Title: Zesting topological order and symmetry-enriched topological order in (2+1)D

Speakers: Colleen Delaney

Collection: Higher Categorical Tools for Quantum Phases of Matter

Date: March 20, 2024 - 2:00 PM

URL: https://pirsa.org/24030086

Abstract: Zesting is a construction that takes a (2+1)D topological order and produces a new one by changing the fusion rules of its anyons. We'll discuss properties of zesting from a physical and computational point of view and explain how the theory produces some closely related families of topological orders, like Kitaev's 16-fold way and modular isotopes. Time permitting we'll cover a generalization of zesting to symmetry-enriched topological order and comment on connections to fusion 2-categories.

I. Zesting topological order*

$$\mathcal{C} \xrightarrow{\text{zesting data } \zeta = (\lambda, \nu, \mathrm{t}, \mathrm{f})} \mathcal{C}^{\zeta}$$

*In this talk, (bosonic) topological order will be synonymous with modular fusion categories.

Compare and contrast of methods for working with (modular) fusion categories

skeletal

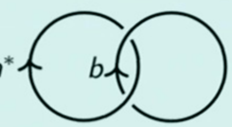
numbers (N, F, R, θ) satisfying pentagon, hexagon, ribbon equations, and $\det(S) = 0$

graphical calculus on trivalent ribbon graphs:

good for concrete examples, necessary for e.g. explicit Hamiltonians, quantum gates

fusion rules

$d_a =$  a

$S_{ab} = \frac{1}{D}$ a^* b 

$\theta_a = \frac{1}{d_a}$ a 

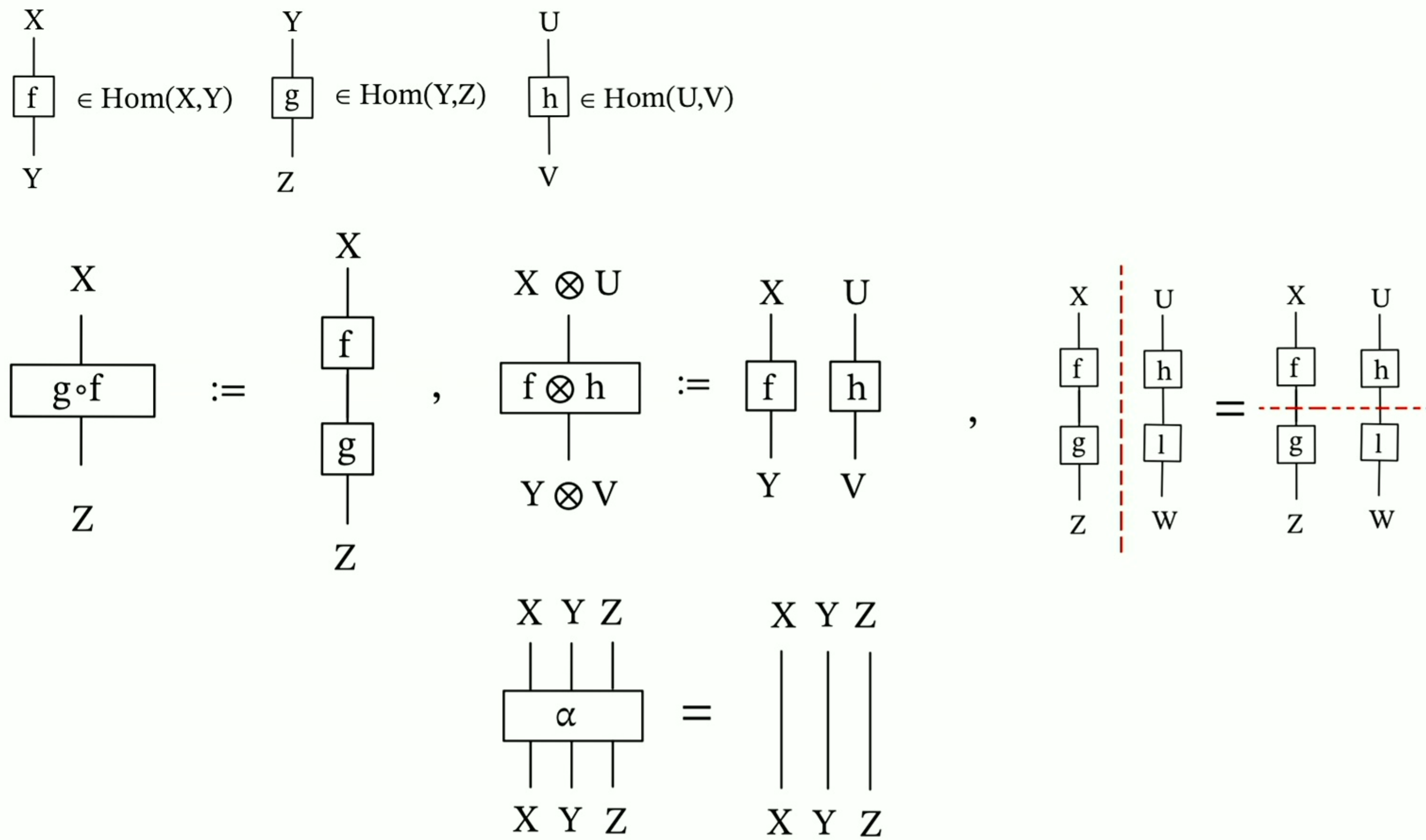
strict

strict modular fusion categories $(C, \otimes, \alpha \equiv \mathrm{id}, \beta, \theta)$ satisfying the pentagon, hexagon, ribbon axioms, and with non-degenerate braiding

graphical calculus on string diagrams

good for studying families, leveraging topology, proving theorems

Review of string diagrams in a strict monoidal category



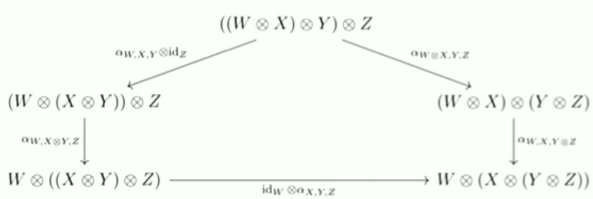
Preliminaries

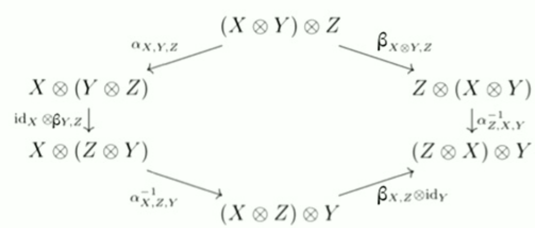
Definition: A modular fusion category $\mathcal{C}$ is a finite semisimple $\mathbb{C}$-linear abelian rigid monoidal category with simple unit and a non-degenerate braiding with ribbon twists

① $\alpha_{X,Y,Z}: (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$

② $\beta_{X,Y}: X \otimes Y \xrightarrow{\sim} Y \otimes X$

③ $\theta_X: X \xrightarrow{\sim} X$

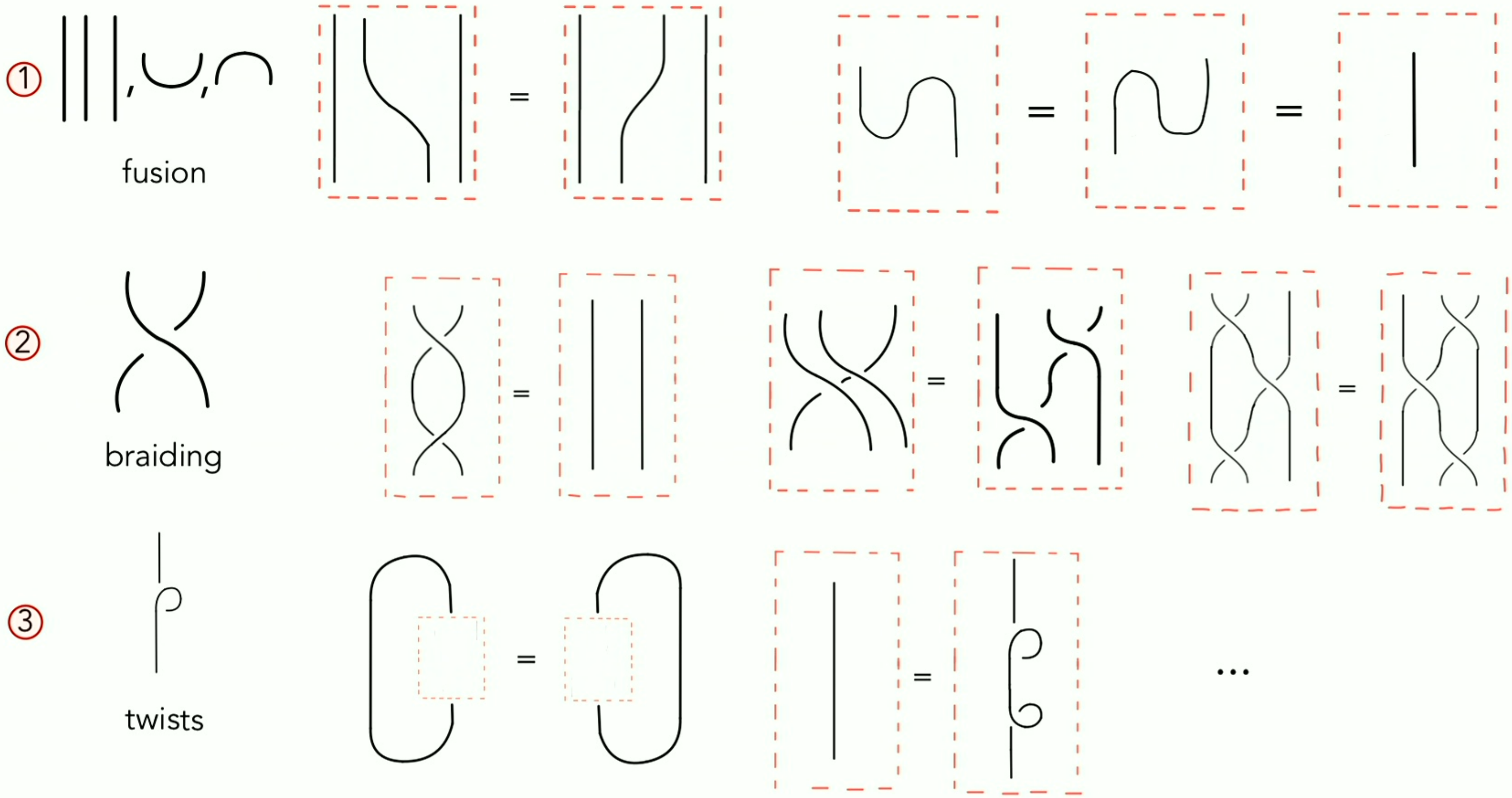




+ similar hexagon diagram

⋮

Review of string diagrams in a strict ribbon fusion category



G-graded fusion categories

Let G be a finite group.

A fusion category C is *G-graded* if

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g \qquad \text{and} \qquad X_g \in \mathcal{C}_g, Y_h \in \mathcal{C}_h \Rightarrow X_g \otimes Y_h \in \mathcal{C}_{gh}$$

Grading is *faithful* if $\mathcal{C}_g \neq 0$ for all $g \in G$

Every fusion category is faithfully graded by its *universal grading group*

Note: If a braided fusion category is G-graded then G is an abelian group because $X_g \otimes Y_h \cong Y_h \otimes X_g \Rightarrow \mathcal{C}_{gh} = \mathcal{C}_{hg}$ for all $g,h \in G$.

We will still use G to denote the abelian group.

Examples of G-graded modular fusion categories

Example: Ising theories have anyons $\{\mathbf{1}, \sigma, \psi\}$ with fusion rules

$$\begin{cases} \sigma \otimes \sigma = \mathbf{1} \oplus \psi \\ \sigma \otimes \psi = \psi \otimes \sigma = \sigma \\ \psi \otimes \psi = \mathbf{1} \end{cases}$$

and are $\mathbb{Z}/2\mathbb{Z}$-graded with Ising = sVec $\bigoplus \{\sigma\}$

Non-Example: Fibonacci theories have anyons $\{\mathbf{1}, \tau\}$ with fusion rule

$$\{ \tau \otimes \tau = 1 \oplus \tau$$

and therefore do not admit a nontrivial grading

Examples of G-graded modular fusion categories

Example: $\mathbb{Z}/2\mathbb{Z}$-toric code topological order has anyons $\{\mathbf{1}, e, m, f\}$

$\otimes$	e	m	f
e	**1**	f	m
m	f	**1**	e
f	m	e	**1**

- *universal grading group* is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

  $D(\mathbb{Z}/2\mathbb{Z}) = \{1\} \bigoplus \{e\} \bigoplus \{m\} \bigoplus \{f\}$

- also admits $\mathbb{Z}/2\mathbb{Z}$-grading

  $D(\mathbb{Z}/2\mathbb{Z}) = \{1, f\} \bigoplus \{e, m\}$

Idea of zesting

Fix a topological order $\mathcal{C}$ with nontrivial G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ and some abelian anyons in the trivial sector $\mathcal{C}_e$

Define a new fusion rule using a 2-cocycle $\lambda \in Z^2(G, \mathrm{Inv}(C_e))$

$$X_g \otimes Y_h \rightsquigarrow X_g \overset{\lambda}{\otimes} Y_h := X_g \otimes Y_h \otimes \underbrace{\lambda(g,h)_e}_{\text{abelian anyon}}$$

Note that while we have fixed the anyon labels, their duals may have changed:

$$X_g{}^* \rightsquigarrow X_g{}^* \otimes \lambda(g,g^{-1})^*$$

Idea of zesting

Having specified a new fusion ring, we now ask whether it categorifies to a

- fusion category
- (pivotal, spherical category)
- braided fusion category
- ribbon fusion category

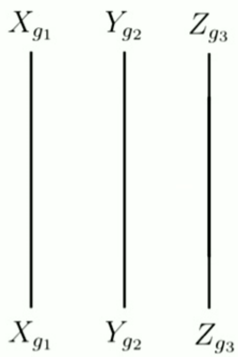
in a “straightforward” way,

and if so, classify the different possible categorifications at each step.

Zested monoidal structure

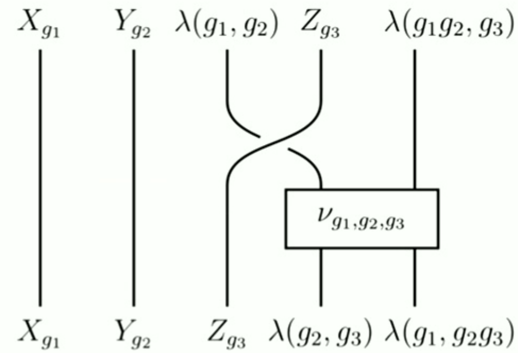
Before:	After:

$\alpha_{X,Y,Z}$ =

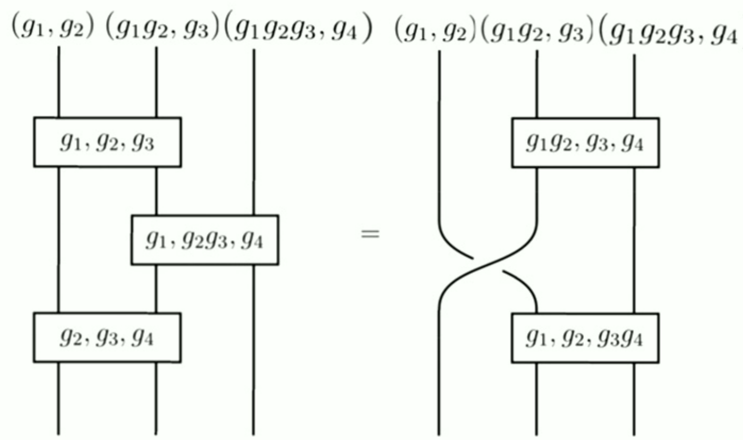


$\rightsquigarrow$

$\alpha^{\lambda,\nu}_{X,Y,Z}$ =



where

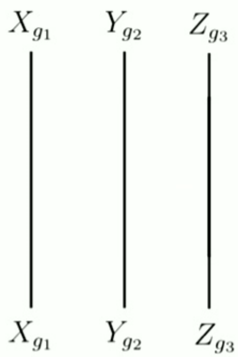


(sometimes I will supress the λ, ν to save space)

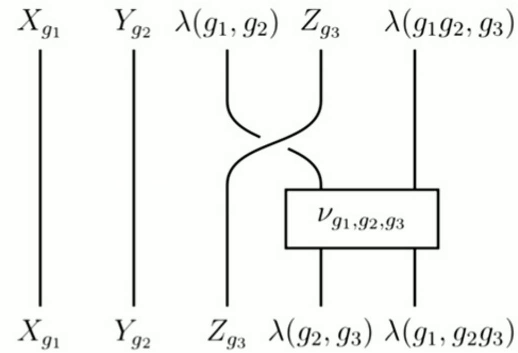
Zested monoidal structure

Before: **After:**

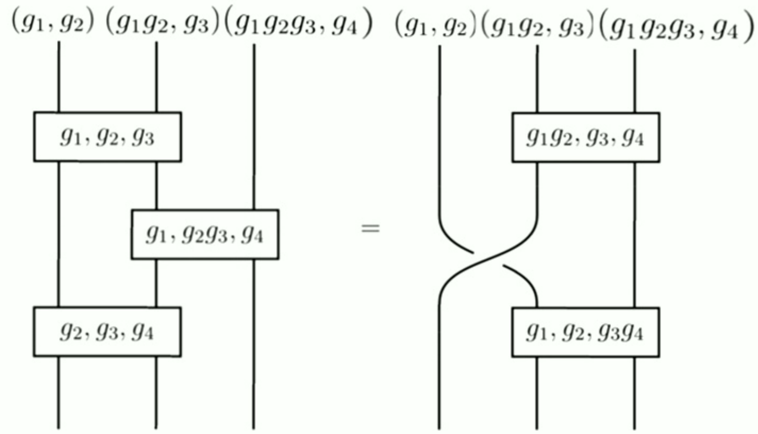
$\alpha_{X,Y,Z} =$



$\rightsquigarrow$ $\alpha^{\lambda,\nu}_{X,Y,Z} =$



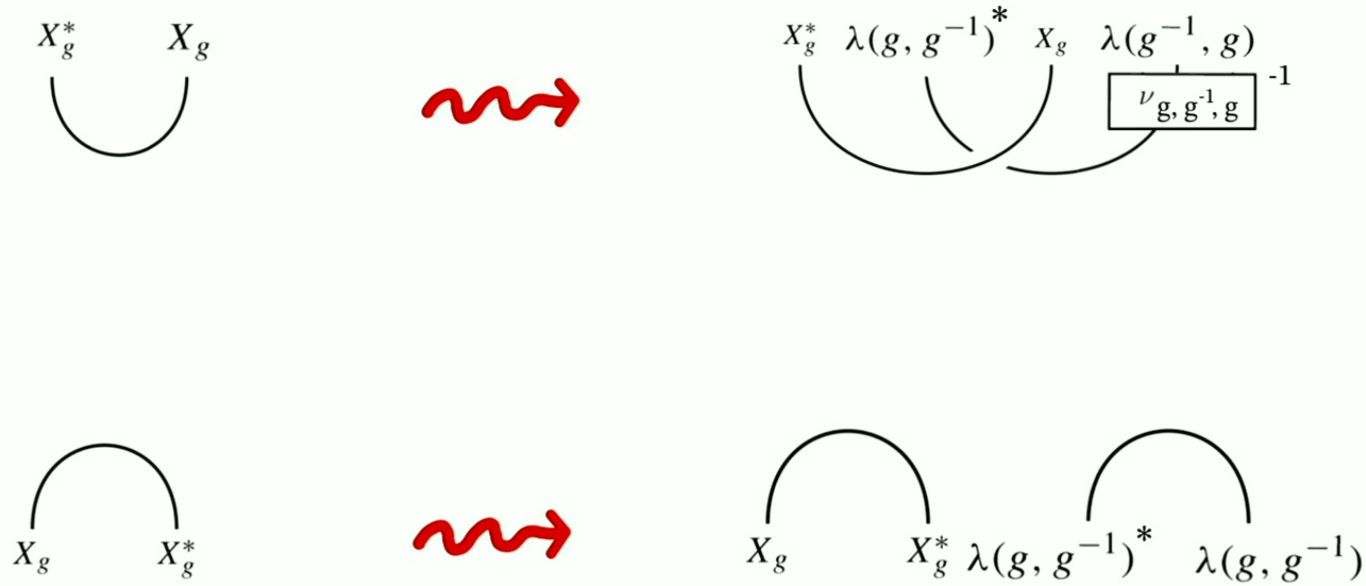
where



(sometimes I will supress the λ, ν to save space)

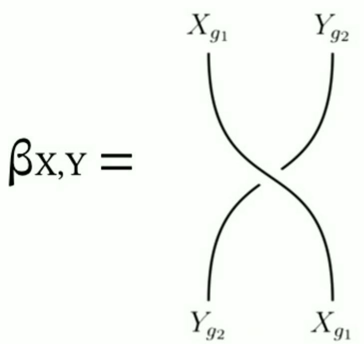
Zested duals and rigidity

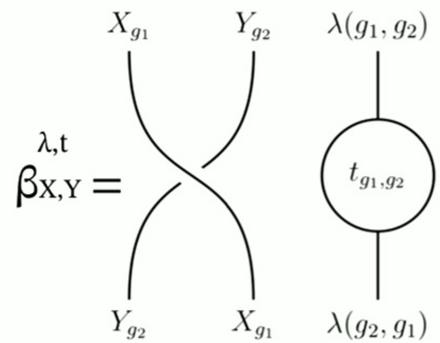
Before: After:



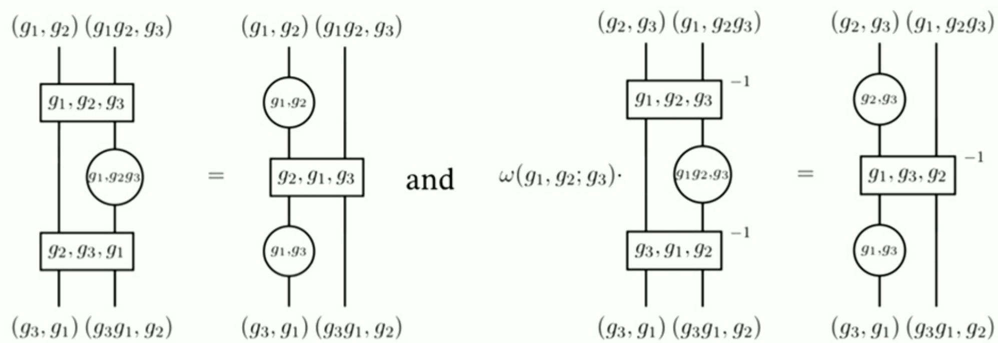
Zested braiding

Before: After:





where



Note: In general, non-degeneracy of β does not imply non-degeneracy of $\beta^{\lambda,t}$

Zesting topological order

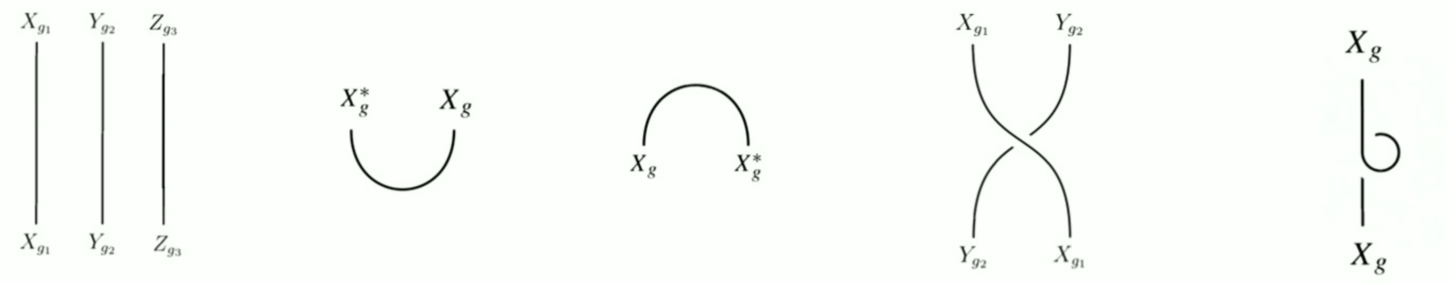
(Theorem:) Let $\mathcal{C}$ be a premodular fusion category and $\zeta = (\lambda,\nu,t,f)$ a choice of data satisfying the associative, braided, and ribbon zesting conditions. Then the category $\mathcal{C}^{\zeta}$ described in the previous slides is a premodular fusion category. **[D, Galindo, Plavnik, Rowell, Zhang (2021)]**

In general, modularity of $\mathcal{C}$ is not enough to guarantee modularity of $\mathcal{C}^{\zeta}$ without imposing an additional condition on the zesting data.

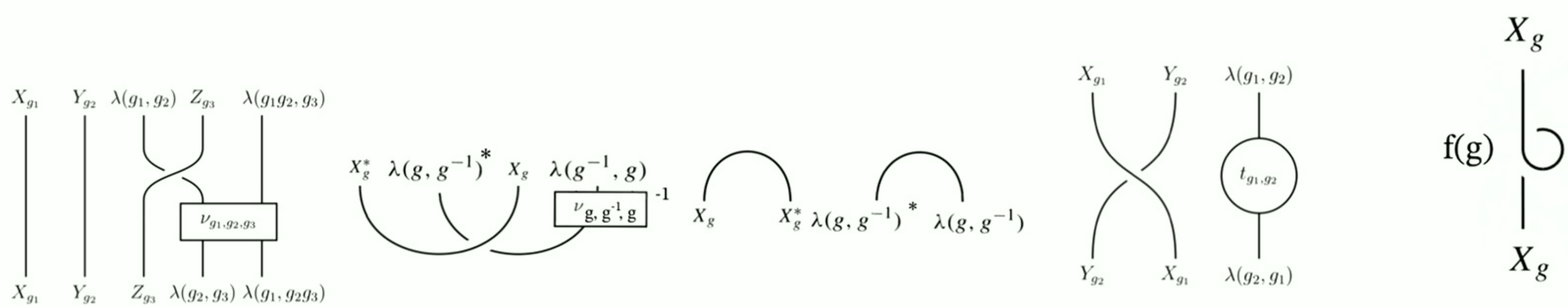
For the rest of this talk we will assume we are working with such modular zestings so that we may interpret zesting as an operation on topological order.

String diagrammatic overview of zesting topological order

Structure morphisms in (strict) $\mathcal{C}$:



Structure morphisms in $\mathcal{C}^{\zeta}$:



Overview of conditions satisfied by zesting data

Let $\mathcal{C}$ be a braided fusion category with G-grading. WLOG can assume $\mathcal{C}$ strict.

"Associative zesting"

1. Pick 2-cocycle $\lambda \in Z^2(G, \mathrm{Inv}(\mathcal{C}_e))$, i.e.

   $\lambda : G \times G \to \mathrm{Inv}(\mathcal{B}_e)$ such that $\lambda(g,h) \otimes \lambda(gh,k) \cong \lambda(h,k) \otimes \lambda(g,hk)$

   $(g, h) \mapsto \lambda(g,h)$

2. Pick 3-cochain $\nu \in C^3(G, k^\times)$ s.t.

   $\nu(g,h,k)\, \nu(g,hk,l)\, \nu(h,k,l) = \beta^{-1}_{\lambda(g,h),\lambda(k,l)}\, \nu(gh,k,l)\nu(g,h,kl)$

"Braided zesting"

3. Pick 2-cochain $t \in C^2(G,k^\times)$ s.t.

   $\nu(g,h,k)t(g,hk)\nu(h,k,g) = t(g,h)\nu(h,g,k)t(g,k)$

   \+ a similar second equation

\+ *"Ribbon zesting"*

Example: Ising topological order

* There are 8 (unitary) modular fusion categories with Ising fusion rules which can be distinguished by the twists $\theta_\sigma^{(r)} = \exp(2\pi i(2r+1)/16)$, r=0,1,…,7
  Let i, j, k ∈ {0,1}, a,b ∈ {0,1} and define

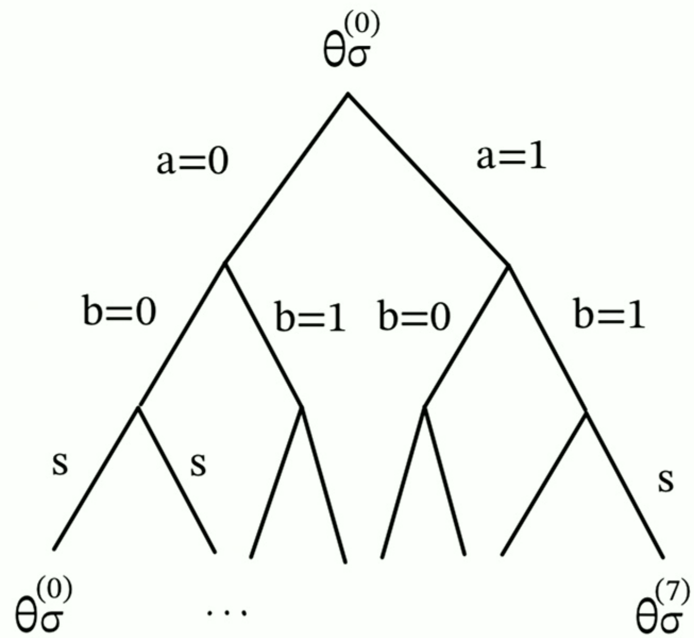
$$\lambda_a(i,j) = \begin{cases} \mathbf{1} & \text{if } i+j < 2 \\ \psi^a & \text{if } i+j \geq 2 \end{cases}$$

$$\nu_b(i,j,k) = \begin{cases} 1 & \text{if } i+j < 2 \\ i^{k(a+2b)} & \text{if } i+j \geq 2 \end{cases}$$

$$t_s(i,j) = s^{-ij} \quad \text{where } s = \pm\sqrt{i^{-(a+2b)}}$$

$$f_s(i) = s^{-i^2}$$

(here i mean the imaginary #)



Example: abelian theories with 4 anyons

* can also realize all the abelian theories of rank 4 through zesting, but need to use non-universal grading groups

$$\mathbb{Z}_4 \rightsquigarrow \begin{matrix} \mathbb{Z}_4 \\ D(\mathbb{Z}_2) \\ \text{Sem} \boxtimes \text{Sem} \\ \overline{\text{Sem}} \boxtimes \overline{\text{Sem}} \\ \vdots \end{matrix}$$

Example: zesting produces modular isotopes

Call inequivalent modular fusion categories with the same S-matrix and T-matrix *modular isotopes*

- Examples of modular isotopes occur among the twisted quantum doubles $D^{\omega}G$ ($\mathcal{Z}(\mathrm{Vec}_G^{\omega})$ or $\mathrm{Rep}(DG^{\omega})$) for $G = \mathbb{Z}_q \rtimes \mathbb{Z}_p$ where p,q are certain odd primes

  [Mignard-Schauenburg]

- These can be constructed by zesting DG **[D, Kim, Plavnik]**

Example: “~~exotic~~” topological order from zesting quantum group MFCs

- $SU(3)_3^{\zeta}$ [D, Galindo, Plavnik, Rowell, Zhang (2021)]

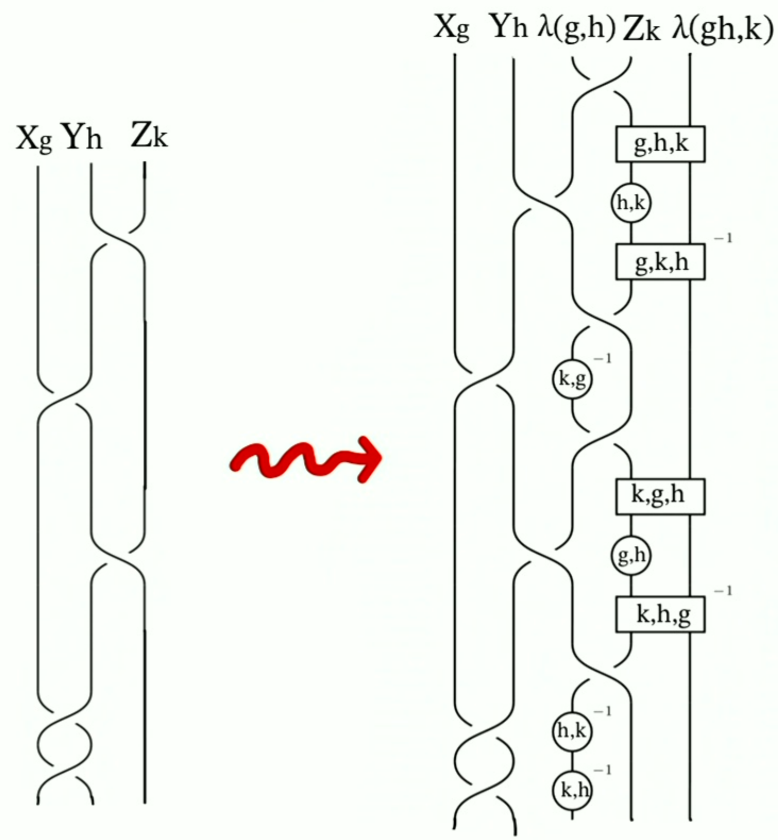
- other examples like this for $SU(2)_k$, $(E6)_3$, $SO(12)_2$ [Galindo, Mora, Rowell]

- shows zesting can take you outside of nice families of topological order

Properties of topological order under zesting

Exchange statistics of anyons

Recall: anyons themselves haven't changed (just how they fuse, braid, and twist)



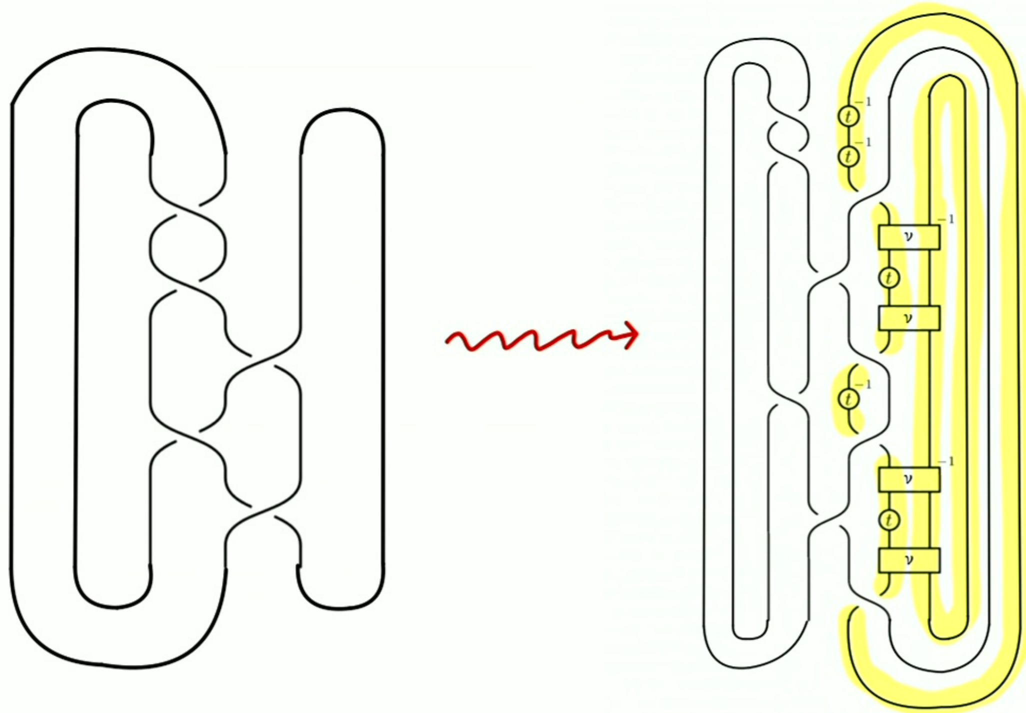
(Theorem:) Braid group representations are (projectively) preserved.

[D, Galindo, Plavnik, Rowell, Zhang (2021)]

Topological invariants

(Theorem:) Framed link invariants factorize, defining a new invariant of framed links colored by G that can be computed in polynomial time in the number of crossings.

[work in progress with McPhail-Snyder]



Example:

$$S^{\lambda}_{X_a, Y_b} = \frac{t^{(2)}(a, -b)\, f(a-b)}{t(a-b, a-b)} \dim(X_a)^{-1} S_{X_a, Y_b} S_{X_a, \lambda(b,-b)} \qquad T^{\lambda}_{X_a, X_a} = f(a) T_{X_a, X_a}$$

Understanding how topological order transforms under zesting

Measure of intrinsic "quantum computational complexity"		**Properties under zesting**
Braid group representations	✓	Projectively preserved [D, Galindo, Plavnik, Rowell, Zhang (2021)]
Mapping class group representations	?	Expect to be controlled by MCG reps of abelian theories
Invariants of framed links	✓	Differ by polynomial time algorithm [work in progress with McPhail-Snyder]
Invariants of closed 3-manifolds	?	Expect to differ FPT algorithm

Zested F-symbols and R-symbols

Given a description of a modular fusion category via fusion rules, F-symbols, and R-symbols it is straightforward to compute the corresponding data for the zested theory using the zesting data $\zeta = (\lambda, \nu, t, f)$

$$\{N, F, R, \theta\} \rightsquigarrow \{N^\zeta, F^\zeta, R^\zeta, \theta^\zeta\}$$

Understanding how topological order transforms under zesting

Measure of intrinsic "quantum computational complexity"		Properties under zesting
Braid group representations	✓	Projectively preserved [D, Galindo, Plavnik, Rowell, Zhang (2021)]
Mapping class group representations	?	Expect to be controlled by MCG reps of abelian theories
Invariants of framed links	✓	Differ by polynomial time algorithm [work in progress with McPhail-Snyder]
Invariants of closed 3-manifolds	?	Expect to differ FPT algorithm

II. "Zesting" symmetry-enriched topological order*

zesting data $\zeta = (\lambda, \nu)$

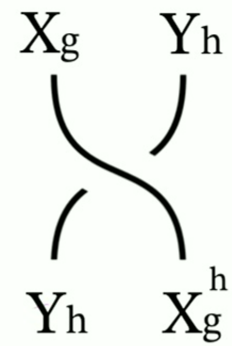
$$\mathcal{C} \rightsquigarrow \mathcal{C}^{\zeta}$$

*In this talk, (bosonic) symmetry-enriched topological (SET) order will be synonymous with G-crossed braided extensions of modular fusion categories. **[Barkeshli, Bonderson, Cheng, Wang]**

G-crossed braided fusion categories

A fusion category C is *G-crossed braided* if it has

1. G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$
2. G-action $T: G \longrightarrow \mathrm{Aut}(\mathcal{C})$
   $g \mapsto T_g$ s.t. $T_g(\mathcal{C}_h) \subset \mathcal{C}_{g^{-1}hg}$
   satisfying coherences
3. G-braiding $\beta_{X_g,Y_h}: X_g \otimes Y_h \longrightarrow Y_h \otimes T_h(X_g)$
   satisfying G-crossed hexagons



Called *G-crossed modular* if $\mathcal{C}_e$ modular

Examples of G-crossed modular fusion categories as SET order

Example: Tambara-Yamagami fusion categories $\mathcal{C}$=TY(A, χ, τ), A odd order

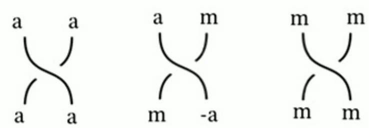
$$\begin{cases} a \otimes m = m \otimes a = m & \text{for all } a \in A \\ m \otimes m = \bigoplus_{a \in A} a \end{cases}$$

"particle-hole symmetry"

$\mathbb{Z}/2\mathbb{Z}$- action $\quad a \mapsto -a,\ m \mapsto m$

$\mathbb{Z}/2\mathbb{Z}$- grading $\quad \{ a \mid a \in A \} \oplus \{ m \}$

$\mathbb{Z}/2\mathbb{Z}$- braiding



Example: extensions of D($\mathbb{Z}$) with respect to e-m duality symmetry,
$\mathcal{C} \boxtimes \mathcal{C}$ with bilayer exchange symmetry

[Barkeshli, Bonderson, Cheng, Wang]

G-crossed braided fusion categories

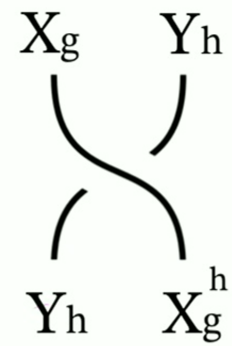
A fusion category C is *G-crossed braided* if it has

1. G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$
2. G-action $T: G \to \mathrm{Aut}(\mathcal{C})$

   $g \mapsto T_g$ s.t. $T_g(\mathcal{C}_h) \subset \mathcal{C}_{g^{-1}hg}$

   satisfying coherences
3. G-braiding $\beta_{X_g, Y_h}: X_g \otimes Y_h \to Y_h \otimes T_h(X_g)$

   satisfying G-crossed hexagons



Called *G-crossed modular* if $\mathcal{C}_e$ modular

Classification of G-crossed braided extensions of braided fusion categories

[Etingof, Nikshych, Ostrik]

A braided fusion category $\mathscr{B}$ has a G-crossed braided extension if it admits a monoidal 2-functor $G \to \mathrm{BrPic}(\mathscr{B})$

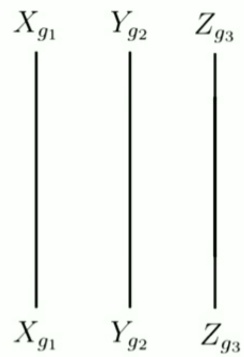
These are classified by (ρ, λ, ω)

* group homomorphism $\rho : G \to \mathrm{Aut}(\mathscr{B})$
* $\lambda \in H^2_\rho(G, \mathrm{Inv}(\mathscr{B}))$
* $\omega \in H^3(G, k^\times)$

Associators of zested defect fusion rule $X_g \otimes Y_h \otimes \lambda(g,h)$

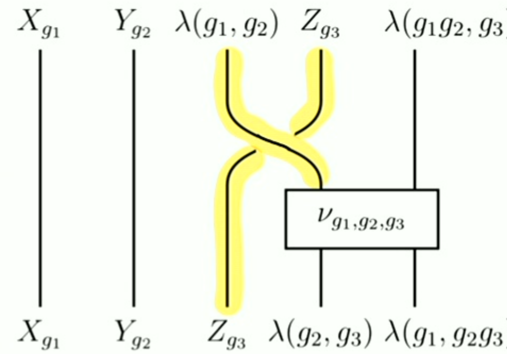
Before:

$\alpha_{X,Y,Z} =$

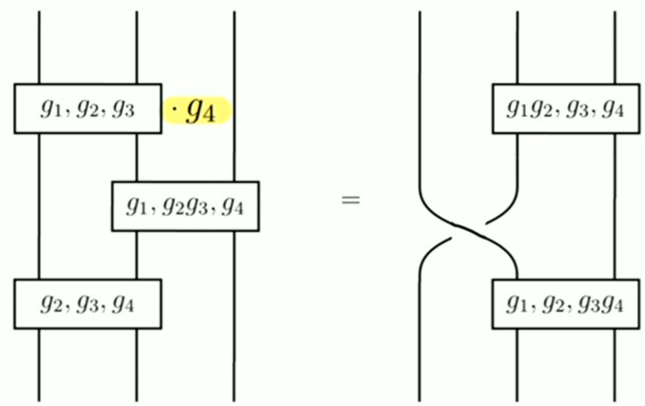


After:

$\alpha^{\lambda,\nu}_{X,Y,Z} =$



where



Zested symmetry defect theory

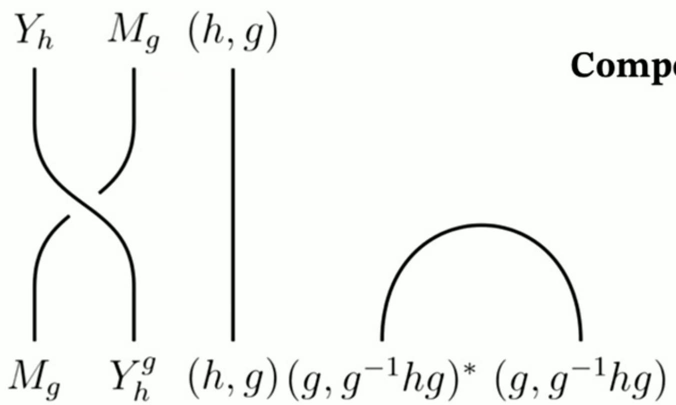
Theorem: zesting an SET order with respect to λ, ν is automatically again an SET order with:

G-symmetry action:

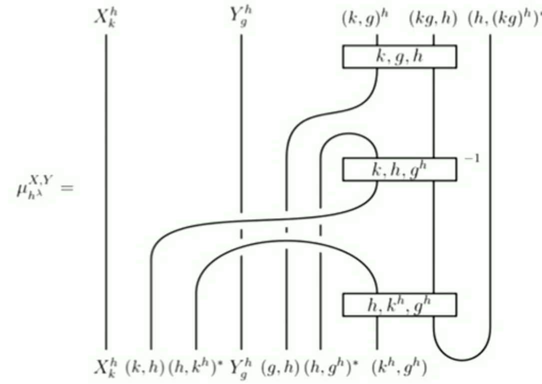
$$T_g^{(\lambda,\nu)} : \mathcal{C}_h^{(\lambda,\nu)} \to \mathcal{C}_{g^{-1}hg}^{(\lambda,\nu)}$$
$$Y_h \mapsto Y_h^g \otimes \lambda(h,g) \otimes \lambda(g, g^{-1}hg)^*$$

Symmetry defect (crossed) braiding:

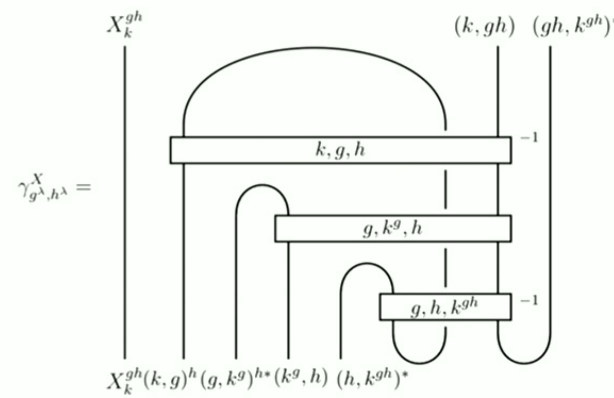
$c_{Y_h,M_g}^{(\lambda,\nu)} =$



Tensorators:



Compositors:



[D., Galindo, Plavnik, Rowell, Zhang (2024)]

Zesting symmetry defects changes their symmetry fractionalization class

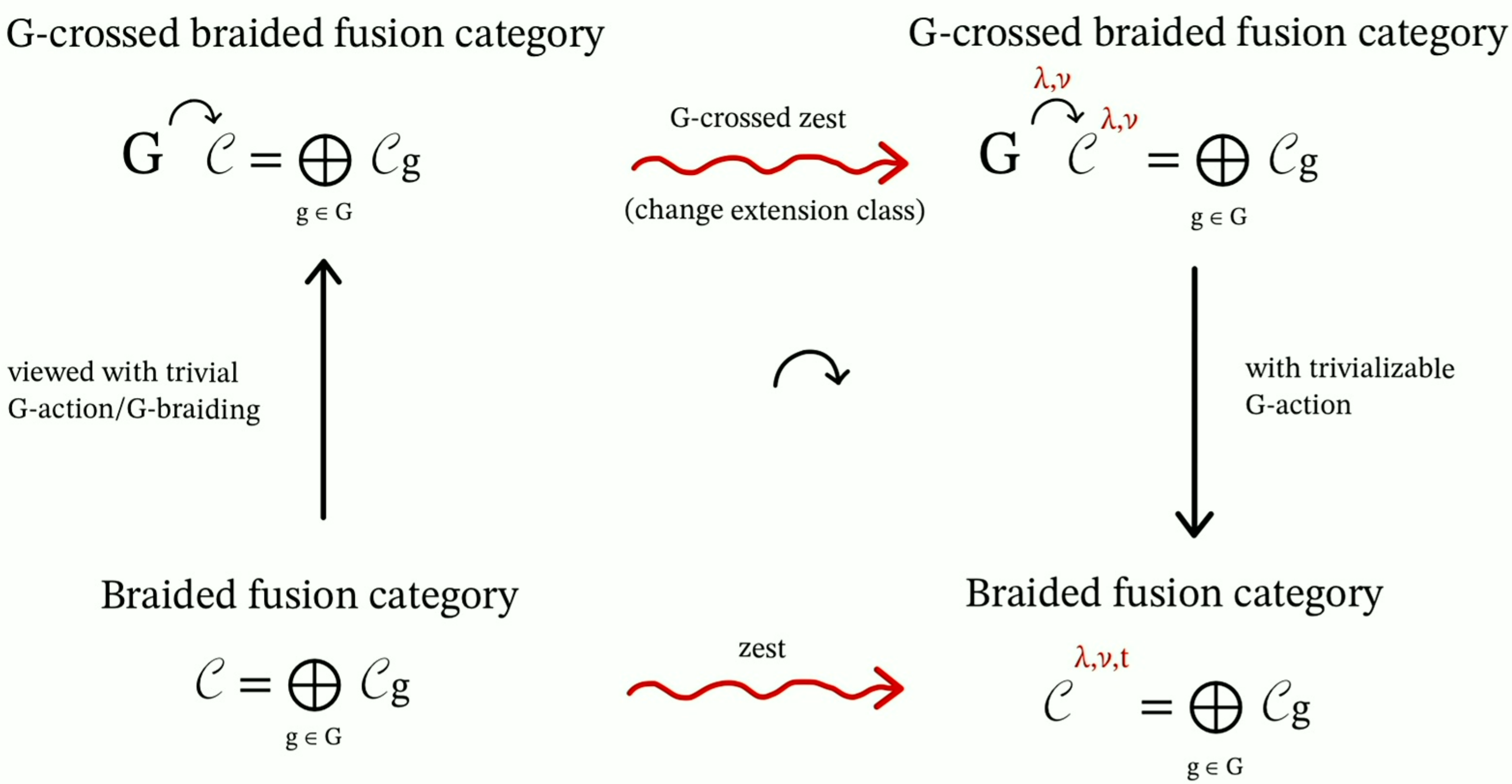
Theorem: Any two G-crossed extensions of a braided fusion category $\mathcal{B}$ with the same group homomorphism $\rho\colon G \to \mathrm{Pic}(\mathcal{B})$ are related by G-crossed braided zesting.

if $\quad \mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g \, , \mathcal{C}_e = \mathcal{B} \qquad\qquad \mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g \, , \mathcal{D}_e = \mathcal{B}$

and $\quad \mathcal{C}_g = \mathcal{D}_g$ as $\mathcal{B}$-module categories for all $g \in G$

then there exists G-crossed zesting data (λ,ν) such that $\mathcal{D} \cong \mathcal{C}^{\zeta}$

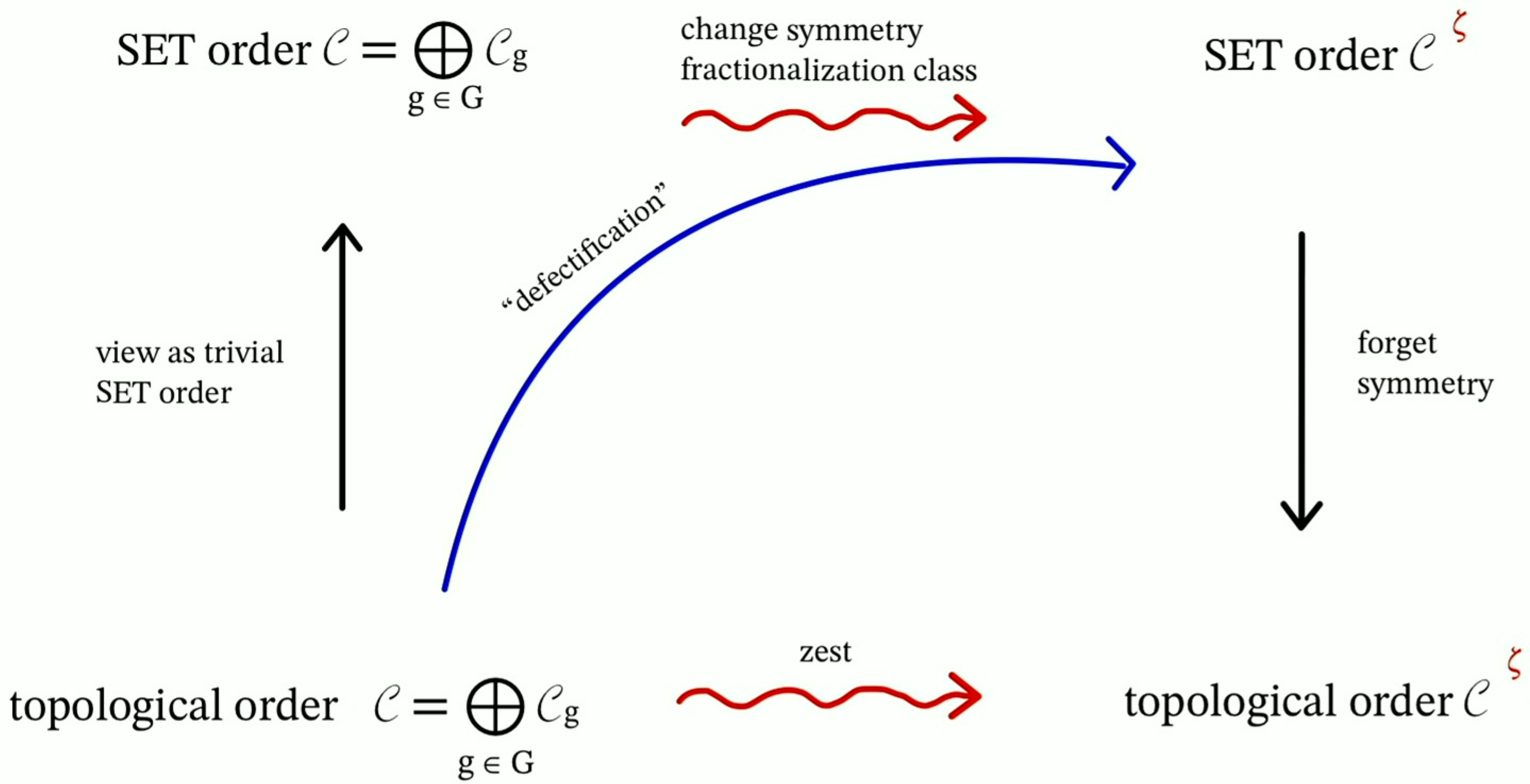
Recovering braided zesting from G-crossed braided zesting*



*technically this only works if λ takes values in a symmetric subcategory

Physics interpretation*

(interpretation requires $\mathcal{C}_e$ modular **and** with local symmetry, plus some other assumptions, so is very limited)



Summary

* zesting is a useful tool in the classification of fusion rings, fusion categories, and topological order

* zesting is connected to G-extension theory, and hence gauging of finite symmetry of topological order [Aasen, Bonderson, Knapp] [D, Galindo, Plavnik, Rowell, Zhang]

* understanding zesting of topological and symmetry-enriched topological order may help us eventually rigorously understand how topological order transforms under gauging

Thanks!