

Title: Weak Hopf symmetric tensor networks

Speakers: Andras Molnar

Collection: Higher Categorical Tools for Quantum Phases of Matter

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# Weak Hopf symmetric tensor networks

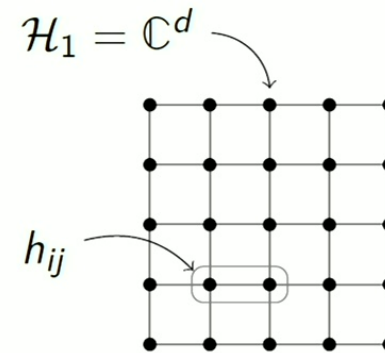
arXiv:2204.05940

Andras Molnar, José Garre Rubio, Alberto Ruiz-de-Alarcón,  
Norbert Schuch, David Pérez-García, Ignacio Cirac



# Many-body problem

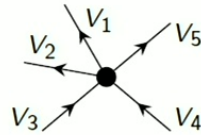
- ▶  $\mathcal{H}_{tot} = \mathcal{H}_1^{\otimes N}$ ,  $H = \sum_{\langle ij \rangle} h_{ij}$
- ▶ Goal: find the GS of  $H$
- ▶ Variational ansatz
  - ▶ Few parameters only
  - ▶ Can calculate  $\langle O_{local} \rangle$
  - ▶ Approximates GS well
- ▶ Tensor Networks: such ansätze



[Cirac:2011.12127]

# Graphical notation of tensor calculus

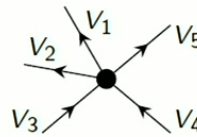
► Tensors



$$A : V_3 \otimes V_4 \rightarrow V_2 \otimes V_1 \otimes V_5$$

# Graphical notation of tensor calculus

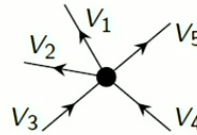
► Tensors



$$A \in V_1 \otimes V_2 \otimes V_3^* \otimes V_4^* \otimes V_5$$

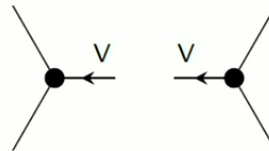
# Graphical notation of tensor calculus

► Tensors



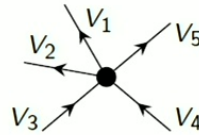
$$A \in V_1 \otimes V_2 \otimes V_3^* \otimes V_4^* \otimes V_5$$
$$A[i, j, k, l, m]$$

► Contraction:  $f \otimes v \mapsto f(v)$



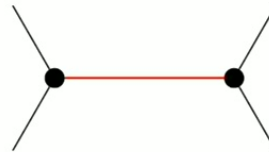
# Graphical notation of tensor calculus

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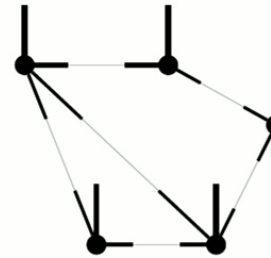
► Contraction:  $f \otimes v \mapsto f(v)$



$$C_{21}(A \otimes B) = \sum_j A_{ijk} \cdot B_{jmn}$$

# Tensor Networks

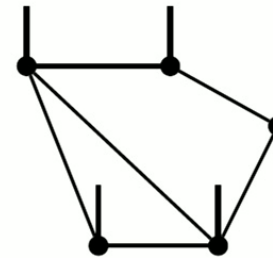
- ▶  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- ▶  $\text{rank}(A_v) = \text{deg}(v) + 1$ 
  - ▶ physical index:  $\mathcal{H}_v$
  - ▶ virtual indices:  $(v, w) \in \mathcal{E}$



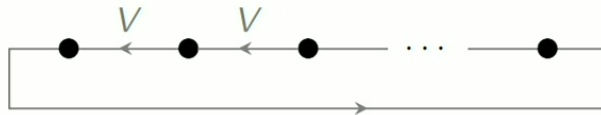


# Tensor Networks

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- ▶  $\text{rank}(A_v) = \text{deg}(v) + 1$ 
  - ▶ physical index:  $\mathcal{H}_v$
  - ▶ virtual indices:  $(v, w) \in \mathcal{E}$
- ▶  $|\Psi\rangle \in \bigotimes_v \mathcal{H}_v$ ,  
 $|\Psi\rangle = \bigotimes_{e \in \mathcal{E}} \mathcal{C}_e \left( \bigotimes_{v \in \mathcal{V}} A_v \right)$

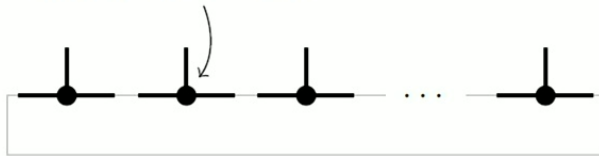


# Matrix product states and operators



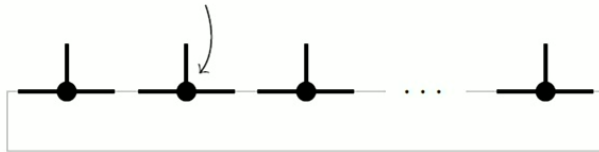
# Matrix product states and operators

$$A \in V \otimes V^* \otimes \mathcal{H}$$

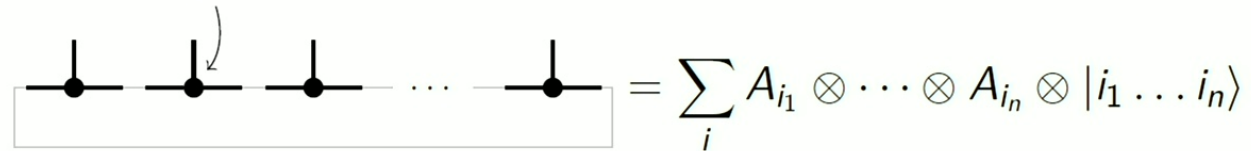


# Matrix product states and operators

$$A \in \text{End}(V) \otimes \mathcal{H}$$



## Matrix product states and operators

$$A = \sum_i A_i \otimes |i\rangle$$

$$= \sum_i A_{i_1} \otimes \dots \otimes A_{i_n} \otimes |i_1 \dots i_n\rangle$$

# Matrix product states and operators

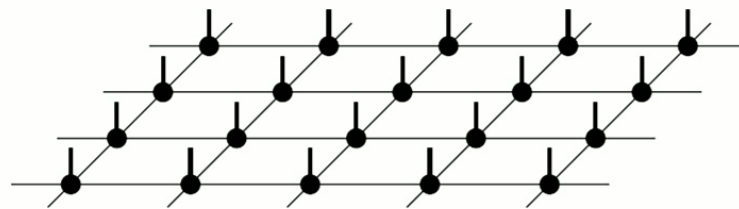
$$A = \sum_i A_i \otimes |i\rangle$$

$$= \sum_i \text{Tr} (A_{i_1} \dots A_{i_n}) |i_1 \dots i_n\rangle$$

$$A = \sum_{ij} A_{ij} \otimes |i\rangle\langle j|$$

$$= \sum_{ij} \text{Tr} (A_{i_1 j_1} \dots A_{i_n j_n}) |i_1 \dots i_n\rangle \langle j_1 \dots j_n|$$

# PEPS: 2D Tensor networks

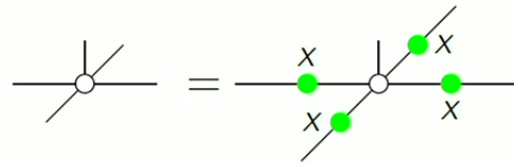


PEPS can describe topological order



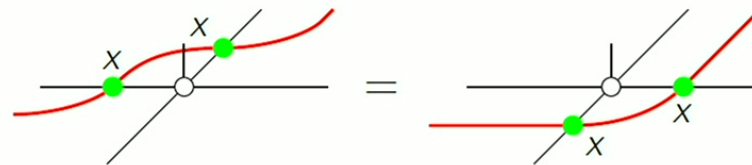
# Topological PEPS

- ▶ PEPS can describe string-nets [Buerschaper:0809.2393]
- ▶ It allows perturbation from RFP, phase transitions
- ▶ Origin of topological properties: symmetries [Sahinoglu:11409.2150]
- ▶ Symmetry is purely virtual, “size-independent”: MPO



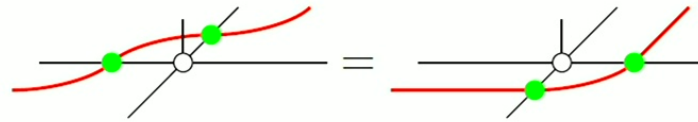
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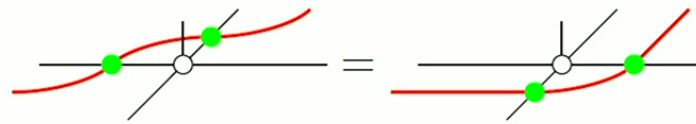
## Size-independence of the symmetry

MPO symmetry of the PEPS tensor:

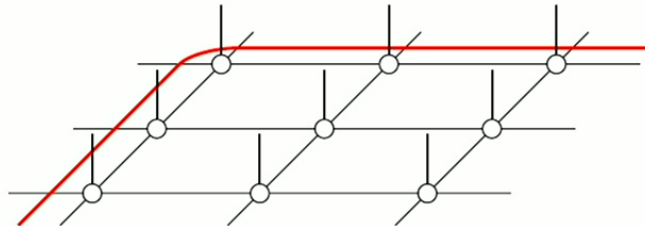


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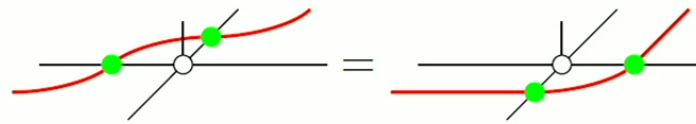


Symmetry of large area:

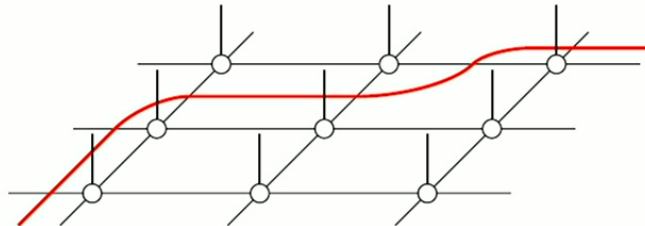


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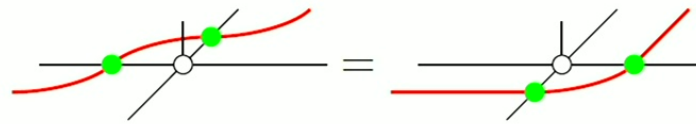


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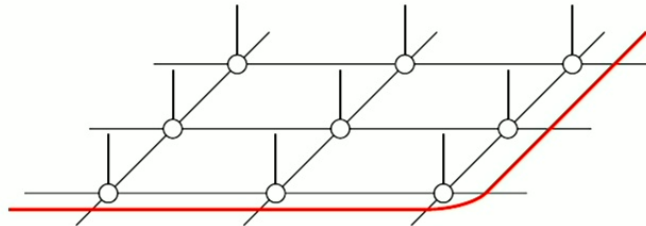


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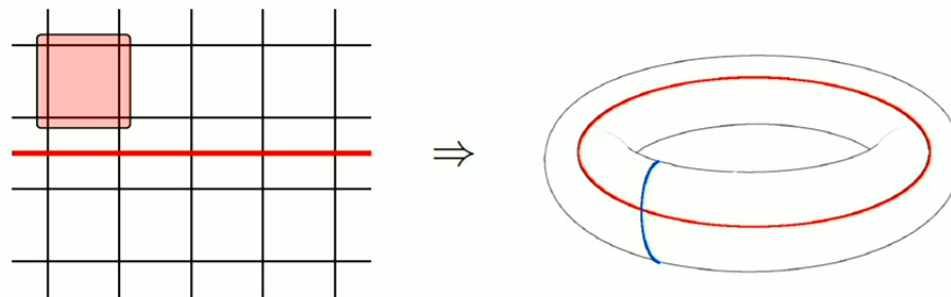


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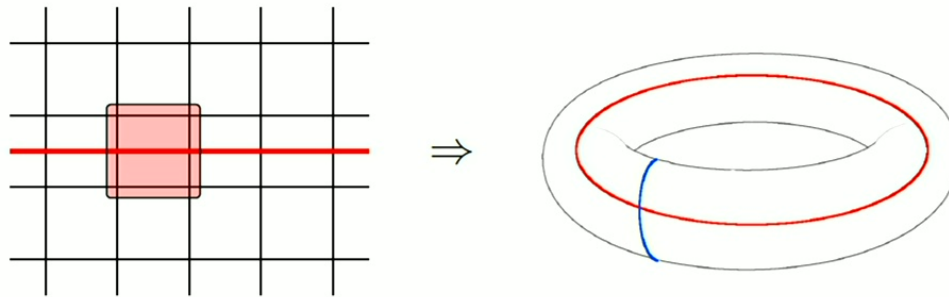
# Topological ground space degeneracy

A local Hamiltonian does not detect the symmetry operator:



# Topological ground space degeneracy

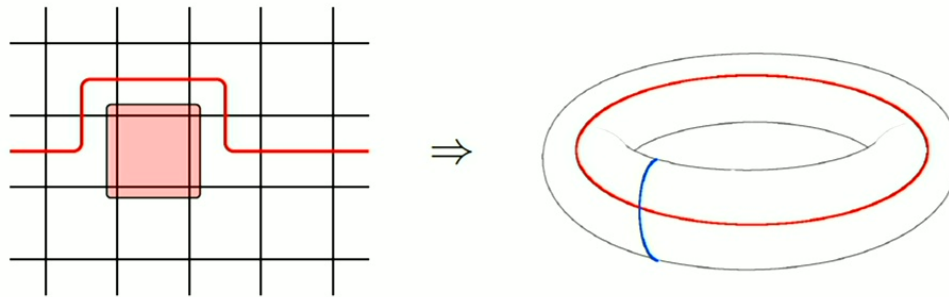
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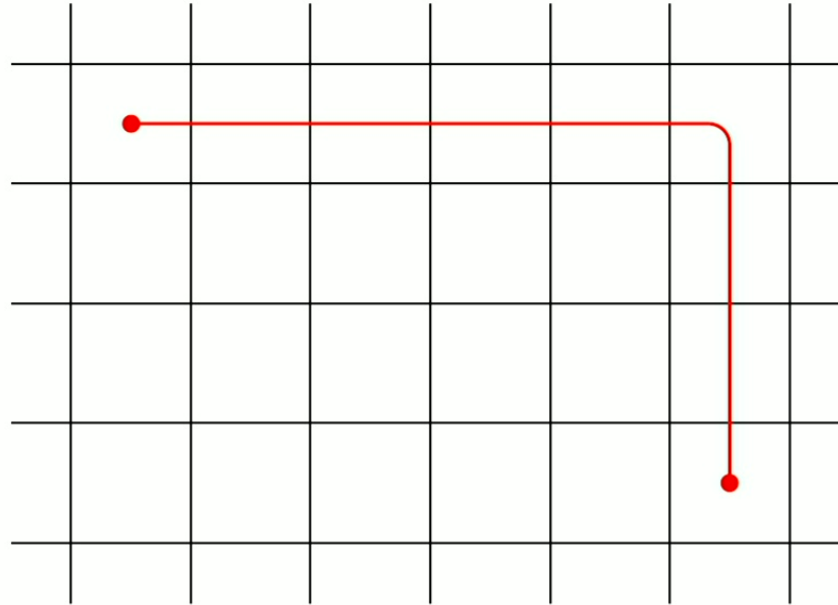


# Topological ground space degeneracy

A local Hamiltonian does not detect the symmetry operator:

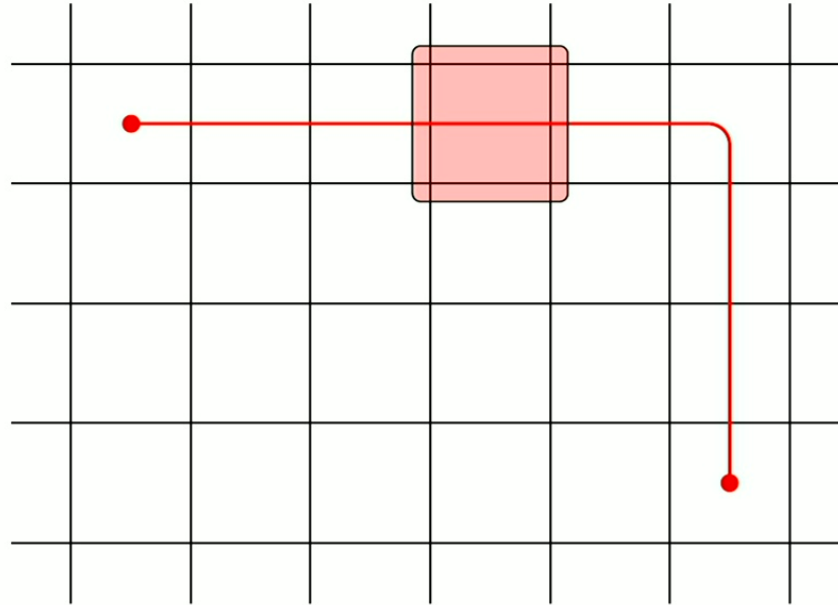


## Excitations – example



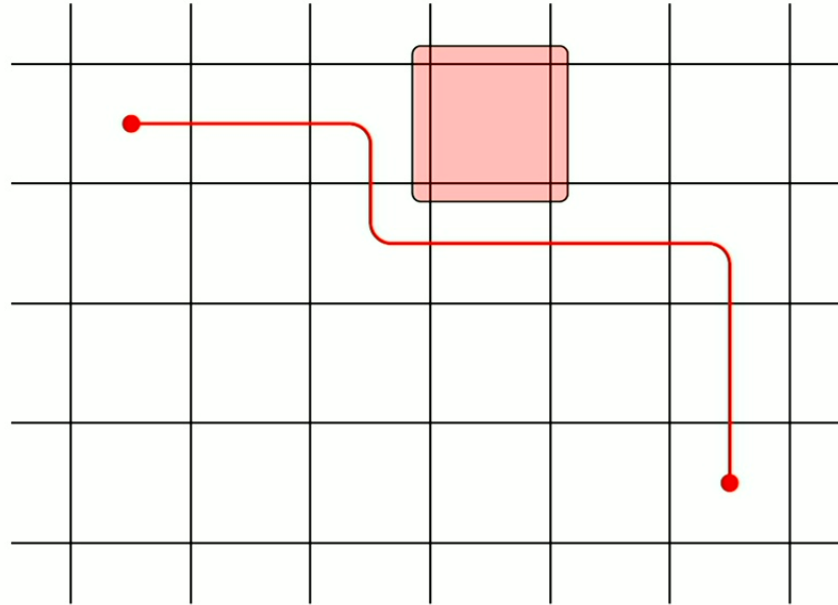
[Bultinck:1511.08090]

## Excitations – example



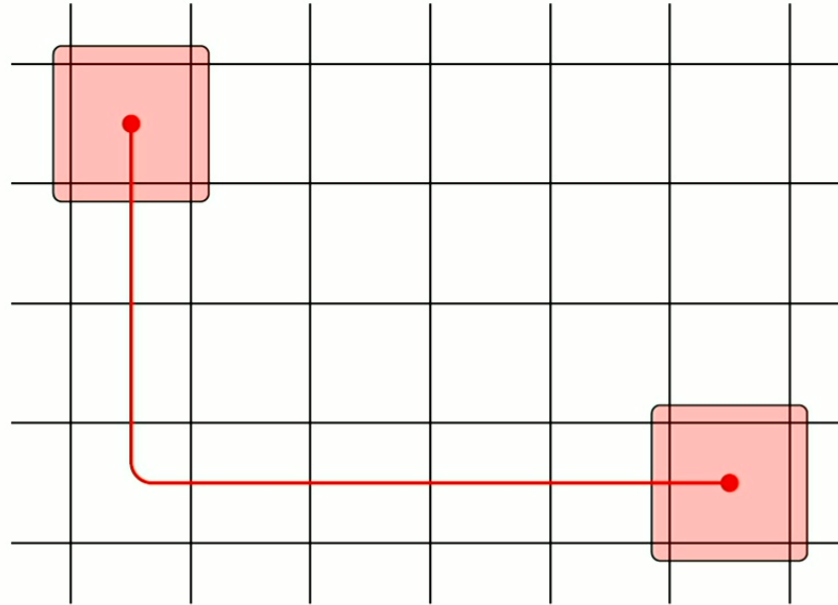
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## Excitations – example



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## Excitations – example



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## The MPO symmetries

Toric code: 2 MPOs,

$$\begin{array}{c} | \\ | \\ \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \\ | \\ | \end{array} \equiv \begin{cases} 1 \otimes 1 \otimes \dots \otimes 1 \\ X \otimes X \otimes \dots \otimes X \end{cases}$$

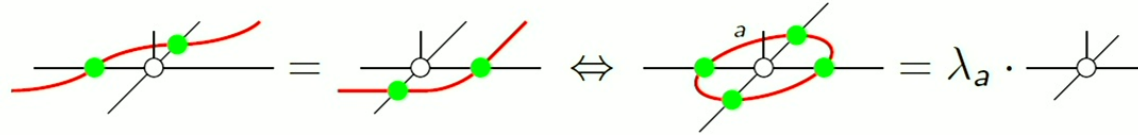
G-injective PEPS: MPOs = elements of  $G$

$$\begin{array}{c} | \\ | \\ \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \\ | \\ | \end{array} \equiv g \otimes g \otimes \dots \otimes g, \quad g \in G$$

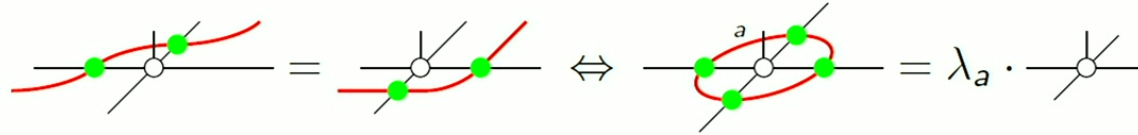
String-net models: MPO = simple objects of fusion category  $\mathcal{C}$

$$\begin{array}{c} | \\ | \\ \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \\ | \\ | \end{array} \equiv a \in \text{Obj}(\mathcal{C})$$

# Algebra of symmetries



## Algebra of symmetries

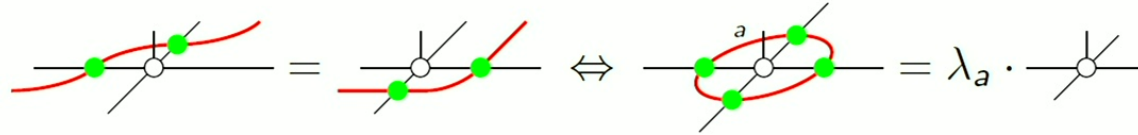


Linear combination and product of MPOs is symmetry:

$$(O_a + O_b)|T\rangle = (\lambda_a + \lambda_b)|T\rangle \quad \text{and} \quad O_a \cdot O_b|T\rangle = \lambda_a \cdot \lambda_b|T\rangle$$



## Algebra of symmetries



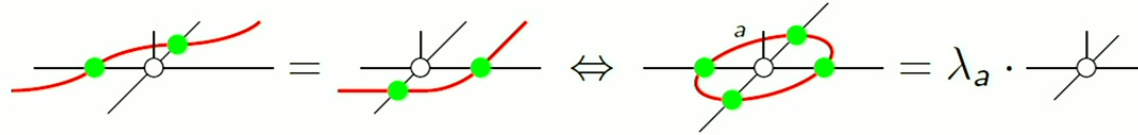
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Algebra of symmetries:

$$\mathcal{A}_{PBC} \equiv \left\{ \begin{array}{c} \lambda \\ \boxed{\bullet \quad \bullet \quad \dots \quad \bullet} \end{array} \middle| \lambda = \bigoplus_a \lambda_a \cdot \mathbb{1}_{D_a} \right\}$$

## Algebra of symmetries



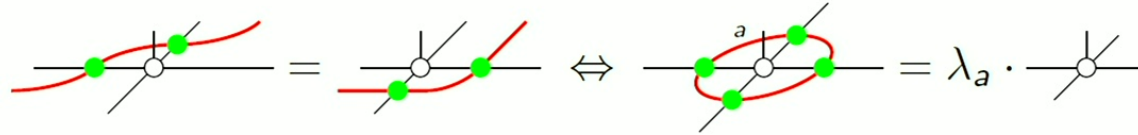
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Algebra of symmetries:

$$\mathcal{A}_{PBC} \equiv \left\{ \sum_a \lambda_a \cdot \boxed{\text{---} \overset{a}{\bullet} \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---}} \mid \lambda_a \in \mathbb{C} \right\}$$

## Algebra of symmetries



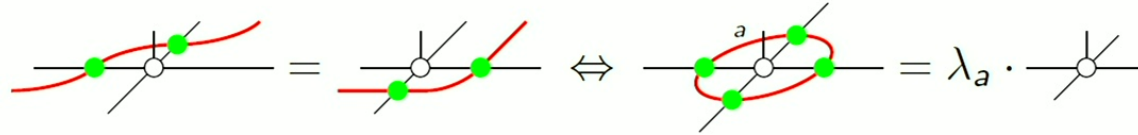
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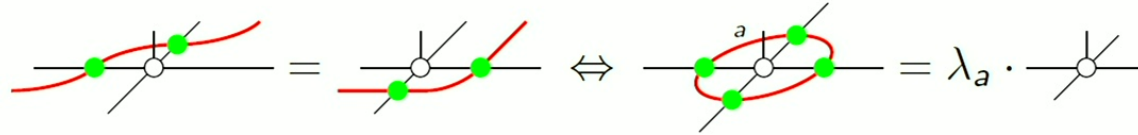
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The product for **groups**:

$$O_g \cdot O_h = O_{gh}$$

## Algebra of symmetries



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The product for **fusion categories**:

$$O_a \cdot O_b = \sum_c N_{ab}^c O_c, \quad N_{ab}^c \in \mathbb{Z}^+$$

# PEPS from MPO symmetries

Special element  $O \in \mathcal{A}_{PBC}$ :

$$O_a \cdot O = d_a O \quad \text{or} \quad \text{Diagram 1} = d_a \cdot \text{Diagram 2}$$

Then a symmetric PEPS tensor is:

$$\text{Diagram 3} \equiv \text{Diagram 4}$$

[Bultinck:1511.08090, Lootens:2008.11187]

# Weak Hopf algebras from MPO symmetries

## Our perspective

Open boundary MPOs are also symmetries:

$$\begin{array}{c} B \\ \bullet \\ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \end{array} \cong \lambda \cdot \begin{array}{c} \text{---} \bullet \text{---} \end{array} \quad B \in \bigoplus_a \mathcal{M}(D_a)$$

Open boundary MPOs form an algebra:  $\forall B, C \exists D$  s.t.

$$\begin{array}{c} C \\ \bullet \\ \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \\ B \\ \bullet \\ \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \end{array} = \begin{array}{c} D \\ \bullet \\ \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \end{array}$$

The algebra of symmetries:

$$\mathcal{A}_{OBC} \equiv \left\{ \begin{array}{c} B \\ \bullet \\ \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \end{array} \mid B \in \bigoplus_a \mathcal{M}(D_a) \right\}$$



# Weak Hopf algebra

► Injective tensors:

$$\mathcal{A} \equiv \left\{ \begin{array}{c} \text{Diagram: A red rectangle with a top horizontal line, a bottom horizontal line, and vertical lines on the left and right. The top line has a black dot labeled 'B' on the left and several green dots on the right. The bottom line is solid red. Ellipses '...' are between the second and third green dots.} \\ \left| B \in \bigoplus_a \mathcal{M}(D_a) \right. \end{array} \right\}$$

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# Weak Hopf algebra

- ▶ Injective tensors:

$$\mathcal{A} \equiv \left\{ \begin{array}{c} B \\ \boxed{\quad \bullet \quad \color{green}\bullet \quad} \\ \end{array} \middle| B \in \bigoplus_a \mathcal{M}(D_a) \right\}$$

- ▶ Growing of MPO:  $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$  "coproduct":

$$\Delta : \begin{array}{c} B \\ \boxed{\quad \bullet \quad \color{green}\bullet \quad} \\ \end{array} \mapsto \begin{array}{c} B \\ \boxed{\quad \bullet \quad \color{green}\bullet \quad \color{green}\bullet \quad} \\ \end{array} \in \mathcal{A} \otimes \mathcal{A}$$

# Weak Hopf algebra

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- ▶  $(\mathcal{A}, \Delta)$  + additional properties = (weak) Hopf algebra

[Bohm:math/9805116, Montgomery: Rep Theory of Semisimple Hopf]

# Weak Hopf algebra

- ▶ Injective tensors:

$$\mathcal{A} \equiv \left\{ \begin{array}{c} \text{Diagram: a red rectangle with a black dot labeled } B \text{ and a green dot on the top edge} \\ \left| B \in \bigoplus_a \mathcal{M}(D_a) \right. \end{array} \right\}$$

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- ▶  $(\mathcal{A}, \Delta)$  + additional properties = (weak) Hopf algebra

[Bohm:math/9805116, Montgomery: Rep Theory of Semisimple Hopf]

- ▶ Fusion category  $\equiv$  Weak Hopf algebras

[Etingof:math/0203060, Kitaev:1104.5047]

## Use of algebraic formulation

- ▶ Transfer operator of topological PEPS: renormalization fixed point MPDO  
[Ruiz-de-Alarcón:2204.06295]
- ▶ Phase classification of RFP MPDO  
[Ruiz-de-Alarcón:2204.06295]
- ▶ Characterization of symmetries in topologically ordered PEPS  
[Molnar: in preparation]
- ▶ Other possibly interesting states with “topological” properties

# MPO symmetries from algebras

## MPOs from coproducts

$\mathcal{A}$  algebra with  $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$  s.t.

- ▶  $(\text{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{Id}) \circ \Delta$
- ▶  $\Delta(xy) = \Delta(x) \cdot \Delta(y)$



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Let  $\mathcal{A}^*$  be set of linear functionals on  $\mathcal{A}$ . Define  $f \star g \in \mathcal{A}^*$  by

$$(f \star g)(x) = (f \otimes g) \circ \Delta(x)$$

## MPOs from coproducts

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$$(f \star g)(x) = (f \otimes g) \circ \Delta(x)$$

Then  $\star$  is a product:

$$(f \star (g \star h))(x) = ((f \star g) \star h)(x)$$

$\mathcal{A}^*$  is an algebra.

## MPOs from coproducts

Fix a representation of  $\mathcal{A}^*$  and of  $\mathcal{A}$ :

$$\psi : f \mapsto \begin{array}{c} f \\ \leftarrow \bullet \leftarrow \end{array} \quad \phi : x \mapsto \begin{array}{c} \uparrow \\ x \bullet \\ \downarrow \end{array}$$

If the representation is injective, then  $\forall x \in \mathcal{A} \exists m(x)$  s.t.

$$f(x) = \text{Tr} \left( m(x) \cdot \psi(f) \right) = \begin{array}{c} m(x) \quad f \\ \bullet \quad \bullet \\ \boxed{\phantom{m(x) \cdot \psi(f)}} \end{array}$$

Fix a basis of  $\mathcal{A}$ ,  $\{x_i\}$ , and dual basis  $\{f_i\}$ , and let

$$\begin{array}{c} \uparrow \\ \leftarrow \bullet \leftarrow \\ \downarrow \end{array} = \sum_i \begin{array}{c} f_i \\ \leftarrow \bullet \leftarrow \end{array} \otimes \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array} x_i .$$

By construction,

$$\phi(x) = \begin{array}{c} m(x) \quad \uparrow \\ \bullet \quad \bullet \\ \boxed{\phantom{m(x) \cdot \psi(f)}} \end{array}$$

# MPOs from coproducts

$$\begin{array}{c} m(y) \\ \bullet \\ \hline \uparrow \quad \uparrow \\ \bullet \quad \bullet \\ \hline \uparrow \quad \uparrow \end{array} = \sum_{ij} \begin{array}{c} m(y) \quad f_i \quad f_j \\ \bullet \quad \bullet \quad \bullet \\ \hline \uparrow \quad \uparrow \quad \uparrow \\ \bullet \quad \bullet \quad \bullet \\ \hline \uparrow \quad \uparrow \quad \uparrow \end{array} \cdot \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \end{array} x_i \otimes \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \end{array} x_i$$

## MPOs from coproducts

$$\begin{aligned}
 \text{Diagram 1} &= \sum_{ij} \text{Diagram 2} \cdot \text{Diagram 3} \\
 &= \sum_{ij} (f_i \star f_j)(y) \cdot \phi(x_i) \otimes \phi(x_j) \\
 &= \sum_{ij} (f_i \otimes f_j) \circ \Delta(y) \cdot \phi(x_i) \otimes \phi(x_j)
 \end{aligned}$$

The diagrammatic equation shows the decomposition of a multiplication map  $m(y)$  into a sum over comultiplication  $\Delta(y)$ . 
   
 - The first diagram shows a red box containing a dot labeled  $m(y)$  followed by two dots, each with a vertical double-headed arrow pointing up and down.
   
 - The second diagram shows a red box containing a dot labeled  $m(y)$ , followed by two dots labeled  $f_i$  and  $f_j$ , each with a vertical double-headed arrow pointing up and down. To the right of this box are two separate vertical double-headed arrows, each labeled  $x_i$ .
   
 - The third diagram is the algebraic expression  $(f_i \star f_j)(y) \cdot \phi(x_i) \otimes \phi(x_j)$ .
   
 - The fourth diagram is the algebraic expression  $(f_i \otimes f_j) \circ \Delta(y) \cdot \phi(x_i) \otimes \phi(x_j)$ .

## MPOs from coproducts

$$\begin{aligned}
 \text{Diagram 1} &= \sum_{ij} \text{Diagram 2} \cdot \text{Diagram 3} \otimes \text{Diagram 4} \\
 &= \sum_{ij} (f_i \star f_j)(y) \cdot \phi(x_i) \otimes \phi(x_j) \\
 &= \sum_{ij} (f_i \otimes f_j) \circ \Delta(y) \cdot \phi(x_i) \otimes \phi(x_j) \\
 &= (\phi \otimes \phi) \circ \Delta(y) = (\phi \boxtimes \phi)(y)
 \end{aligned}$$

Similarly:

$$\text{Diagram 5} = \phi^{\otimes n} \circ \Delta^n(x)$$

## Example: $\mathbb{C}[G]$

Basis of  $\mathbb{C}[G]$ :  $\{g\}_{g \in G}$ . Dual basis in  $(\mathbb{C}[G])^*$ :  $\{\delta_g\}_{g \in G}$

$$\delta_g(h) = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{else} \end{cases}$$

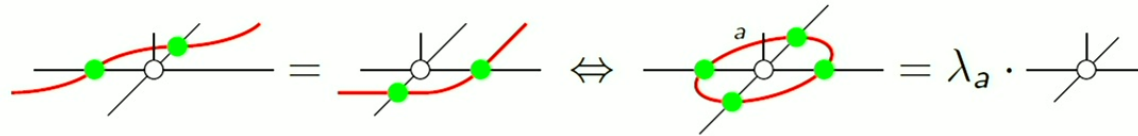
Product in  $(\mathbb{C}[G])^*$ :

$$\delta_g \star \delta_h = \begin{cases} \delta_g & \text{if } g = h \\ 0 & \text{else} \end{cases}$$

All irreps are 1D; left regular:  $g \mapsto |g\rangle\langle g|$

$$\begin{array}{c} \updownarrow \\ \bullet \\ \leftarrow \bullet \leftarrow \\ \updownarrow \end{array} = \sum_g \begin{array}{c} \leftarrow |g\rangle \\ \bullet \\ \updownarrow \\ g \langle g| \leftarrow \end{array} \Rightarrow \begin{array}{c} m(x) \\ \bullet \\ \updownarrow \\ \bullet \cdots \bullet \\ \updownarrow \quad \updownarrow \\ \boxed{\phantom{\bullet \cdots \bullet}} \end{array} = \sum_g x_g \cdot g \otimes \cdots \otimes g$$

## WHA-symmetric PEPS





## Example: G-injective PEPS

$$\begin{array}{c} \text{---} \circ \text{---} \\ \nearrow \quad \searrow \\ \text{---} \end{array} \equiv \frac{1}{|G|} \sum_g \begin{array}{c} g \bullet \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ g^{-1} \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \bullet \end{array} \quad \phi(g^{-1}) = (\bar{\phi}(g))^T
 \end{array}$$

# Example: G-injective PEPS

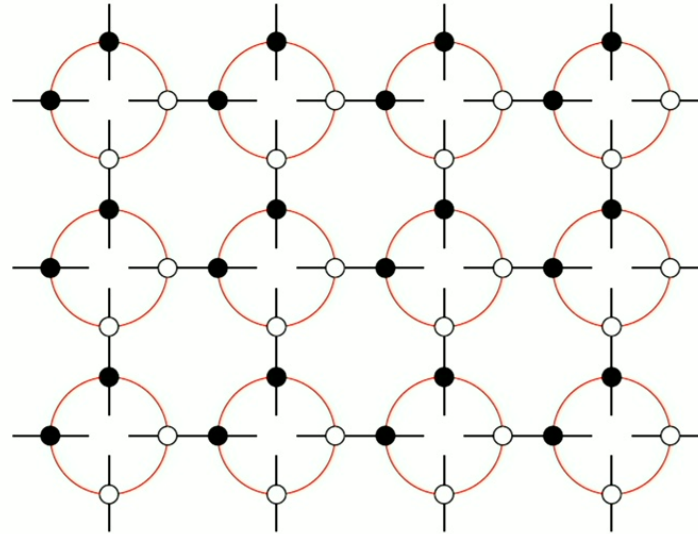
$$\begin{array}{c} \leftarrow \\ \circ \\ \rightarrow \end{array} \equiv \frac{1}{|G|} \sum_g \begin{array}{c} g \\ \bullet \\ \hline \circ \\ \hline g^{-1} \\ \circ \\ \hline g^{-1} \end{array} = \begin{array}{c} \bullet \\ \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \bullet \end{array} \quad \phi(g^{-1}) = (\bar{\phi}(g))^T$$

Then the pulling-through equation is:

$$\begin{array}{c} h \\ \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \end{array} \equiv \begin{array}{c} \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \end{array} \quad .$$

$$\sum_g hg \otimes g^{-1} = \sum_g g \otimes g^{-1}h$$

## G-injective PEPS



- ▶ Global symmetry
- ▶ Regrouping DoF: Kitaev model

## Algebraic structure in $\mathbb{C}[G]$

- ▶  $\mathbb{C}[G]$  is algebra and coalgebra ( $\Delta : g \mapsto g \otimes g$ )
- ▶ Symmetrization: projector onto trivial irrep:

$$\Lambda = \frac{1}{|G|} \sum_g g$$

- ▶  $\Delta(\Lambda)$  is symmetric:

$$(1 \otimes h) \cdot \Delta(\Lambda) = \frac{1}{|G|} \sum_g g \otimes hg = \frac{1}{|G|} \sum_g h^{-1} g \otimes g = (S(h) \otimes 1) \cdot \Delta(\Lambda),$$

with  $S(h) = h^{-1}$  for  $h \in G$ .

- ▶  $\phi$  representation on  $V \Rightarrow \phi^T \circ S$  is representation on  $V^*$ .

## Semisimple Hopf algebras

Bialgebra +  $\exists \Lambda$  non-degenerate s.t.

$$\begin{aligned}\Delta(\Lambda) &= \Delta_{op}(\Lambda) \\ (1 \otimes x) \cdot \Delta(\Lambda) &= (S(x) \otimes 1) \cdot \Delta(\Lambda) \\ \Delta \circ S &= (S \otimes S) \circ \Delta_{op}\end{aligned}$$

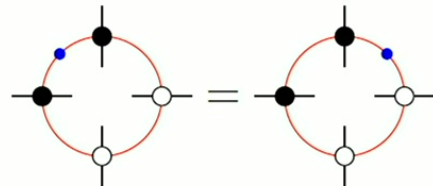
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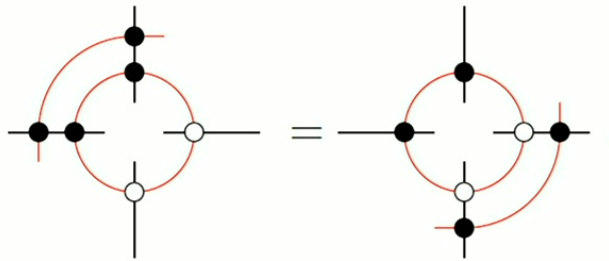
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## Semisimple Hopf algebras

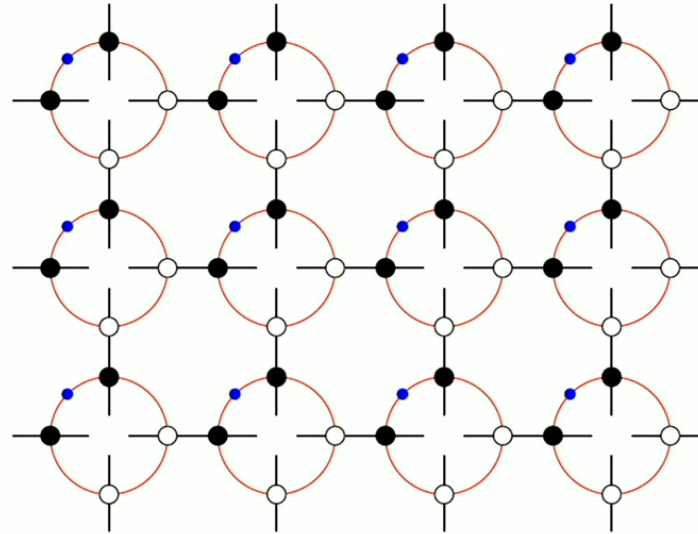
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Compatibility of plaquettes



## Hopf-injective PEPS



- ▶ Global symmetry
- ▶ Regrouping DoF: Kitaev model

## Weak Hopf algebras

Bialgebra +  $\exists \Lambda \in \mathcal{A}$  non-degenerate s.t.

$$\Delta(\Lambda) = \Delta_{op}(\Lambda)$$

$$(1 \otimes x) \cdot \Delta(\Lambda) = (\hat{S}(x) \otimes 1) \cdot \Delta(\Lambda)$$

$$\Delta \circ \hat{S} = (\hat{S} \otimes g \otimes \hat{S}) \circ \Delta_{op}$$

$$\Delta_{\mathcal{A}^*}(g) = \Delta_{\mathcal{A}^*}(\epsilon) \cdot (g \otimes g) = (g \otimes g) \cdot \Delta_{\mathcal{A}^*}(\epsilon)$$

# Weak Hopf algebras

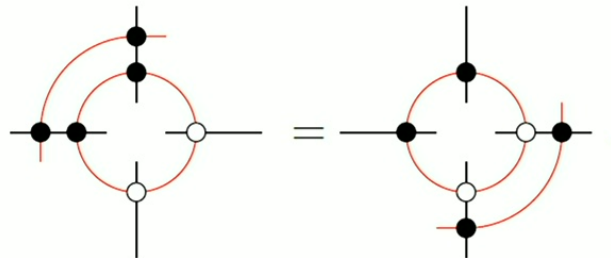
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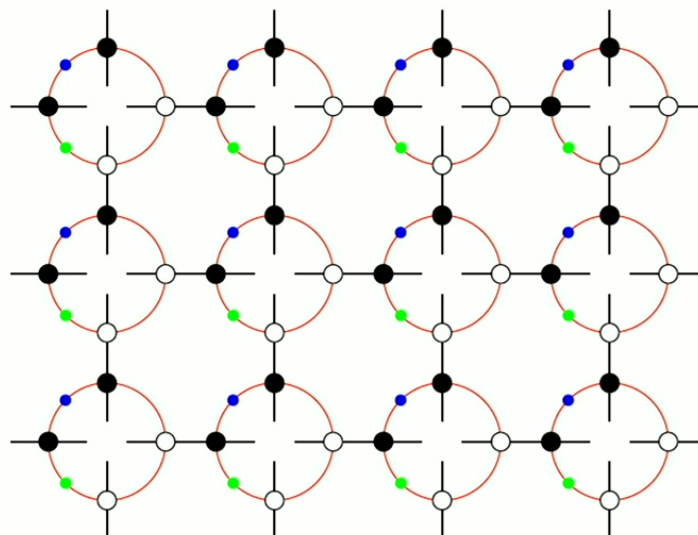
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Pivotal non-semisimple Hopf algebras (e.g. Taft): same

## Weak Hopf-injective PEPS



- ▶ Global symmetry
- ▶ Regrouping DoF: Kitaev model

## Taft-Hopf-injective PEPS

- ▶ The PEPS is non-zero on finite region, torus
- ▶ The PEPS is zero on a sphere
- ▶ The GS of the parent Hamiltonian is the expected one
- ▶ Non-zero correlation length
- ▶ Nicer Hamiltonian? Excitations?
- ▶ Connection to non-semisimple TQFTs?

## Conclusion

- ▶ Tensor networks can describe top. order
- ▶ MPO symmetries
- ▶ MPO from fusion categories
- ▶ Alternative formulation: weak Hopf algebra