Title: Weak Hopf symmetric tensor networks

Speakers: Andras Molnar

Collection: Higher Categorical Tools for Quantum Phases of Matter

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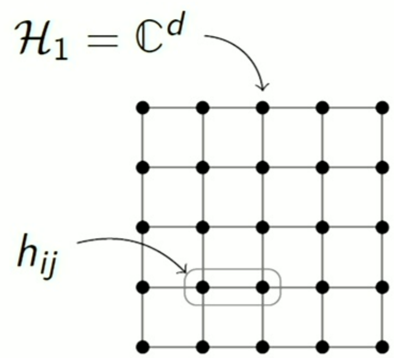
Weak Hopf symmetric tensor networks

arXiv:2204.05940

Andras Molnar, José Garre Rubio, Alberto Ruiz-de-Alarcón,
Norbert Schuch, David Pérez-García, Ignacio Cirac

Many-body problem

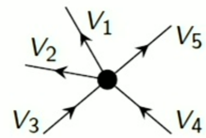
- $\mathcal{H}_{tot} = \mathcal{H}_1^{\otimes N}$, $H = \sum_{\langle ij \rangle} h_{ij}$
- Goal: find the GS of H
- Variational ansatz
  - Few parameters only
  - Can calculate $\langle O_{local} \rangle$
  - Approximates GS well
- Tensor Networks: such ansätze



[Cirac:2011.12127]

Graphical notation of tensor calculus

- Tensors



$$A : V_3 \otimes V_4 \to V_2 \otimes V_1 \otimes V_5$$

Graphical notation of tensor calculus

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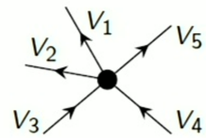

$$A \in V_1 \otimes V_2 \otimes V_3^* \otimes V_4^* \otimes V_5$$

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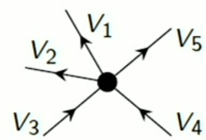
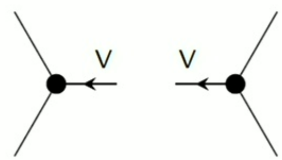


$$A \in V_1 \otimes V_2 \otimes V_3^* \otimes V_4^* \otimes V_5$$

$$A[i, j, k, l, m]$$

- Contraction: $f \otimes v \mapsto f(v)$



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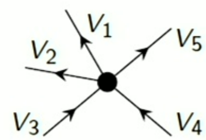

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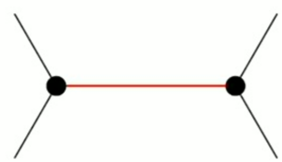
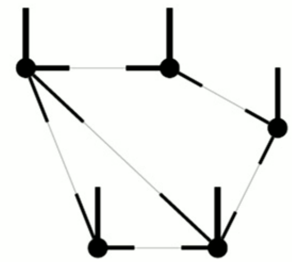


$$\mathcal{C}_{21}\left(A \otimes B\right) = \sum_j A_{ijk} \cdot B_{jmn}$$

Tensor Networks

- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- $rank(A_v) = deg(v) + 1$
  - physical index: $\mathcal{H}_v$
  - virtual indices: $(v, w) \in \mathcal{E}$



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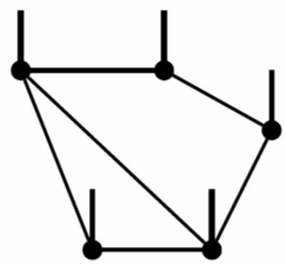
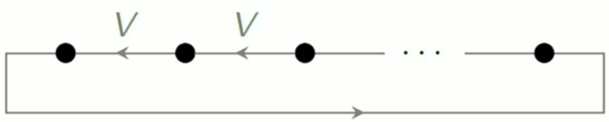
  - virtual indices: $(v, w) \in \mathcal{E}$
- $|\Psi\rangle \in \bigotimes_v \mathcal{H}_v$,
  $|\Psi\rangle = \bigotimes_{e \in \mathcal{E}} \mathcal{C}_e \left(\bigotimes_{v \in \mathcal{V}} A_v \right)$



Matrix product states and operators



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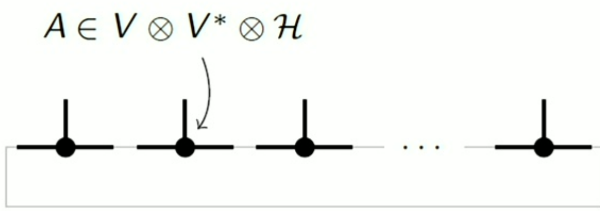

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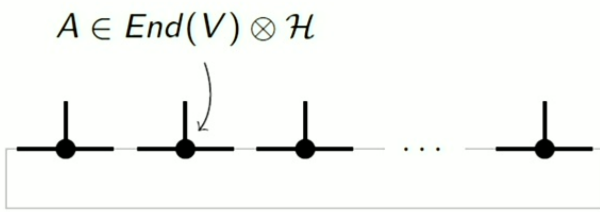

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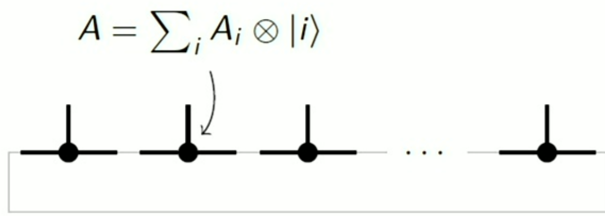
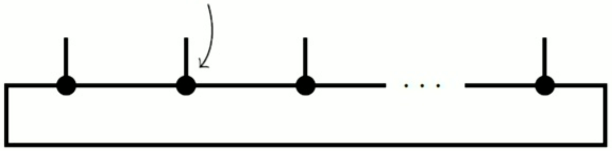


$$= \sum_i A_{i_1} \otimes \cdots \otimes A_{i_n} \otimes |i_1 \dots i_n\rangle$$

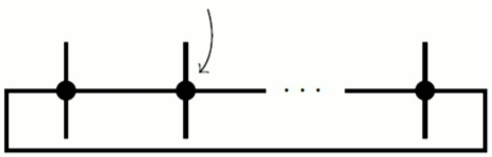
Matrix product states and operators

$A = \sum_i A_i \otimes |i\rangle$



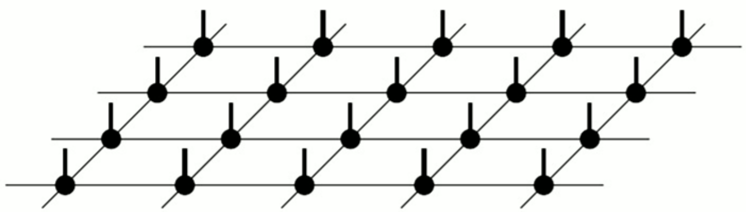
$$= \sum_i \mathrm{Tr}\left(A_{i_1} \dots A_{i_n}\right) |i_1 \dots i_n\rangle$$

$A = \sum_i A_{ij} \otimes |i\rangle\langle j|$



$$= \sum_{ij} \mathrm{Tr}\left(A_{i_1 j_1} \dots A_{i_n j_n}\right) |i_1 \dots i_n\rangle\langle j_1 \dots j_n|$$

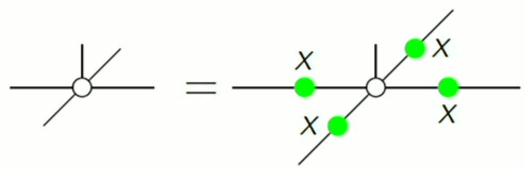
PEPS: 2D Tensor networks



PEPS can describe topological order

Topological PEPS

- PEPS can describe string-nets [Buerschaper:0809.2393]
- It allows perturbation from RFP, phase transitions
- Origin of topological properties: symmetries [Sahinoglu:11409.2150]
- Symmetry is purely virtual, "size-independent": MPO



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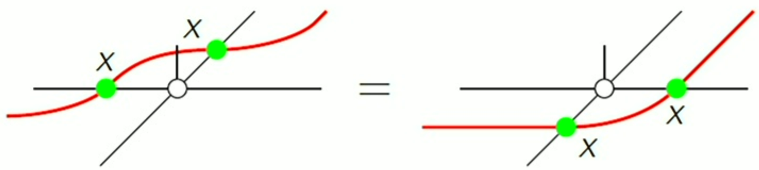
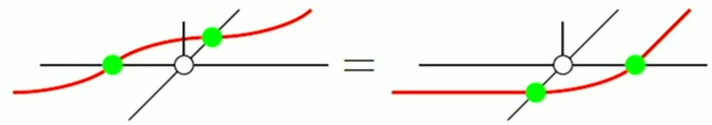


Size-independence of the symmetry

MPO symmetry of the PEPS tensor:



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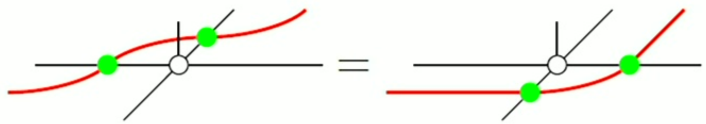
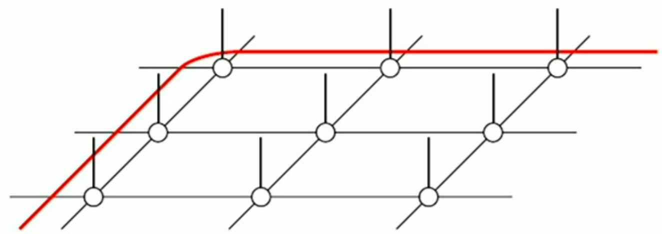


Symmetry of large area:



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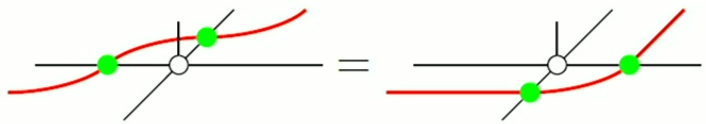

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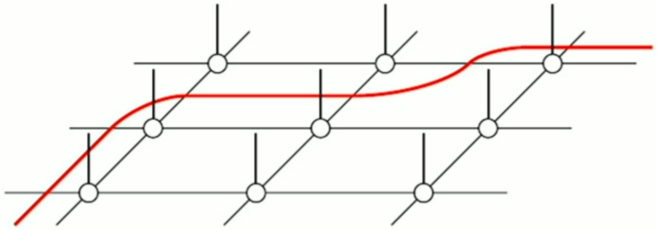

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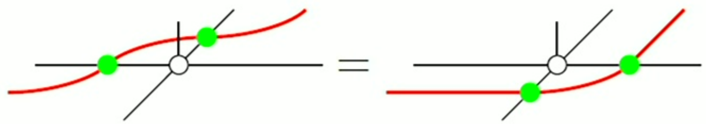

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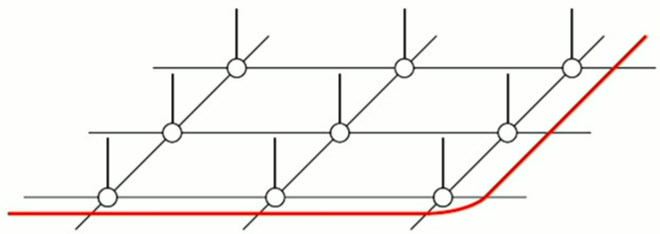
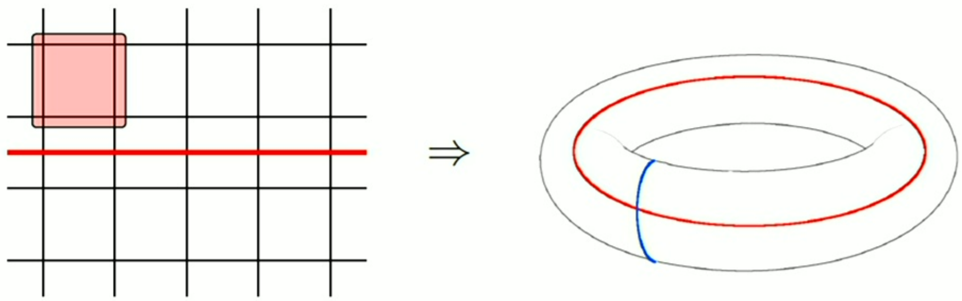


Topological ground space degeneracy

A local Hamiltonian does not detect the symmetry operator:



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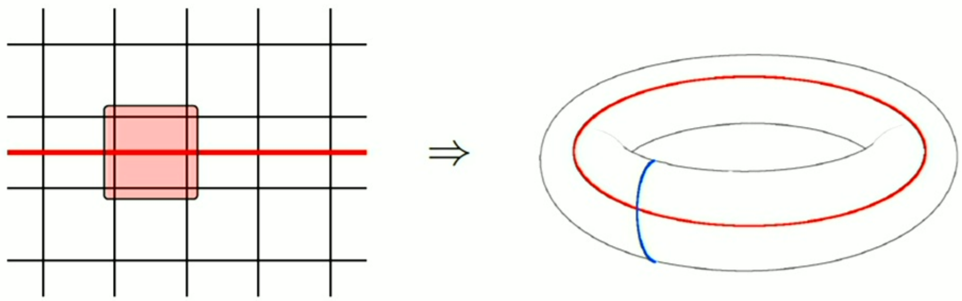

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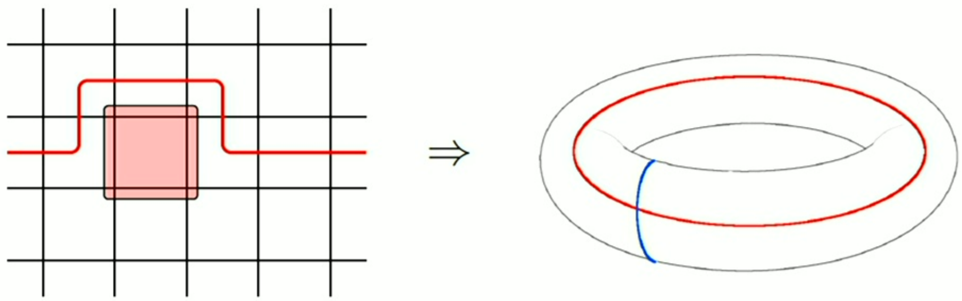
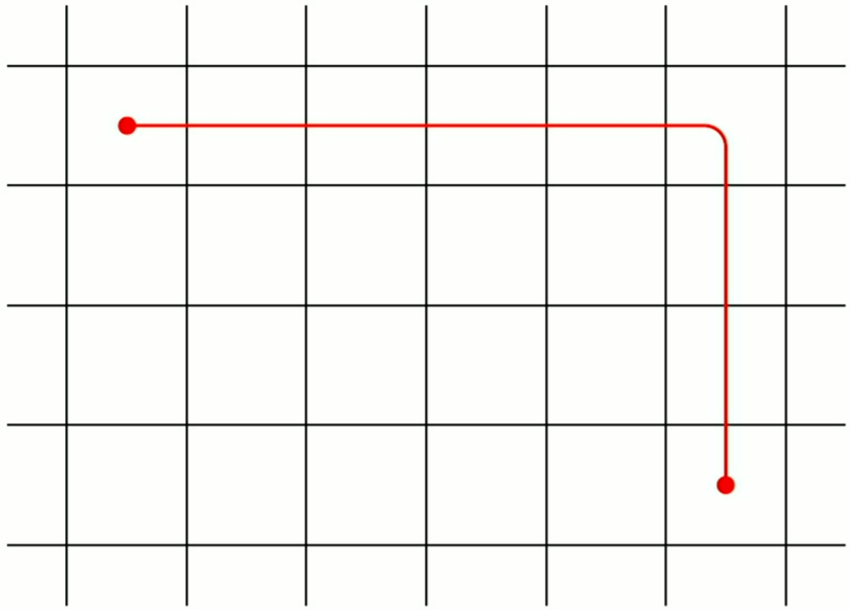


Excitations – example



[Bultinck:1511.08090]

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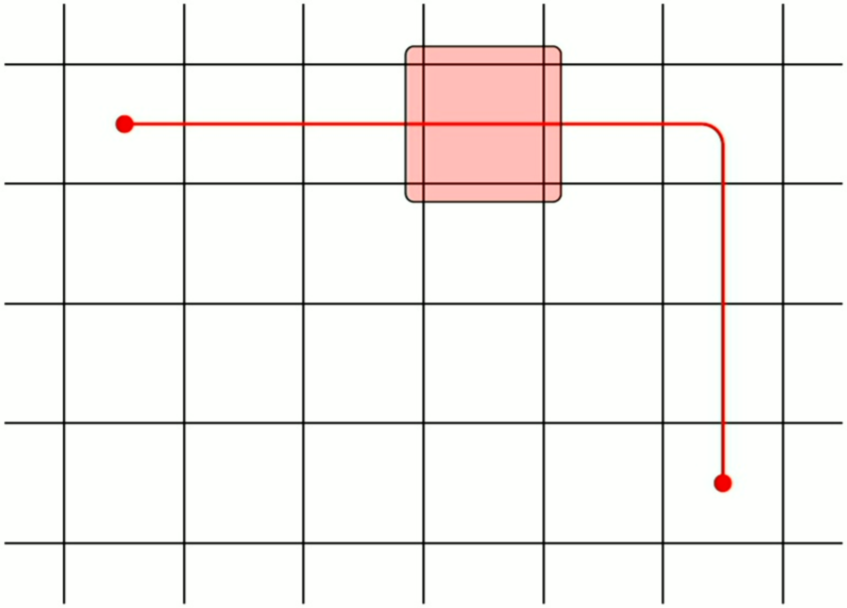

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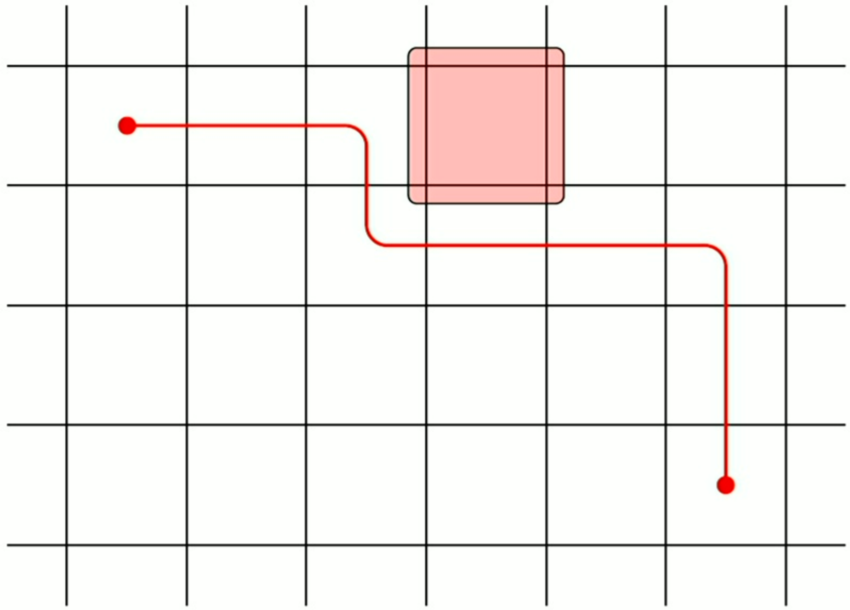

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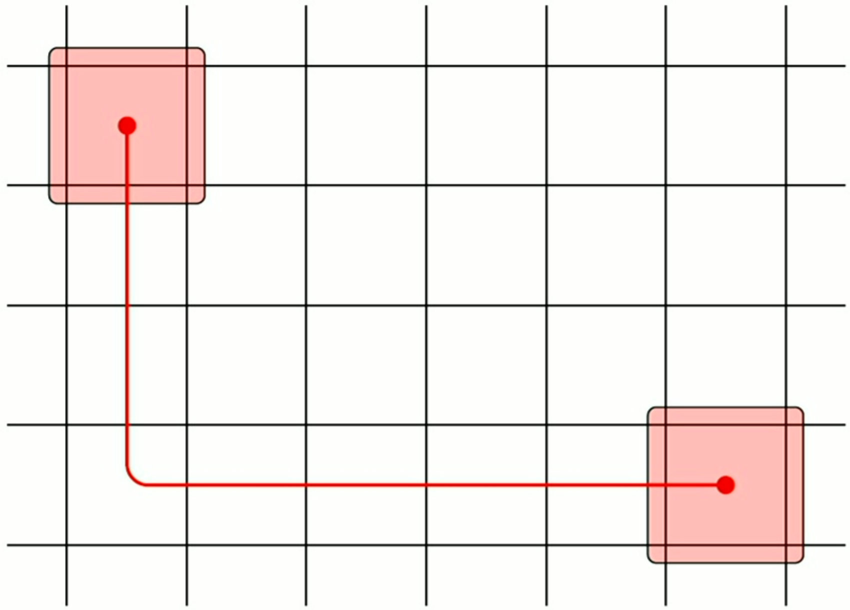
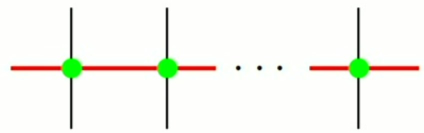


[Bultinck:1511.08090]

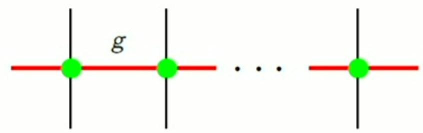
The MPO symmetries

Toric code: 2 MPOs,



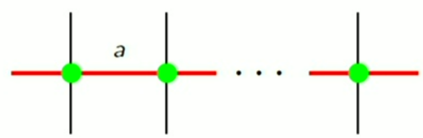
$$\equiv \begin{cases} 1 \otimes 1 \otimes \cdots \otimes 1 \\ X \otimes X \otimes \cdots \otimes X \end{cases}$$

G-injective PEPS: MPOs = elements of G



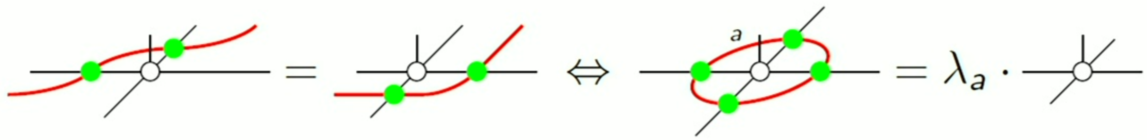
$$\equiv g \otimes g \otimes \cdots \otimes g, \quad g \in G$$

String-net models: MPO = simple objects of fusion category $\mathcal{C}$



$$\equiv a \in Obj(\mathcal{C})$$

Algebra of symmetries



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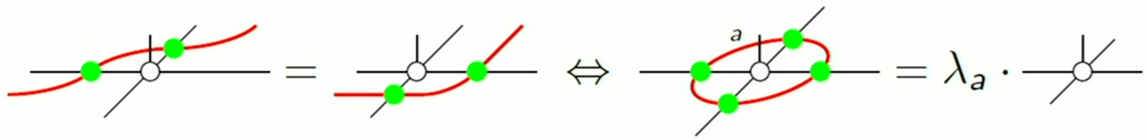

Linear combination and product of MPOs is symmetry:

$$(O_a + O_b)|T\rangle = (\lambda_a + \lambda_b)|T\rangle \quad \text{and} \quad O_a \cdot O_b|T\rangle = \lambda_a \cdot \lambda_b|T\rangle$$

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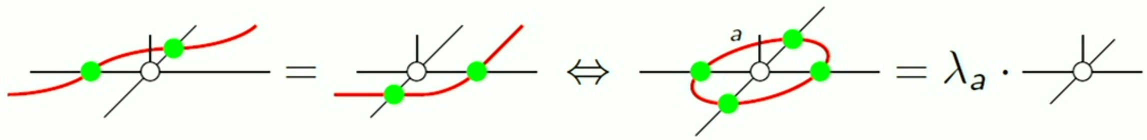

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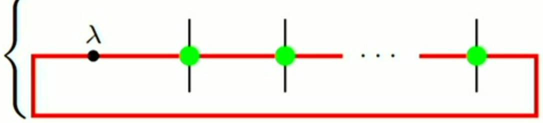
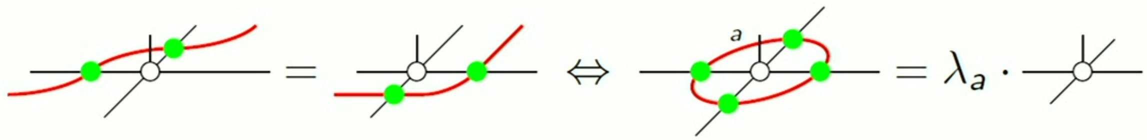


$\Big| \lambda = \bigoplus_a \lambda_a \cdot \mathbb{1}_{D_a} \Big\}$

Algebra of symmetries



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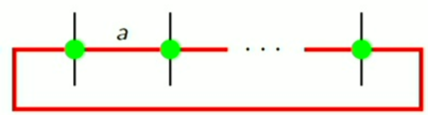
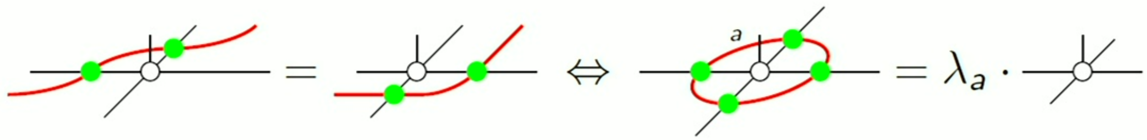


$$\left| \lambda_a \in \mathbb{C} \right\}$$

Algebra of symmetries



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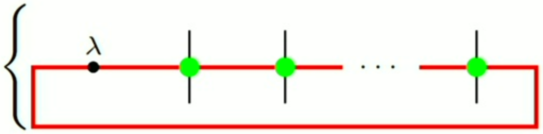
$\mathcal{A}_{PBC} \equiv \Big\{$  $\Big| \lambda = \bigoplus_a \lambda_a \cdot \mathbb{1}_{D_a} \Big\}$

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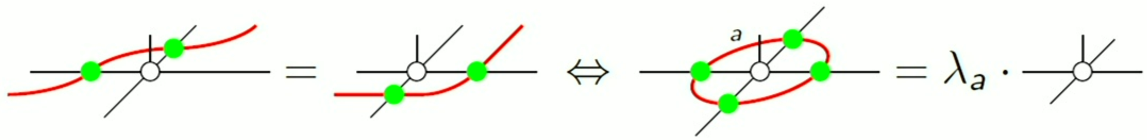

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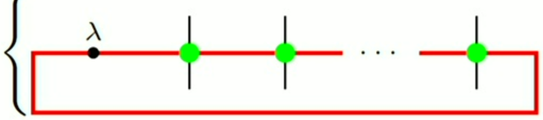

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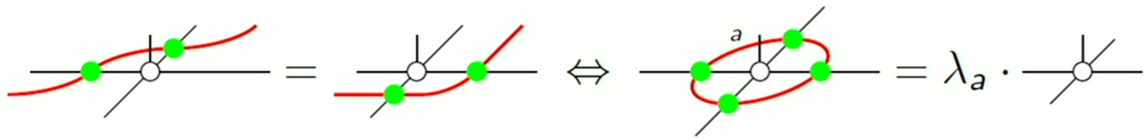
The product for groups:

$$O_g \cdot O_h = O_{gh}$$

Algebra of symmetries



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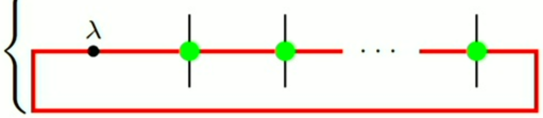


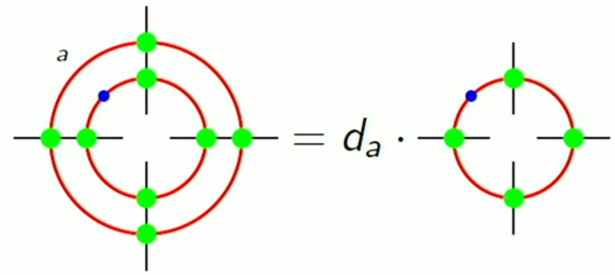
$\Bigg| \lambda = \bigoplus_a \lambda_a \cdot \mathbb{1}_{D_a} \Bigg\}$

The product for fusion categories:

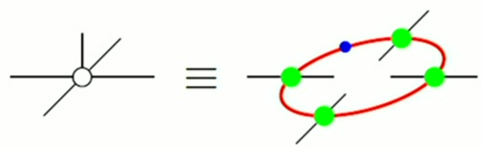
$$O_a \cdot O_b = \sum_c N_{ab}^c O_c, \quad N_{ab}^c \in \mathbb{Z}^+$$

PEPS from MPO symmetries

Special element $O \in \mathcal{A}_{PBC}$:

$O_a \cdot O = d_a O$ or 

Then a symmetric PEPS tensor is:

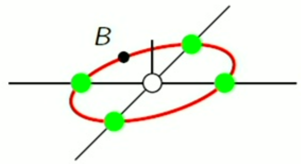


[Bultinck:1511.08090, Lootens:2008.11187]

Weak Hopf algebras from MPO symmetries

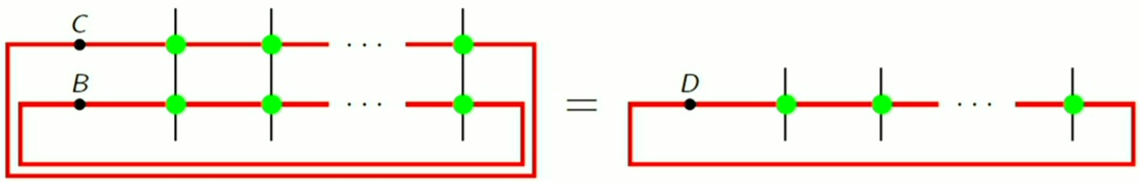
Our perspective

Open boundary MPOs are also symmetries:



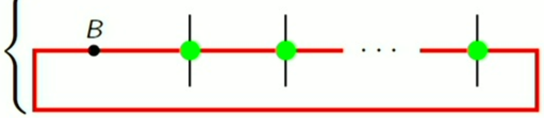
$$\cong \lambda \cdot \qquad B \in \bigoplus_a \mathcal{M}(D_a)$$

Open boundary MPOs form an algebra: $\forall B, C \; \exists D$ s.t.



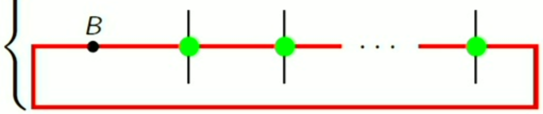
The algebra of symmetries:

$$\mathcal{A}_{OBC} \equiv \left\{ \text{[diagram]} \;\middle|\; B \in \bigoplus_a \mathcal{M}(D_a) \right\}$$



Weak Hopf algebra

- Injective tensors:

$\mathcal{A} \equiv \Big\{$  $\Big| B \in \bigoplus_a \mathcal{M}(D_a) \Big\}$

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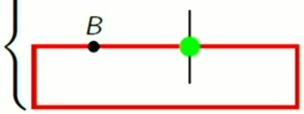

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Weak Hopf algebra

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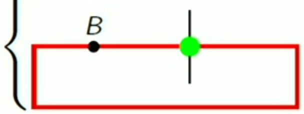
$$\mathcal{A} \equiv \Bigg\{$$ 

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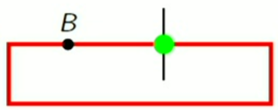
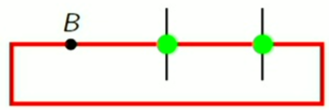
$$\Delta :$$ 

$$\mapsto$$ 

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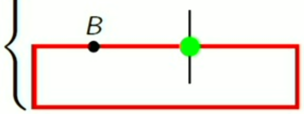
$\mathcal{A} \equiv \Big\{$ 

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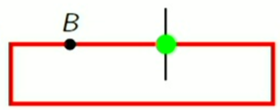
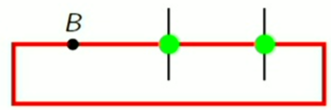
$\Delta :$ 

$\mapsto$ 

$\in \mathcal{A} \otimes \mathcal{A}$

- $(\mathcal{A}, \Delta)$ + additional properties = (weak) Hopf algebra

[Bohm:math/9805116, Montgomery: Rep Theory of Semisimple Hopf]

Weak Hopf algebra

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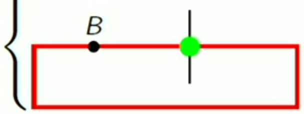
$$\mathcal{A} \equiv \Bigg\{$$ 

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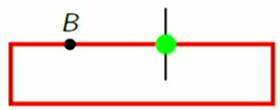
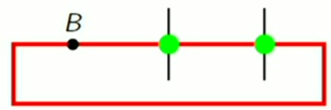
$$\Delta :$$ 

$$\mapsto$$ 

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  [Bohm:math/9805116, Montgomery: Rep Theory of Semisimple Hopf]

- Fusion category $\equiv$ Weak Hopf algebras

  [Etingof:math/0203060, Kitaev:1104.5047]

Use of algebraic formulation

- Transfer operator of topological PEPS: renormalization fixed point MPDO

  [Ruiz-de-Alarcón:2204.06295]
- Phase classification of RFP MPDO

  [Ruiz-de-Alarcón:2204.06295]
- Characterization of symmetries in topologically ordered PEPS

  [Molnar: in preparation]
- Other possibly interesting states with "topological" properties

MPO symmetries from algebras

MPOs from coproducts

$\mathcal{A}$ algebra with $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ s.t.

- $(\mathrm{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta$
- $\Delta(xy) = \Delta(x) \cdot \Delta(y)$

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Let $\mathcal{A}^*$ be set of linear functionals on $\mathcal{A}$. Define $f \star g \in \mathcal{A}^*$ by

$$(f \star g)(x) = (f \otimes g) \circ \Delta(x)$$

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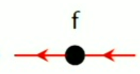
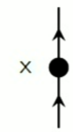
Then $\star$ is a product:

$$(f \star (g \star h))(x) = ((f \star g) \star h)(x)$$

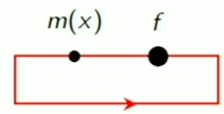
$\mathcal{A}^*$ is an algebra.

MPOs from coproducts

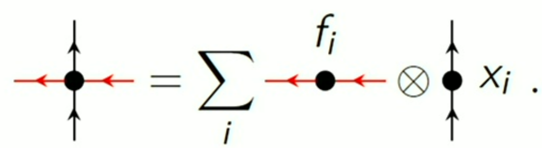
Fix a representation of $\mathcal{A}^*$ and of $\mathcal{A}$:

$\psi : f \mapsto$  $\phi : x \mapsto$ 

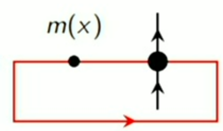
If the representation is injective, then $\forall x \in \mathcal{A}\ \exists m(x)$ s.t.

$f(x) = \mathrm{Tr}\left(m(x) \cdot \psi(f)\right) =$ 

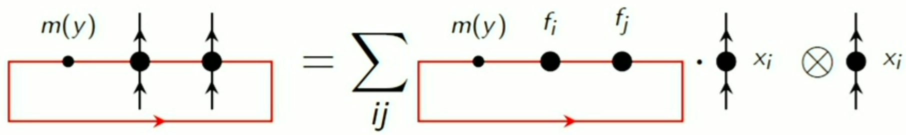
Fix a basis of $\mathcal{A}$, $\{x_i\}$, and dual basis $\{f_i\}$, and let



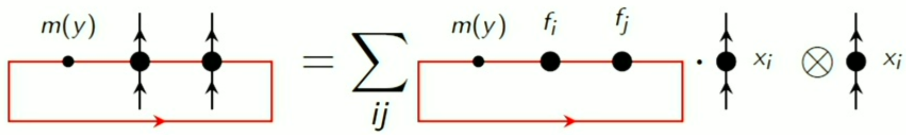
By construction,

$\phi(x) =$ 

MPOs from coproducts



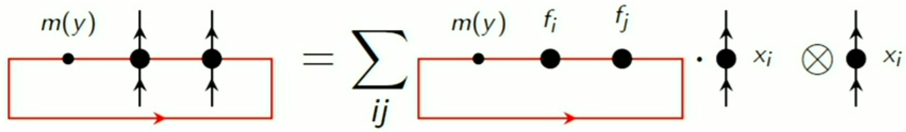
MPOs from coproducts



$$= \sum_{ij} (f_i \star f_j)(y) \cdot \phi(x_i) \otimes \phi(x_j)$$

$$= \sum_{ij} (f_i \otimes f_j) \circ \Delta(y) \cdot \phi(x_i) \otimes \phi(x_j)$$

MPOs from coproducts



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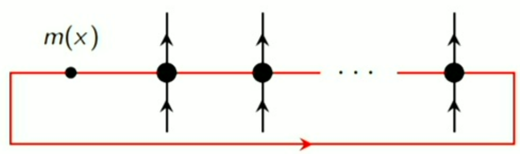
$$= (\phi \otimes \phi) \circ \Delta(y) = (\phi \boxtimes \phi)(y)$$

Similarly:



$$= \phi^{\otimes n} \circ \Delta^n(x)$$

Example: $\mathbb{C}[G]$

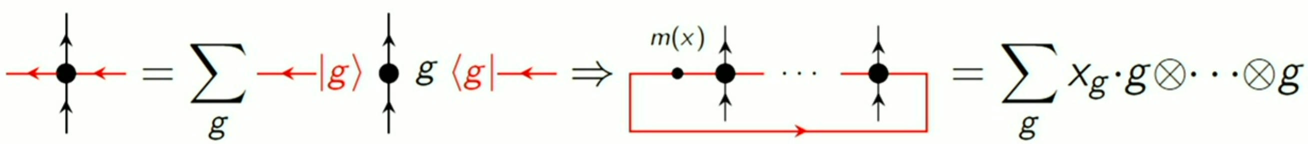
Basis of $\mathbb{C}[G]$: $\{g\}_{g\in G}$. Dual basis in $(\mathbb{C}[G])^*$: $\{\delta_g\}_{g\in G}$

$$\delta_g(h) = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{else} \end{cases}$$

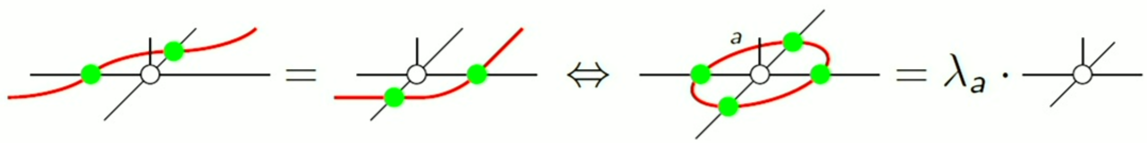
Product in $(\mathbb{C}[G])^*$:

$$\delta_g \star \delta_h = \begin{cases} \delta_g & \text{if } g = h \\ 0 & \text{else} \end{cases}$$

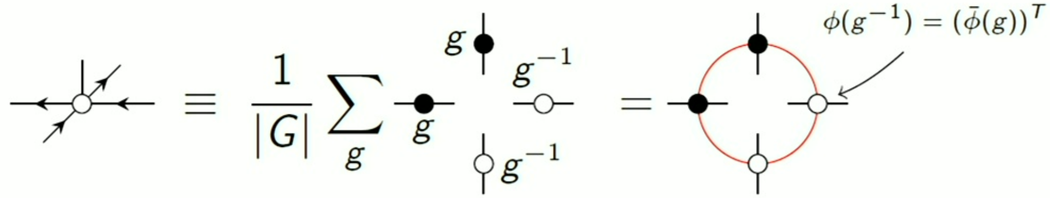
All irreps are $1D$; left regular: $g \mapsto |g\rangle\langle g|$



WHA-symmetric PEPS



Example: G-injective PEPS



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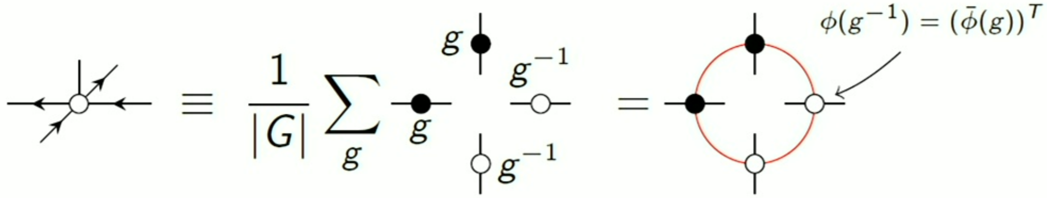
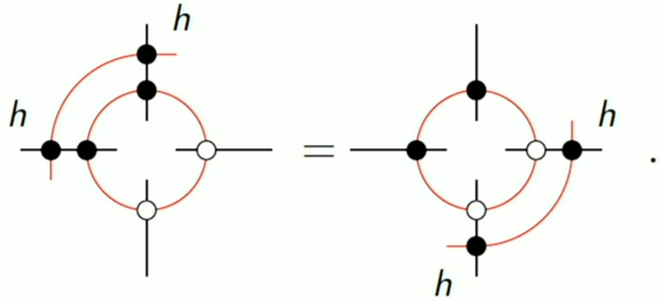
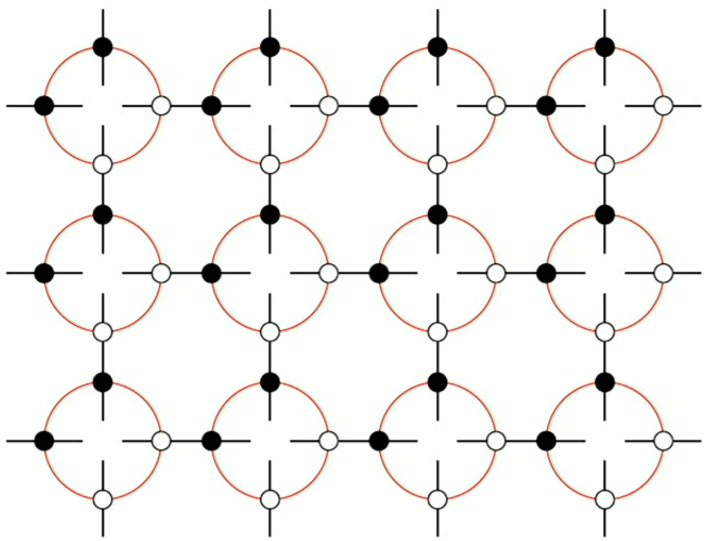


Then the pulling-through equation is:



$$\sum_g hg \otimes g^{-1} = \sum_g g \otimes g^{-1}h$$

G-injective PEPS



- Global symmetry
- Regrouping DoF: Kitaev model

Algebraic structure in $\mathbb{C}[G]$

- $\mathbb{C}[G]$ is algebra and coalgebra ($\Delta : g \mapsto g \otimes g$)
- Symmetrization: projector onto trivial irrep:

$$\Lambda = \frac{1}{|G|} \sum_g g$$

- $\Delta(\Lambda)$ is symmetric:

$$(1 \otimes h) \cdot \Delta(\Lambda) = \frac{1}{|G|} \sum_g g \otimes hg = \frac{1}{|G|} \sum_g h^{-1} g \otimes g = (S(h) \otimes 1) \cdot \Delta(\Lambda),$$

  with $S(h) = h^{-1}$ for $h \in G$.
- ϕ representation on $V \Rightarrow \phi^T \circ S$ is representation on V^*.

Semisimple Hopf algebras

Bialgebra + $\exists\Lambda$ non-degenerate s.t.

$$\Delta(\Lambda) = \Delta_{op}(\Lambda)$$
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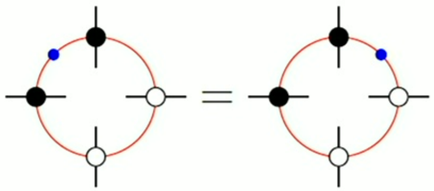

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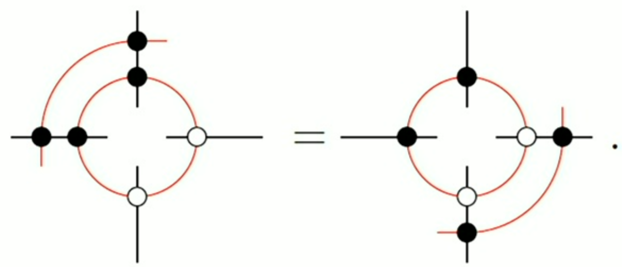

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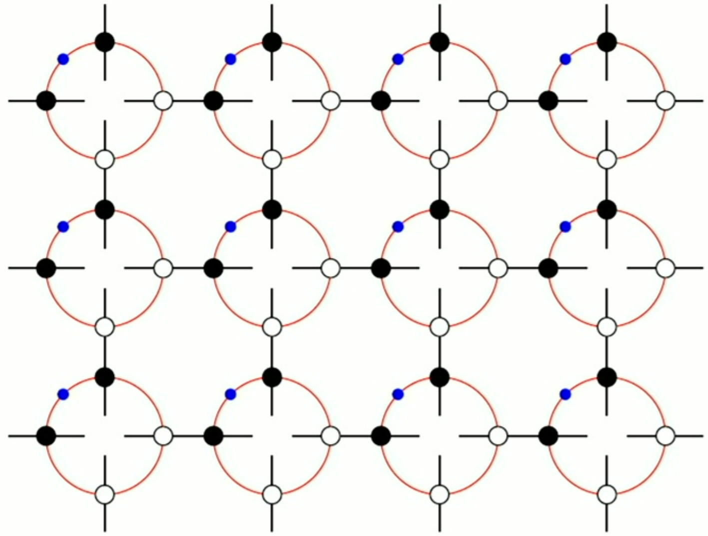
Compatibility of plaquettes

Hopf-injective PEPS



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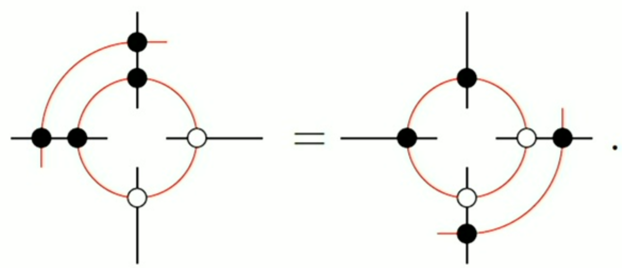

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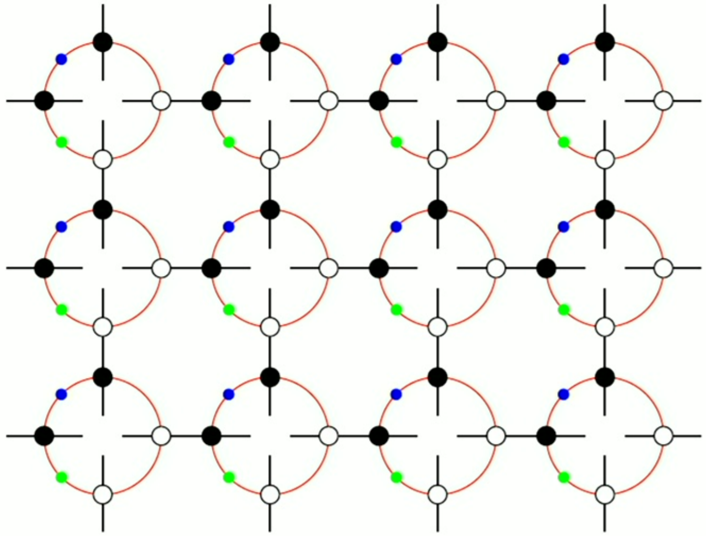
Pivotal non-semisimple Hopf algebras (e.g. Taft): same

Weak Hopf-injective PEPS



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Taft-Hopf-injective PEPS

- The PEPS is non-zero on finite region, torus
- The PEPS is zero on a sphere
- The GS of the parent Hamiltonian is the expected one
- Non-zero correlation length
- Nicer Hamiltonian? Excitations?
- Connection to non-semisimple TQFTs?

Conclusion

- Tensor networks can describe top. order
- MPO symmetries
- MPO from fusion categories
- Alternative formulation: weak Hopf algebra