t)} \circ f_1^{-1}. \quad (1)$$

Obtaining new flows from old

Proposition
Let M be a manifold. There exists a composition

$$\begin{aligned} *\colon \mathrm{Flow}_{\underline{M}} \times \mathrm{Flow}_{\underline{M}} &\to \mathrm{Flow}_{\underline{M}} \\ (f, g) &\mapsto g * f \end{aligned}$$

where

$$(g * f)_t = \begin{cases} f_{2t} & 0 \leq t \leq 1/2, \\ g_{2(t-1/2)} \circ f_1 & 1/2 \leq t \leq 1. \end{cases} \tag{2}$$

For a pair $\underline{M} = (M, A)$, $(\mathrm{Flow}_{\underline{M}}, *)$ is a magma.

Obtaining new pre-motions from old

Proposition
Let M be a manifold. There is an associative composition

$$\cdot : \mathrm{Flow}_{\underline{M}} \times \mathrm{Flow}_{\underline{M}} \to \mathrm{Flow}_{\underline{M}}$$
$$(f, g) \mapsto g \cdot f$$

where $(g \cdot f)_t = g_t \circ f_t$.

NOTE: Again proof uses that $\mathrm{H}_{\underline{M}}$ is a topological group.

Lemma
For a manifold M, $(\mathrm{Flow}_{\underline{M}}, \cdot)$ is a group, with identity Id_M and inverse map $(f^{-1})_t = (f_t)^{-1}$.

Lemma
For $f, g \in \mathrm{Flow}_{\underline{M}}$, $f^{-1} \overset{p}{\sim} \bar{f}$ and $g \cdot f \overset{p}{\sim} g * f$.

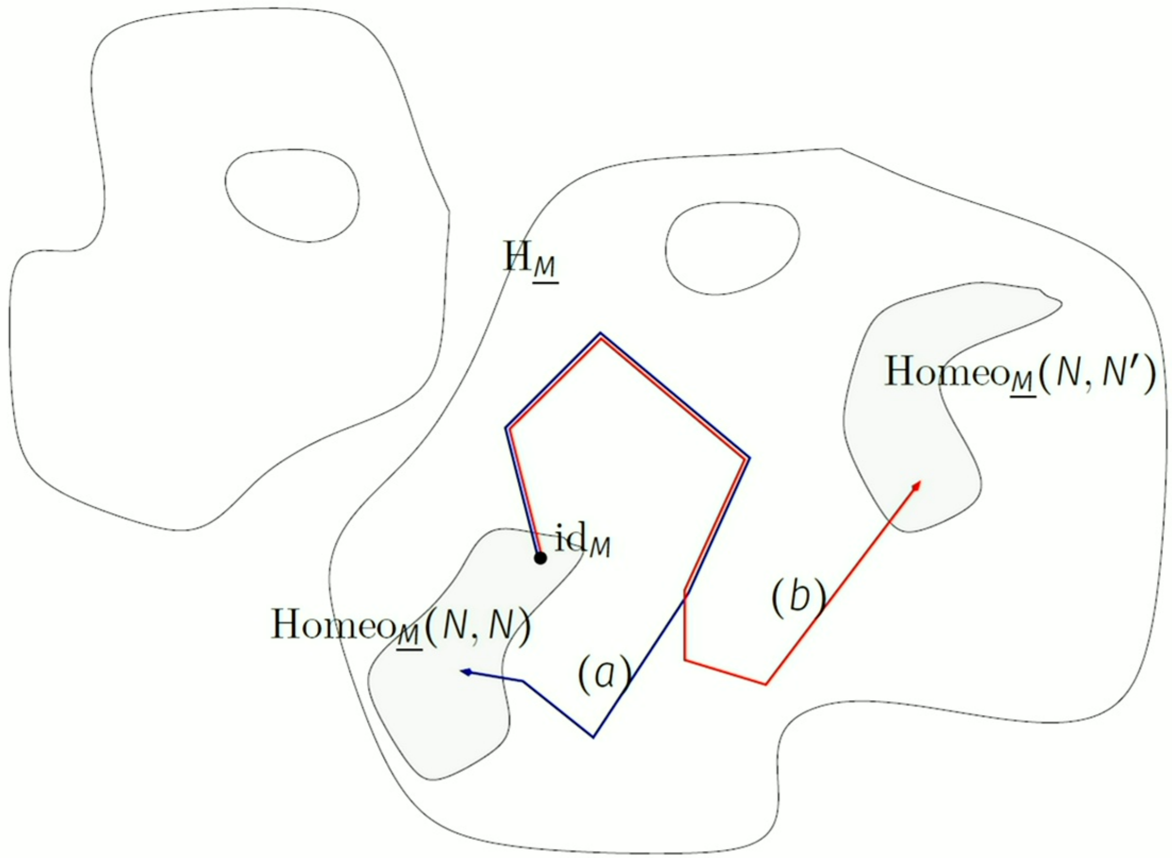
Motions

Definition
Fix a $\underline{M} = (M, A)$. A motion in M is a triple $(f, N, f_1(N))$ consisting of a flow $f \in \mathrm{Flow}_{\underline{M}}$, a subset $N \subseteq M$ and the image of N at the endpoint of f, $f_1(N)$.

We will denote such a triple by $f\colon N \rightsquigarrow N'$ where $f_1(N) = N'$, and say it is a motion from N to N'.

$$Mt_M(N, N') = \{\text{motions } f\colon N \rightsquigarrow N'\}$$

Motions



$\mathrm{H}_{\underline{M}}$
$\mathrm{Homeo}_{\underline{M}}(N, N')$
id_M
(b)
$\mathrm{Homeo}_{\underline{M}}(N, N)$
(a)

Motions

For any $N \subset M$, $Id_M\colon N \hookrightarrow N$ is a motion. Let $f\colon N \hookrightarrow N'$ and $g\colon N' \hookrightarrow N''$ be motions in M, then $g \cdot f\colon N \hookrightarrow N''$ ($(g \cdot f)_t = g_t \circ f_t$) is a motion.

Lemma
There is a group action of $(\mathrm{Flow}_{\underline{M}}, \cdot)$ on $\mathcal{P}M$, thus there is an action groupoid

$$\mathrm{Mt}^{\cdot}_{\underline{M}} = (\mathcal{P}M, \mathrm{Mt}_{\underline{M}}(N, N'), \cdot, \mathrm{Id}_M, f^{-1}).$$

Similarly $g * f\colon N \hookrightarrow N''$ is a motion.

Lemma
There is a magma action of $(\mathrm{Flow}_{\underline{M}}, *)$ on $\mathcal{P}M$ we obtain an action magmoid

$$\mathrm{Mt}^{*}_{\underline{M}} = (\mathcal{P}M, \mathrm{Mt}_{\underline{M}}(N, N'), *).$$

Motions as maps $M \times \mathbb{I} \to M \times \mathbb{I}$

Definition
Let $\underline{M} = (M, A)$ and $\underline{M \times \mathbb{I}} = (M \times \mathbb{I}, A \times \mathbb{I})$, and $N, N' \subset M$ subsets. Let

$$\mathrm{Mt}_{\underline{M}}^{hom}(N, N') \subset \mathrm{H}_{\underline{M \times \mathbb{I}}}$$

denote the subset of homeomorphisms $g \in \mathrm{H}_{\underline{M \times \mathbb{I}}}$ such that

(I) $g(m, 0) = (m, 0)$ for all $m \in M$,

(II) $g(M \times \{t\}) = M \times \{t\}$ for all $t \in \mathbb{I}$, and

(III) $g(N \times \{1\}) = N' \times \{1\}$.

Theorem (T., Faria Martins, Martin)
Let M be a manifold and $N, N' \subset M$. There is a bijection

$$\begin{aligned} \Theta \colon \mathrm{Mt}_{\underline{M}}(N, N') &\to \mathrm{Mt}_{\underline{M}}^{hom}(N, N'), \\ f &\mapsto ((m, t) \mapsto (f_t(m), t)). \end{aligned}$$

MOTIONS AS MAPS $M \times \mathbb{I} \to M \times \mathbb{I}$

Theorem
Let M be a manifold and $N, N' \subseteq M$. There is a bijection

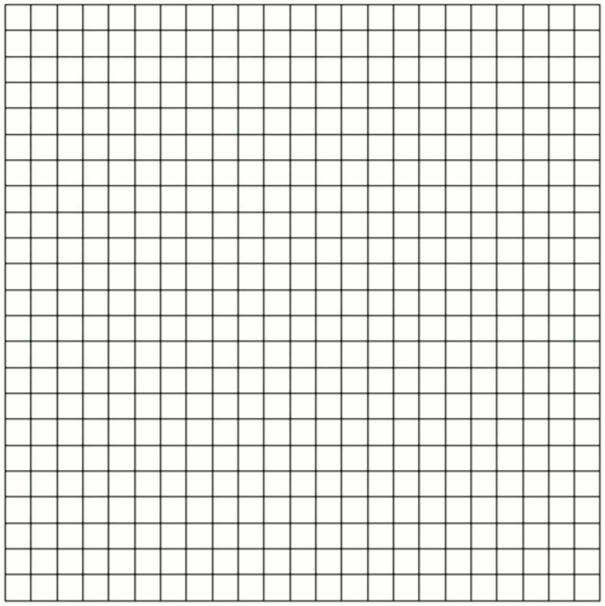
$$\Theta\colon \mathrm{Mt}_{\underline{M}}(N, N') \to \mathrm{Mt}^{hom}_{\underline{M}}(N, N'),$$
$$f \mapsto ((m,t) \mapsto (f_t(m), t)).$$

Idea of proof
(e.g. Hatcher) As M is locally compact, Hausdorff, there is a bijection

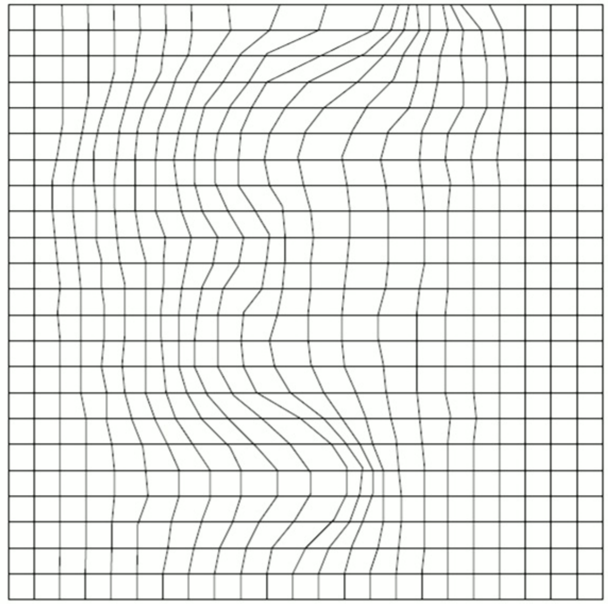
$$\Phi\colon \mathbf{Top}(\mathbb{I}, \mathbf{TOP}(M, M)) \to \mathbf{Top}(M \times \mathbb{I}, M).$$

(Coming from an adjunction between the product functor $M \times -$ and the hom functor $\mathbf{TOP}(M, -)$). It follows that the image is continuous. To show that the image is a homeomorphism we need that $\mathbf{TOP}^h(M, M)$ is a topological group.

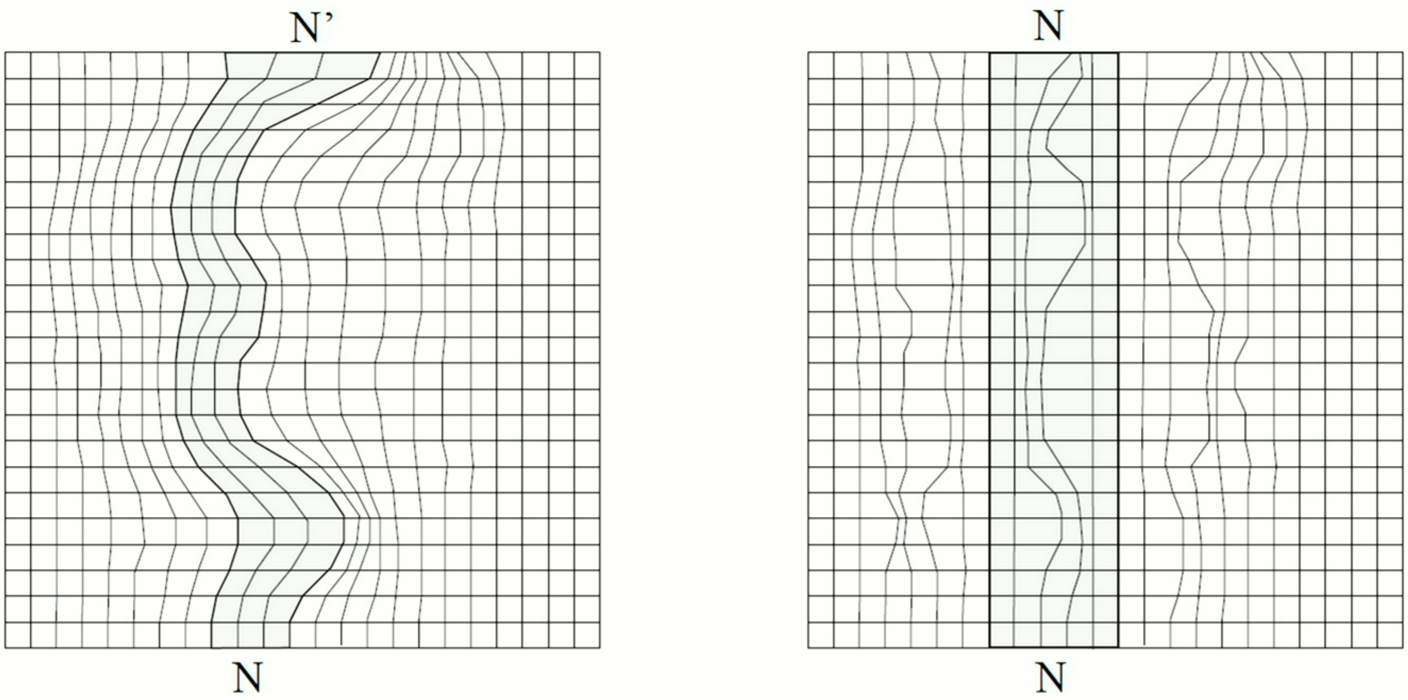
$M = \mathbb{I}$



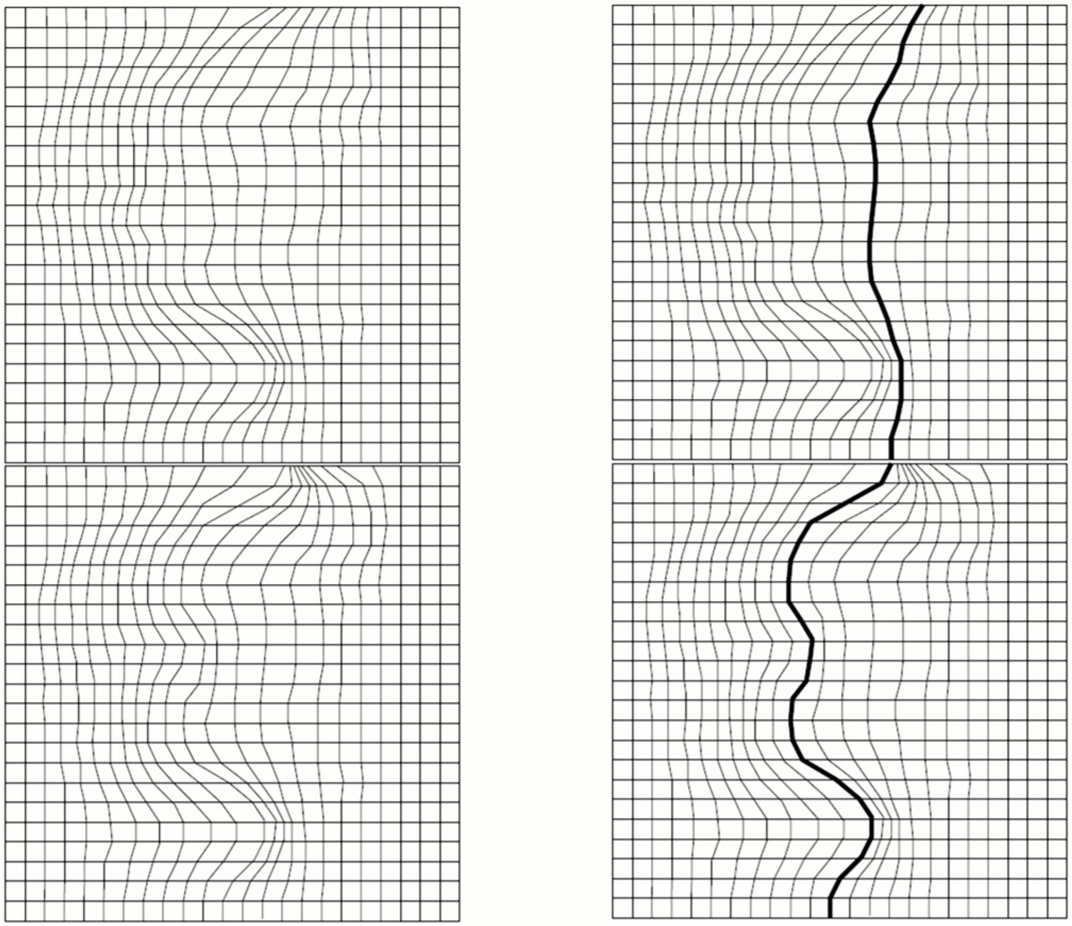
$\mapsto$



$M = \mathbb{I}$



⋆ COMPOSITION WHEN $M = \mathbb{I}$



Congruence by set-stationary motions

Definition
Let $\underline{M} = (M, A)$ be a manifold, subset pair and $N \subset M$ a subset. A motion $f: N \rightharpoonup N$ in $\underline{M}$ is said to be N-stationary if $f_t(N) = N$ for all $t \in \mathbb{I}$. Define

$$\mathrm{SetStat}_{\underline{M}}^{N} = \left\{ f: N \rightharpoonup N \in \mathrm{Mt}_{\underline{M}}(N, N) \mid f_t(N) = N \text{ for all } t \in \mathbb{I} \right\}.$$

Example
Let $M = D^2$ and let $\tau_{2\pi}$ denote a flow such that $(\tau_{2\pi})_t$ is a $2\pi t$ rotation of the disk. Now let N be a circle centred on the centre of the disk. Then $\tau_{2\pi}: N \rightharpoonup N$ is N-stationary.

Example
Let $M = D^2$, the 2-disk and let $N \subset M$ be a finite set of points. Then a motion $f: N \rightharpoonup N$ is N-stationary if and only if $f_t(x) = x$ for all $x \in N$ and $t \in \mathbb{I}$. More generally this holds if N is a totally disconnected subspace of M, e.g. $\mathbb{Q}$ in $\mathbb{R}$.

Congruence by set-stationary motions

Lemma
For $N, N' \subset M$, denote by $\overset{m}{\sim}$ the relation

$$f\colon N \rightsquigarrow N' \overset{m}{\sim} g\colon N \rightsquigarrow N' \quad \text{if} \quad \overline{g} * f \in [\mathrm{SetStat}_{\underline{M}}^{N}]_{\mathrm{p}}$$

on $\mathrm{Mt}_{\underline{M}}(N, N')$. This is an equivalence relation.
We call this <u>motion-equivalence</u> and denote by $[f\colon N \rightsquigarrow N']_{\mathrm{m}}$ the motion-equivalence class of $f\colon N \rightsquigarrow N'$.

Idea of proof
Quotient first by path-homotopy. Then classes which intersect $\mathrm{SetStat}_{\underline{M}}^{N}(N, N)$ form a totally disconnected normal subgroupoid. Can be proved in general that for any totally disconnected, normal subgroupoid $\mathcal{H}$ of a groupoid $\mathcal{G}$ there is a congruence given by the relation $g_1 \sim g_2$ if $g_2^{-1} *_{\mathcal{G}} g_1 \in \mathcal{H}$. This leads to an equivalent relation to the given relation.

Motion groupoid

Theorem
Let $\underline{M} = (M, A)$ where M is a manifold and $A \subset M$ a subset. There is a groupoid

$$\mathrm{Mot}_{\underline{M}} = (\mathcal{P}M,\ \mathrm{Mt}_{\underline{M}}(N, N')/\overset{m}{\sim}, *, [\mathrm{Id}_M]_{\mathrm{m}},\ [f]_{\mathrm{m}} \mapsto [\bar{f}]_{\mathrm{m}})$$

where

(I) objects are subsets of M;

(II) morphisms between subsets N, N' are motion-equivalence classes $[f: N \rightsquigarrow N']_{\mathrm{m}}$ of motions;

(III) composition of morphisms is given by

$$[g: N' \rightsquigarrow N'']_{\mathrm{m}} * [f: N \rightsquigarrow N']_{\mathrm{m}} = [g * f: N \rightsquigarrow N'']_{\mathrm{m}}.$$

(IV) the identity at each object N is the motion-equivalence class of $\mathrm{Id}_M: N \rightsquigarrow N$, $(\mathrm{Id}_M)_t(m) = m$ for all $m \in M$;

(V) the inverse for each morphism $[f: N \rightsquigarrow N']_{\mathrm{m}}$ is the motion-equivalence class of $\bar{f}: N' \rightsquigarrow N$ where $\bar{f}_t = f_{(1-t)} \circ f_1^{-1}$.

Proposition
Let $\underline{M} = (M, A)$ where M is a manifold and $A \subset M$ a subset, then

$$\mathrm{Mot}_{\underline{M}} = (\mathcal{P}M,\ \mathrm{Mt}_{\underline{M}}(N, N')/\overset{m}{\sim}, \cdot, [\mathrm{Id}_M]_{\mathrm{m}},\ [f]_{\mathrm{m}} \mapsto [f^{-1}]_{\mathrm{m}}).$$

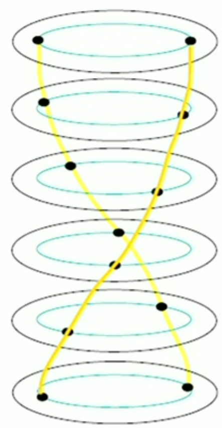
Proof
It is sufficient to observe that motions which are path equivalent are motion equivalent. Let g, f be flows satisfying $f \overset{p}{\sim} g$, then $\bar{g} * f \overset{p}{\sim} g^{-1} \cdot f \overset{p}{\sim} g^{-1} \cdot g$, using that $\bar{g} \overset{p}{\sim} g^{-1}$, and $g * f \overset{p}{\sim} g \cdot f$. Then for all $t \in \mathbb{I}$, $(g^{-1} \cdot g)_t(N) = N$, hence it is stationary.

On $\mathrm{Mot}_{\mathbb{I}}$

Suppose $N \subset \mathbb{I} \setminus \{0, 1\}$ is a compact subset with a finite number of connected components i.e. N is a union of points and closed intervals.
We can assign a word in $\{a, b\}$ to N by representing each point in N by a and each interval by b, ordered in the obvious way using the natural ordering on $\mathbb{I}$.
Let $N' \subset \mathbb{I} \setminus \{0, 1\}$ be another subset defined in the same way. If the word assigned to N and N' is the same, $|\mathrm{Mot}_{\mathbb{I}}(N, N')| = 1$. Otherwise $\mathrm{Mot}_{\mathbb{I}}(N, N') = \varnothing$.

Braid groups and loop braid groups

Theorem (T., Faria Martins, Martin)
Let n be a positive integer. Consider $M = D^2$. Given any finite subset K, with n elements, in the interior of D^2, then $\mathrm{Mot}_{D^2}(K, K)$ is isomorphic to the braid group in n strands (as in 'Theory of Braids', Artin). In particular the image of the class of a motion which moves points as below is an elementary braid on two strands.



Also if $\underline{D^3} = (D^3, \partial D^3)$ and $L \subset D^3$ is an unlink in the interior with n components, then $\mathrm{Mot}_{\underline{D^3}}(L, L)$ is isomorphic to the extended loop braid group (as in 'A journey through loop braid groups', Damiani).

Relating motion groupoids

Lemma
Let (M, A) and (M', A') be pairs such that there exists a homeomorphism $\psi: M \to M'$ satisfying $\psi(A) = A'$. Then there is a isomorphism of categories

$$\Psi: \mathrm{Mot}_{\underline{M}} \to \mathrm{Mot}_{\underline{M'}}$$

defined as follows. On objects $N \subset M$, $\Psi(N) = \psi(N)$. For a motion $f: N \rightsquigarrow N'$ in M, let $(\psi \circ f \circ \psi^{-1})_t = \psi \circ f_t \circ \psi^{-1}$. Then Ψ sends the equivalence class $[f: N \rightsquigarrow N']_{\mathrm{m}}$ to the equivalence class $[\psi \circ f \circ \psi^{-1}: \psi(N) \to \psi(N')]_{\mathrm{m}}$.

Relating automorphism groups

Proposition
For any pair (M, A) and subset $N \subseteq M$ there is an involutive endofunctor on $\mathrm{Mot}_{\underline{M}}$ defined by

$$\mathrm{Mot}_{\underline{M}}(N, N) \cong \mathrm{Mot}_{\underline{M}}(M \setminus N, M \setminus N),$$
$$f\colon N \hookrightarrow N' \mapsto f\colon M \setminus N \hookrightarrow M \setminus N'.$$

Notice that generally these automorphism groups are not connected *in* the motion groupoid - this would imply N homeomorphic to $M \setminus N$.

ALTERNATIVE EQUIVALENCE RELATIONS ON THE MOTION GROUPOID

Definition

The worldline of a motion $f\colon N \rightsquigarrow N'$ in a manifold M is

$$\mathsf{W}\,(f\colon N \rightsquigarrow N') := \bigcup_{t\in[0,1]} f_t(N) \times \{t\} \subseteq M \times \mathbb{I}.$$

Proposition

Let $f, g\colon N \rightsquigarrow N'$ be motions with the same worldline, so we have

$$\mathsf{W}(f\colon N \rightsquigarrow N') = \mathsf{W}(g\colon N \rightsquigarrow N').$$

Then $f\colon N \rightsquigarrow N'$ and $g\colon N \rightsquigarrow N'$ are motion equivalent.

Proof

For all $t \in \mathbb{I}$, $(g^{-1} \cdot f)_t(N) = g_t^{-1} \circ g_t(N) = N$. Thus $g^{-1} \cdot f$ is N-stationary, and hence $\bar{g} * f$ path-homotopic to a stationary motion.

Worldlines of motions

Theorem (T., Faria Martins, Martin)
Let $\underline{M} = (M, A)$ where M is a manifold and $A \subset M$ a subset. Two motions $f, f' : N \hookrightarrow N'$ in $\mathrm{Mt}_{\underline{M}}$ are motion equivalent if, and only if, their worldlines are level preserving ambient isotopic, relative to $(M \times (\{0,1\})) \cup (A \times \mathbb{I})$, pointwise.

Relative path-equivalence

Definition
Fix a pair (M, A). Define a relation on $\mathrm{Mt}_{\underline{M}}(N, N')$ as follows. Let $f: N \rightarrowtail N' \overset{rp}{\sim} g: N \rightarrowtail N'$ if the motions $f: N \rightarrowtail N'$ and $g: N \rightarrowtail N'$ are relative path-homotopic. This means there exists a continuous map

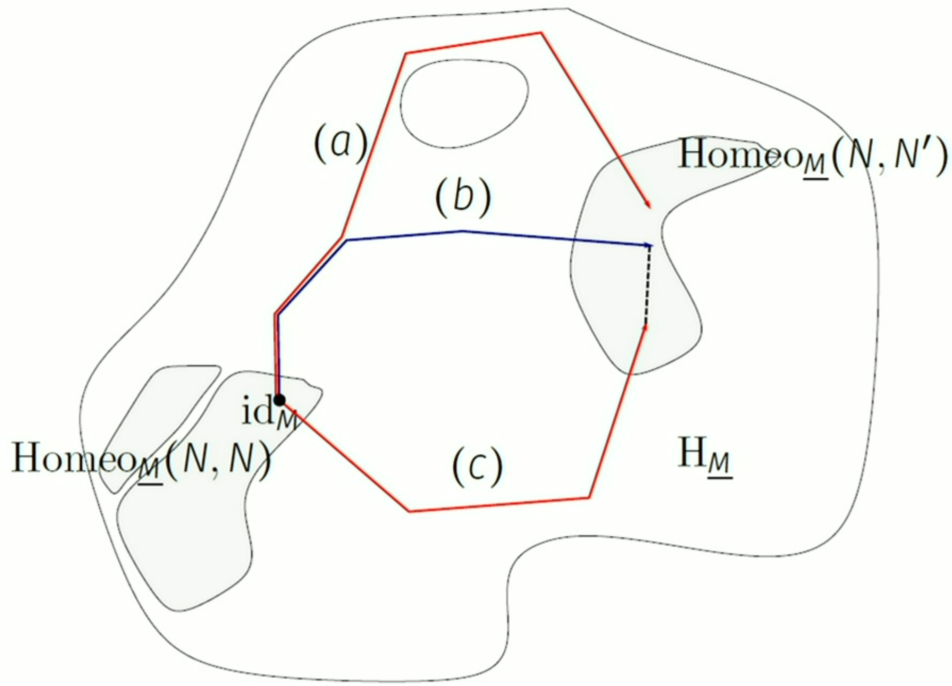
$$H: \mathbb{I} \times \mathbb{I} \to \mathrm{H}_{\underline{M}}$$

such that

- for any fixed $s \in \mathbb{I}$, $t \mapsto H(t, s)$ is a motion from N to N',
- for all $t \in \mathbb{I}$, $H(t, 0) = f_t$, and
- for all $t \in \mathbb{I}$, $H(t, 1) = g_t$.

We call such a homotopy a relative path-homotopy.

Relative path-equivalence



Relative path-equivalence

Theorem (T. , Faria Martins, Martin)
For a pair $\underline{M} = (M, A)$ and a motion $f: N \rightsquigarrow N'$ in $\underline{M}$ we have

$$[f: N \rightsquigarrow N']_{\mathrm{rp}} = [f: N \rightsquigarrow N']_{\mathrm{m}}.$$

Key ingredients of proof
Direct construction of appropriate homotopies. Uses normality of stationary motions.

Relative path equivalence is precisely the equivalence relation in the relative fundamental group, hence

$$\mathrm{Mot}_{\underline{M}}(N, N) = \pi_1(\mathrm{Homeo}_{\underline{M}}(\varnothing, \varnothing), \mathrm{Homeo}_{\underline{M}}(N, N), \mathrm{id}_M)$$

We will need this later!

Mapping class groupoid

Recall that for a pair $\underline{M} = (M, A)$ and for subsets $N, N' \subset M$, morphisms in $\mathrm{Homeo}_{\underline{M}}(N, N')$ are triples denoted $\mathfrak{f}\colon N \rightsquigarrow N'$ where $\mathfrak{f} \in \mathrm{H}_{\underline{M}}$ and $\mathfrak{f}(N) = N'$. We also think of the elements of $\mathrm{Homeo}_{\underline{M}}(N, N')$ as the projection to the first coordinate of each triple i.e. $\mathfrak{f} \in \mathrm{H}_{\underline{M}}$ such that $\mathfrak{f}(N) = N'$.

Definition
Let $N, N' \subset M$. For any $\mathfrak{f}\colon N \rightsquigarrow N'$ and $\mathfrak{g}\colon N \rightsquigarrow N'$ in $\mathrm{Homeo}_{\underline{M}}(N, N')$, $\mathfrak{f}\colon N \rightsquigarrow N'$ is said to be isotopic to $\mathfrak{g}\colon N \rightsquigarrow N'$, denoted by $\overset{i}{\sim}$, if there exists a continuous map

$$H\colon M \times \mathbb{I} \to M$$

such that

- for all fixed $s \in \mathbb{I}$, the map $m \mapsto H(m, s)$ is in $\mathrm{Homeo}_{\underline{M}}(N, N')$,
- for all $m \in M$, $H(m, 0) = \mathfrak{f}(m)$, and
- for all $m \in M$, $H(m, 1) = \mathfrak{g}(m)$.

We call such a map an isotopy from $\mathfrak{f}\colon N \rightsquigarrow N'$ to $\mathfrak{g}\colon N \rightsquigarrow N'$.

Mapping class groupoids

Lemma
The family of relations $(\mathrm{Homeo}_{\underline{M}}(N,N'), \overset{i}{\sim})$ for all pairs $N, N' \subseteq M$ are a congruence on $\mathrm{Homeo}_{\underline{M}}$.

Theorem (T., Faria Martins, Martin)
Let $\underline{M} = (M, A)$ be a manifold submanifold pair. There is a groupoid

$$\mathrm{MCG}_{\underline{M}} = (\mathcal{P}M, \mathrm{Homeo}_{\underline{M}}(N,N')/\overset{i}{\sim}, \circ, [\mathrm{id}_M]_i, [\mathfrak{f}]_i \mapsto [\mathfrak{f}^{-1}]_i).$$

We call this the mapping class groupoid of M.

Mapping class groupoids

Using bijection

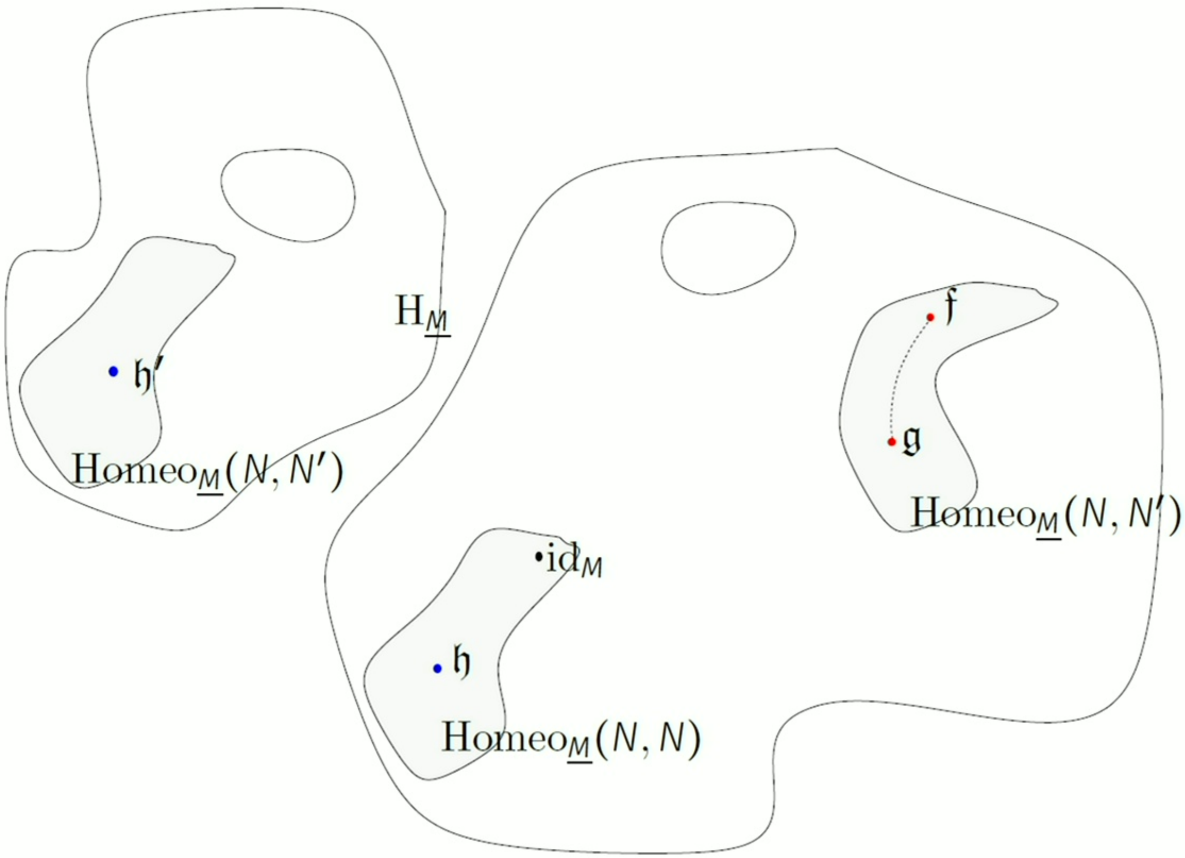
$$\Phi\colon \mathsf{Top}(\mathbb{I}, \mathsf{TOP}(M, M)) \to \mathsf{Top}(M \times \mathbb{I}, M),$$

a continuous map $M \times \mathbb{I} \to M$ which is an isotopy corresponds to a path $\mathbb{I} \to \mathrm{Homeo}_{M}(N, N')$ from $\mathfrak{f}$ to $\mathfrak{g}$. Hence

Lemma
Let M be a manifold. We have that as sets

$$\mathrm{MCG}_{\underline{M}}(N, N') = \pi_0(\mathrm{Homeo}_{\underline{M}}(N, N')).$$

Mapping class groupoids



Mapping class groupoid, $M = S^1$

Example
If $\underline{S^1} = (S^1, \varnothing)$, we have

$$\mathrm{MCG}_{\underline{S^1}}(\varnothing, \varnothing) = \mathbb{Z}/2\mathbb{Z}.$$

H_{S^1} has two path-components, containing respectively the orientation preserving and the orientation reversing homeomorphisms from S^1 to itself. Each is homotopic to S^1 (Hamstrom). Therefore the homomorphism $\pi_0(\mathrm{Homeo}_{\underline{S^1}}(\varnothing, \varnothing)) \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ induced by the degree homomorphism $\deg\colon \mathrm{H}_{S^1} = \mathrm{Homeo}_{\underline{S^1}}(\varnothing, \varnothing) \to \{\pm 1\}$ is an isomorphism.

Example

Proposition

Let $\underline{D^2} = (D^2, \partial D^2)$. The morphism group $\mathrm{MCG}_{\underline{D^2}}(\varnothing, \varnothing)$ is trivial.

Proof

(This follows from the Alexander trick.) Suppose we have $\mathfrak{f} \colon \varnothing \rightsquigarrow \varnothing$ in $\underline{D^2}$. Define

$$f_t(x) = \begin{cases} t\,\mathfrak{f}(x/t) & 0 \leq |x| \leq t, \\ x & t \leq |x| \leq 1. \end{cases}$$

Notice that $f_0 = \mathrm{id}_{D^2}$ and $f_1 = \mathfrak{f}$ and each f_t is continuous. Moreover:

$$H \colon D^2 \times \mathbb{I} \to D^2,$$
$$(x, t) \mapsto f_t(x)$$

is a continuous map. So we have constructed an isotopy from any boundary preserving self-homeomorphism of D^2 to id_{D^2}.

Functor from the motion groupoid to the mapping class groupoid

Functor $\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$

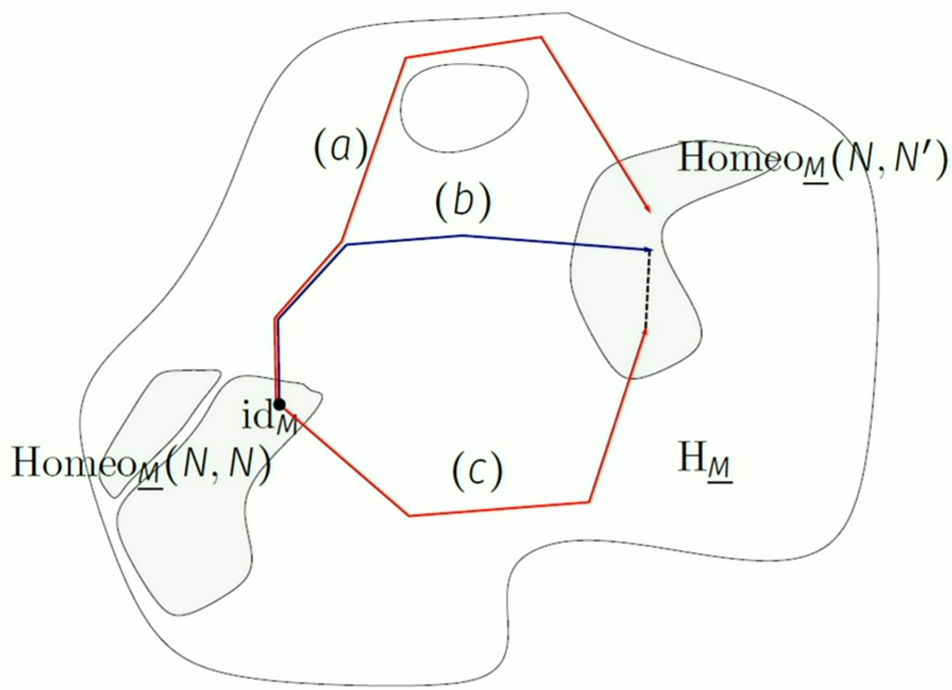
Theorem (T., Faria Martins, Martin)
Let $\underline{M} = (M, A)$. There is a functor

$$\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$$

which is the identity on objects and on morphisms we have

$$\mathsf{F}\left([f\colon N \rightarrowtail N']_{\mathrm{m}}\right) = [f_1\colon N \rightsquigarrow N']_{\mathrm{i}}.$$

Well definedness of F



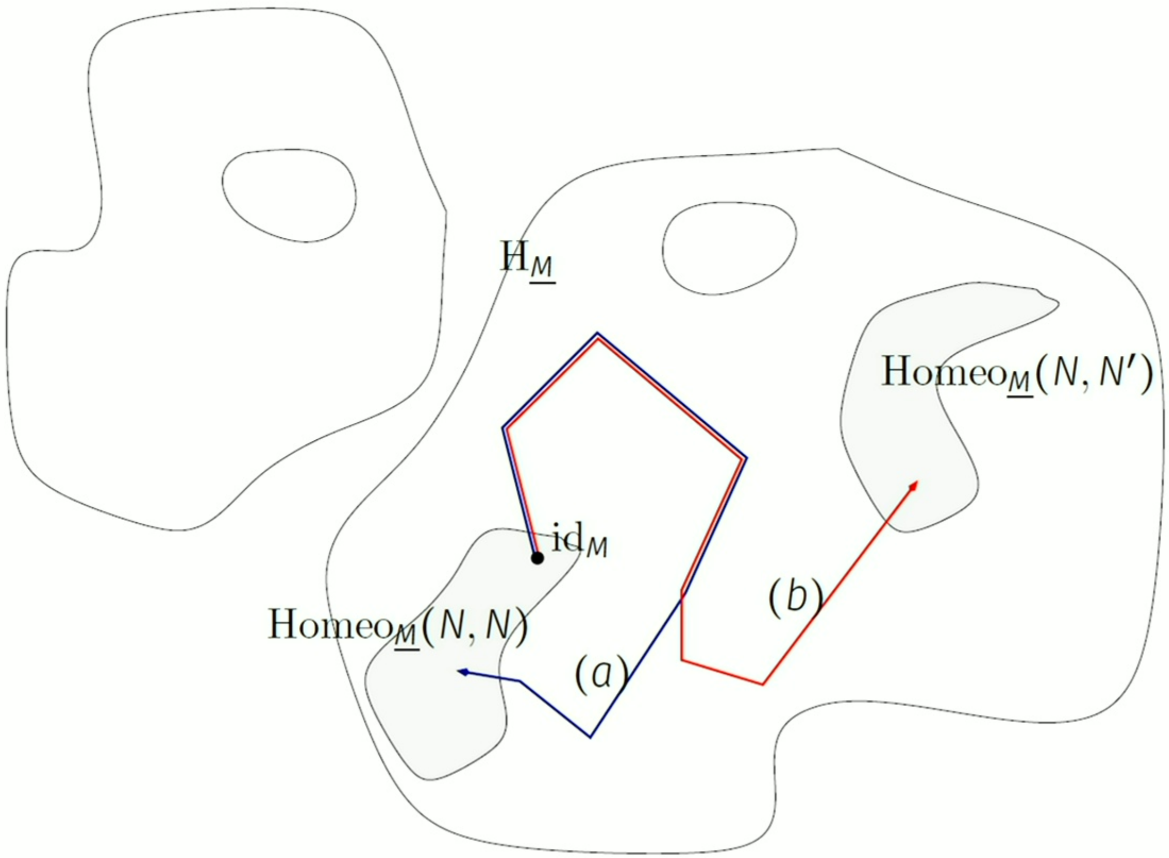
Functor $\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$

Lemma
The functor

$$\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$$

is full if and only if $\pi_0(\mathrm{Homeo}_M(\varnothing, \varnothing), \mathrm{id}_M)$ is trivial.

FUNCTOR $F\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$



FUNCTOR $F\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$

(Hatcher) Let X be a space, $Y \subset X$ a subspace and $x_0 \in Y$ a basepoint. There is a long exact sequence:

$$\ldots \to \pi_n(Y,\{x_0\}) \xrightarrow{i_*^n} \pi_n(X,\{x_0\}) \xrightarrow{j_*^n} \pi_n(X,Y,\{x_0\}) \xrightarrow{\partial^n} \pi_{n-1}(Y,\{x_0\}) \xrightarrow{i_*^{n-1}} \ldots \xrightarrow{i_*^0} \pi_0(X,\{x_0\}).$$

Maps i and j are inclusions. Maps ∂ are restrictions to single face, in particular

$$\partial^1\colon \pi_1(X,A,\{x_0\}) \to \pi_0(A,\{x_0\}),$$
$$[\gamma]_{\mathrm{rp}} \mapsto [\gamma(1)]_{\mathrm{p}}.$$

Functor $\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$

Recall $\mathrm{Mot}_{\underline{M}}(N,N) = \pi_1(\mathrm{Homeo}_M(\varnothing,\varnothing), \mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M)$ and $\mathrm{MCG}_{\underline{M}}(N,N) = \pi_0(\mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M)$.

Lemma
Let $\underline{M} = (M, A)$ be a manifold, subset pair, and fix a subset $N \subset M$. Then we have a long exact sequence

$$\begin{aligned}
\ldots \to \pi_n(\mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M) &\xrightarrow{i_*^n} \pi_n(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing), \mathrm{id}_M) \xrightarrow{j_*^n} \\
\pi_n(\mathrm{Homeo}_M(\varnothing,\varnothing), \mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M) &\xrightarrow{\partial^n} \pi_{n-1}(\mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M) \xrightarrow{i_*^{n-1}} \\
\ldots \xrightarrow{\partial^2} \pi_1(\mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M) &\xrightarrow{i_*^1} \pi_1(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing), \mathrm{id}_M) \\
\xrightarrow{j_*^1} \mathrm{Mot}_{\underline{M}}(N,N) &\xrightarrow{\mathsf{F}} \mathrm{MCG}_{\underline{M}}(N,N) \xrightarrow{i_*^0} \pi_0(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing), \mathrm{id}_M)
\end{aligned}$$

where all maps are group maps and F is the appropriate restriction of the functor $\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$.

Functor $\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$

Lemma
Suppose

- $\pi_1(\mathrm{Homeo}_{\underline{M}}(\varnothing, \varnothing), \mathrm{id}_M)$ is trivial, and
- $\pi_0(\mathrm{Homeo}_{\underline{M}}(\varnothing, \varnothing), \mathrm{id}_M)$ is trivial.

Then there is a group isomorphism

$$\mathsf{F}\colon \mathrm{Mot}_{\underline{M}}(N, N) \xrightarrow{\sim} \mathrm{MCG}_{\underline{M}}(N, N).$$

Functor $\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$

Theorem (T., Faria Martins, Martin)
Let M be a manifold. If

- $\pi_1(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing),\mathrm{id}_M)$ is trivial, and
- $\pi_0(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing),\mathrm{id}_M)$ is trivial,

the functor

$$\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}},$$

is an isomorphism of categories.

Functor $\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$

Proof

Suppose $\pi_1(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing),\mathrm{id}_M)$ and $\pi_0(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing),\mathrm{id}_M)$ are trivial. Already proved F is full. We check F is faithful. Let $[f\colon N \rightsquigarrow N']_{\mathrm{m}}$ and $[f'\colon N \rightsquigarrow N']_{\mathrm{m}}$ be in $\mathrm{Mot}_{\underline{M}}(N,N')$. If $\mathsf{F}\left([f\colon N \rightsquigarrow N']_{\mathrm{m}}\right) = \mathsf{F}\left([f'\colon N \rightsquigarrow N']_{\mathrm{m}}\right)$, then

$$\begin{aligned}
[\mathrm{id}_M\colon N \rightsquigarrow N]_{\mathrm{i}} &= \mathsf{F}([f'\colon N \rightsquigarrow N']_{\mathrm{m}})^{-1} \circ \mathsf{F}([f\colon N \rightsquigarrow N']_{\mathrm{m}}) \\
&= \mathsf{F}([f'\colon N \rightsquigarrow N']_{\mathrm{m}}^{-1} * [f\colon N \rightsquigarrow N']_{\mathrm{m}}) \\
&= \mathsf{F}([\bar{f'} * f\colon N \rightsquigarrow N]_{\mathrm{m}}).
\end{aligned}$$

By group isomorphism this is true if and only if

$$[\bar{f'} * f\colon N \rightsquigarrow N]_{\mathrm{m}} = [\mathrm{Id}_M\colon N \rightsquigarrow N]_{\mathrm{m}}$$

which is equivalent to saying $\mathrm{Id}_M * (\bar{f'} * f)$ is path-equivalent to a stationary motion, and hence that $\bar{f'} * f$ is path-equivalent to the stationary motion (since $Id_M * (\bar{f'} * f) \overset{p}{\sim} \bar{f'} * f$). So we have $[f\colon N \rightsquigarrow N']_{\mathrm{m}} = [f'\colon N \rightsquigarrow N']_{\mathrm{m}}$.

EXAMPLES: $M = D^n$

Proposition
Let D^n be the n-disk, and $\underline{D^n} = (D^n, \partial D^n)$. Then we have an isomorphism

$$\mathsf{F}\colon \mathrm{Mot}_{\underline{D^n}} \to \mathrm{MCG}_{\underline{D^n}}.$$

Examples: $M = D^n$

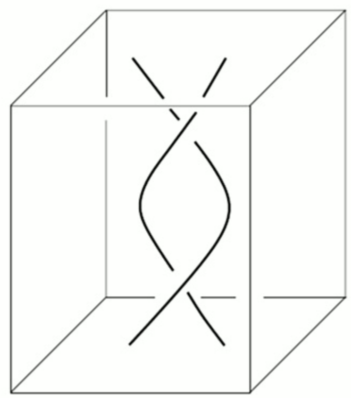
Proposition
Let D^n be the n-disk, and $\underline{D^n} = (D^n, \partial D^n)$. Then we have an isomorphism

$$\mathsf{F}\colon \mathrm{Mot}_{\underline{D^n}} \to \mathrm{MCG}_{\underline{D^n}}.$$

Idea of proof
We proved that $\mathrm{MCG}_{\underline{D^2}}(\varnothing, \varnothing) = \pi_0(\mathrm{Homeo}_{\underline{D^2}}(\varnothing, \varnothing), \mathrm{id}_M)$ is trivial. Alexander trick gives same result for all n. Also $\mathrm{Homeo}_{\underline{D^n}}(\varnothing, \varnothing)$ is contractible (Hamstrom).

EXAMPLES: $M = D^2$

Suppose we don't fix the boundary. Let $P_2 \subset D^2$ be a subset consisting of two points equidistant from the centre of the disk. Let τ_π be the path in $\mathsf{TOP}^h(D^2, D^2)$ such that $\tau_{\pi t}$ is a πt rotation of the disk.
The motion $\tau_\pi \colon P_2 \rightarrowtail P_2$ represents a non-trivial equivalence class in Mot_{D^2}, and its end point also represents a non trivial element of MCG_{D^2}. Now consider the motion $\tau_\pi * \tau_\pi \colon P_2 \rightarrowtail P_2$.



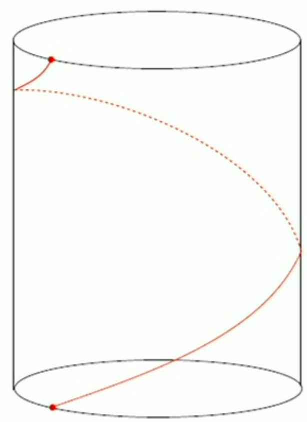
Examples: $M = D^2$

In fact, the map $F\colon \mathrm{Mot}_{D^2} \to \mathrm{MCG}_{D^2}$ is neither full nor faithful. The space Homeo_{D^2} is homotopy equivalent to $S^1 \sqcup S^1$, where the first connected component corresponds to orientation preserving homeomorphisms and the second orientation reversing (Hamstrom). Hence we have that $\pi_1(\mathrm{Homeo}_{D^2}(\varnothing, \varnothing), \mathrm{id}_{D^2}) = \mathbb{Z}$ where the single generating element corresponds to the 2π rotation. And $\pi_0(\mathrm{Homeo}_{D^2}(\varnothing, \varnothing), \mathrm{id}_{D^2}) = \mathbb{Z}/2\mathbb{Z}$. So we have an exact sequence:

$$\ldots \to \pi_1(\mathrm{Homeo}_{D^2}(N, N), \mathrm{id}_{D^2}) \xrightarrow{i_*^1} \mathbb{Z} \to \mathrm{Mot}_{D^2}(N, N) \to \mathrm{MCG}_{D^2}(N, N) \to \mathbb{Z}/2\mathbb{Z}.$$

EXAMPLES: $M = S^1$

Let $P \subset S^1$ be a subset containing a single point in S^1. Similarly to the disk, there is a non-trivial morphism in $\mathrm{Mot}_{\underline{S^1}}(P, P)$ represented by a 2π rotation of the circle.



EXAMPLES: $M = S^1$

Note that the connected component containing id_{S^1} of $\mathrm{Homeo}_{S^1}(P,P)$ is contractible, (Hamstrom). In particular $\pi_1(\mathrm{Homeo}_{S^1}(P,P), \mathrm{id}_{S^1})$ is trivial. We also have that $S^1 \sqcup S^1$ is a strong deformation retract of $\mathrm{Homeo}_{S^1}(\varnothing, \varnothing)$, with the first copy of S^1 corresponding to orientation preserving homeomorphisms and the second to orientation reversing. Hence the sequence becomes

$$\ldots \to \{1\} \to \mathbb{Z} \to \mathrm{Mot}_{S^1}(P,P) \to \mathrm{MCG}_{S^1}(P,P) \to \mathbb{Z}/2\mathbb{Z}.$$

The exact sequence gives an injective map $\mathbb{Z} \cong \pi_1(\mathrm{Homeo}_{\underline{S^1}}(\varnothing,\varnothing), \mathrm{id}_{S^1}) \to Mot_{S^1}(P,P)$, sending $n \in \mathbb{Z}$ to the equivalence class of the flow tracing a $2n\pi$ rotation of the circle S^1. The space $\mathrm{Homeo}_{\underline{S^1}}(P,P)$ only has two connected components, consisting of orientations preserving and orientation reversing homeomorphisms of S^1 fixing P. Hence the exact sequence becomes:

$$\ldots \to \{1\} \to \mathbb{Z} \xrightarrow{\cong} Mot_{S^1}(P,P) \xrightarrow{0} MCG_{S^1}(P,P) \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z}.$$

Motion groupoids

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