Title: Douglas-Reutter 4d TQFT as a generalised orbifold

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Collection: Higher Categorical Tools for Quantum Phases of Matter

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Abstract: The state-sum invariants of 4d manifolds obtained from spherical fusion 2-categories due to Douglas-Reutter offer an exciting entrypoint to the study of 4d TQFTs.
In this talk we will argue that these invariants arise from a TQFT, obtained by filling the trivial 4d TQFT with a defect foam.
Such construction is known as a generalised orbifold, the Turaev-Viro-Barrett-Westbury (i.e. 3d state-sum) models are also known to arise in this way from the defects in the trivial 3d TQFT (a result by Carqueville-Runkel-Schaumann).
Advantages of this point of view offer e.g. realisations of state-spaces, examples of domain walls and commuting-projector realisations of (3+1)-dimensional topological phases.
Based on a joint project with Nils Carqueville and Lukas Müller.

Douglas–Reutter TQFT as generalised orbifold

Vincentas Mulevičius (Vilnius University)
joint work in progress with Nils Carqueville (Vienna) and Lukas Müller (PI)

Higher Categorical Tools for Quantum Phases of Matter
Perimeter Institute for Theoretical Physics
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Plan

1. Defect TQFTs
2. Generalised orbifolds
3. State-sum constructions (for dim $\leqslant 4$, 4d = Douglas–Reutter'18)
4. Some implications

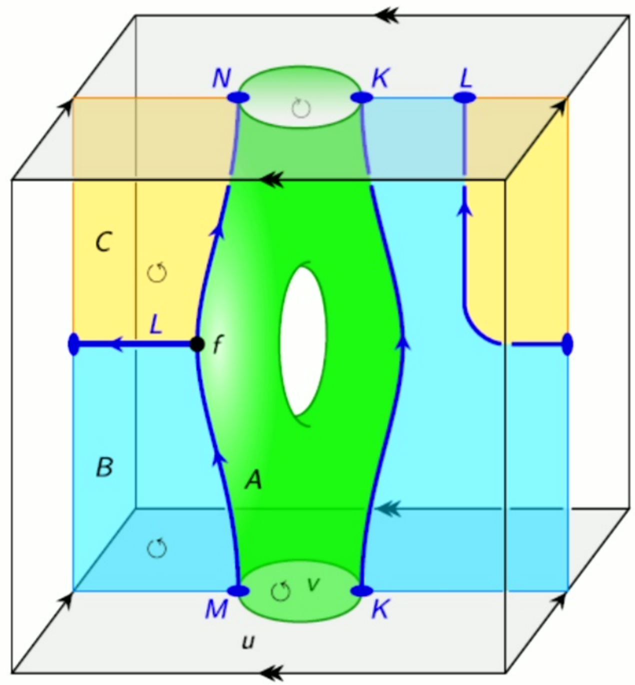
Defect TQFTs

[Davydov–Kong–Runkel'11]
[Carqueville–Runkel–Schaumann'17]

... are symmetric monoidal functors

$$Z\colon\ \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}_{\Bbbk}\,,$$

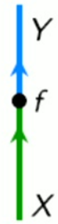
where $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ is the category of *defect bordisms*
- stratified with labelled strata:

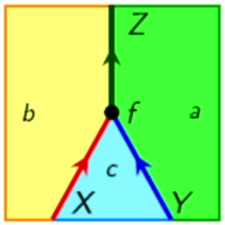


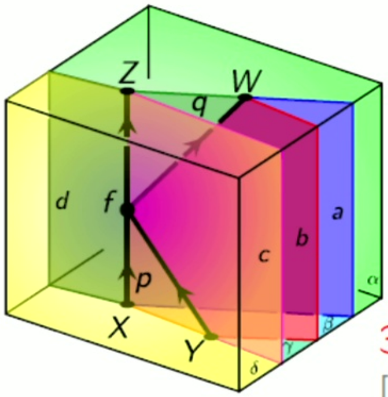
Defect data

The defect datum $\mathbb{D}$ of the category $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ encodes the labelling convention label sets $D_n, D_{n-1}, \ldots, D_0$ for n-, $(n-1)$-, 0-strata and their adjacency.

An n-category $\mathcal{X}$ gives an n-dim. defect datum $\mathbb{D}^{\mathcal{X}}$, $D_k^{\mathcal{X}} = \{(n-k)\text{-morphisms}\}$:

- $n = 1$: 

, $X, Y \in D_1^{\mathcal{X}}$, $f \in D_0^{\mathcal{X}}$
- $n = 2$: 

, $a, b, c \in D_2^{\mathcal{X}}$, $X, Y, Z \in D_1^{\mathcal{X}}$, $f \in D_0^{\mathcal{X}}$
- $n = 3$: 

$\alpha, \beta, \gamma, \delta \in D_3^{\mathcal{X}}$, $a, b, c, d, p, q \in D_2^{\mathcal{X}}$, $X, Y, Z, W \in D_1^{\mathcal{X}}$, $f \in D_0^{\mathcal{X}}$

3d graphical calculus!
[Barrett–Meusburger–Schaumann'12]

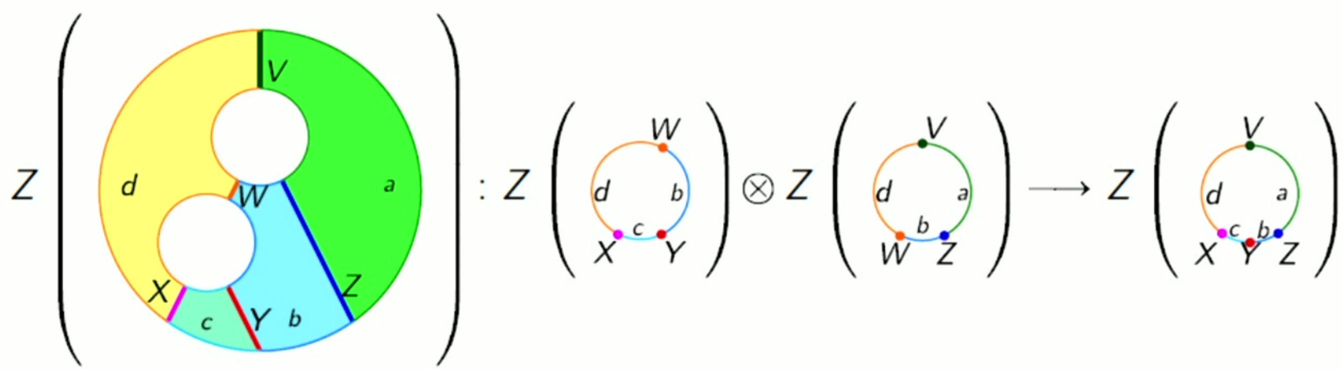
Associated higher categories

[Kapustin'10]
[Carqueville–Meusburger–Schaumann'17]

From a defect TQFT $Z\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ one obtains a (maximally strict) n-category $\mathcal{D}^Z$,

- objects – bulk theories (n-dim. defects)
- 1-morphisms – domain walls ($(n-1)$-dim. defects)
- ...
- n-morphisms – obtained from evaluating stratified $(n-1)$-spheres.

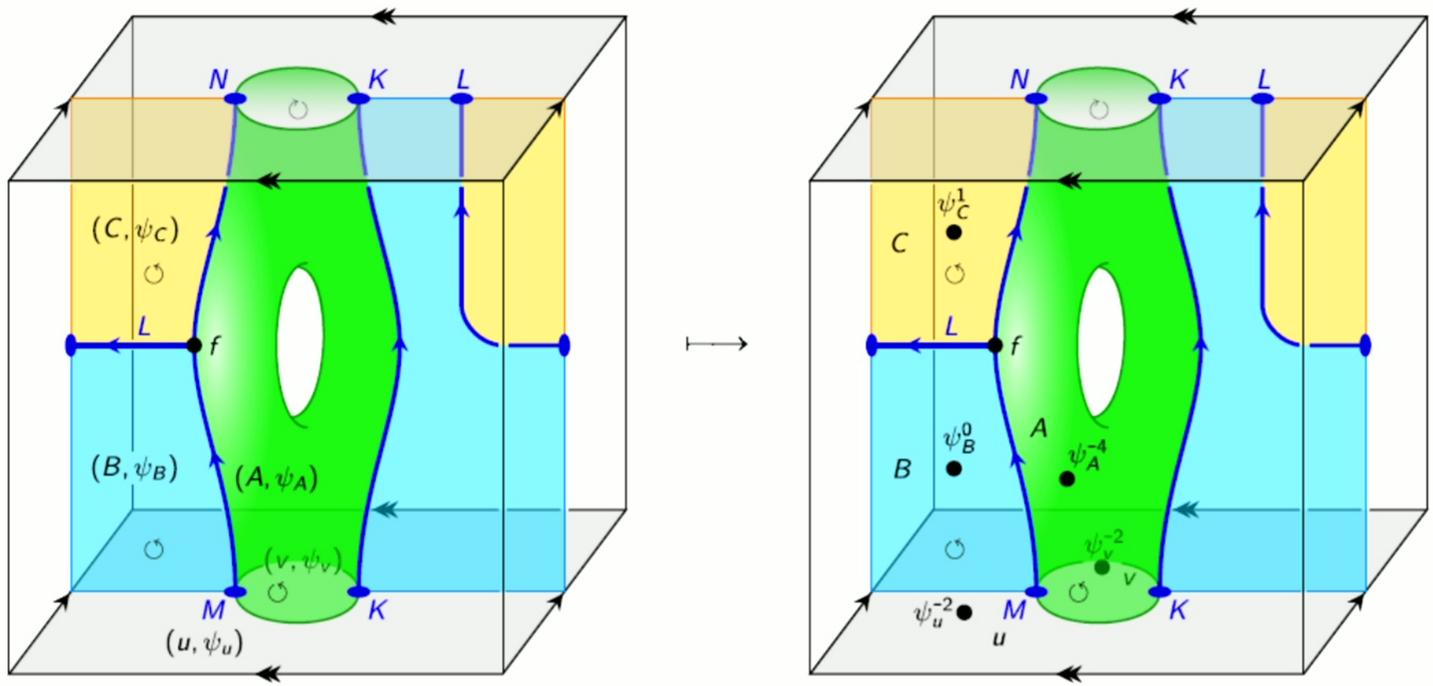
E.g. composition of 2-morphisms in 2d:



For $\mathbb{D}^{\mathcal{X}}$ the n-category $\mathcal{D}^Z$ need not be $\mathcal{X}$ - take constant trivial defect TQFT

Euler completions of defect data

Having a defect TQFT $Z\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$,
can get a new defect datum $\mathbb{D}^{\mathrm{eu}}$, where $D_k^{\mathrm{eu}} \subseteq D_k \times D_0$, $k \geqslant 2$
and one has a functor $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\mathrm{eu}}) \longrightarrow \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$, e.g. in 3d:



i.e. each k-stratum s receives $\underbrace{\chi_{\mathrm{sym}}(s) = 2\chi(s) - \chi(\partial s)}_{\text{symmetric Euler characteristic}}$ point insertions.

Orbifold data

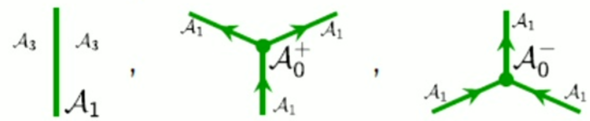
Let $\mathbb{D}$ be an n-dim. defect datum

Definition. A *candidate orbifold datum* is a sub-defect datum $\mathbb{A} \subseteq \mathbb{D}$ with

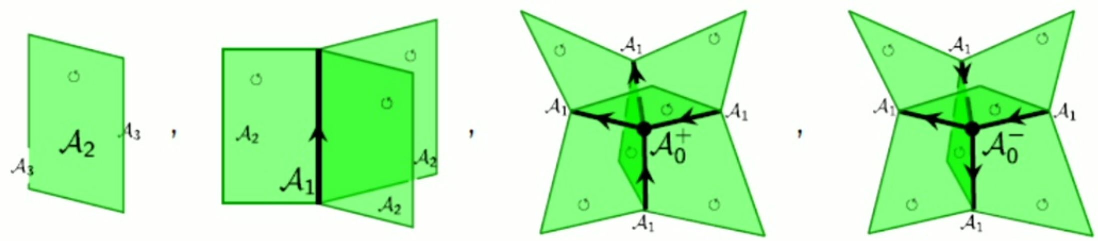
$$A_n = \{\mathcal{A}_n\}\,, \quad \dots \quad , A_1 = \{\mathcal{A}_1\}\,, \quad A_0 = \{\mathcal{A}_0^+, \mathcal{A}_0^-\}\,,$$

such that $\mathcal{A}_n, \dots, \mathcal{A}_1, \mathcal{A}_0^{\pm}$ can be used to label dual triangulations.

Example in 2d:



Example in 3d



Let $Z\colon \operatorname{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \operatorname{Vect}_{\Bbbk}$ be a defect TQFT.

Definition. An *orbifold datum* is a candidate orbifold datum $\mathbb{A} \subseteq \mathbb{D}$ such that all dual-triangulated spheres $[\varnothing \to (S^{n-1}, s)] \in \operatorname{Bord}_n^{\mathrm{def}}(\mathbb{A})$ evaluate to the same vector $\Bbbk \to Z(S^{n-1}, s)$.

$\Rightarrow$ can perform the dual Pachner moves away from boundary.

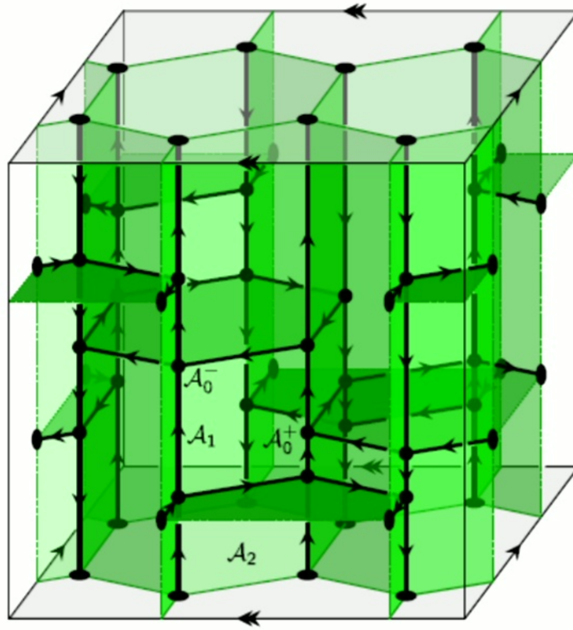
[Carqueville–Runkel–Schaumann'17]

Generalised orbifold TQFT

Proposition. Let $\mathbb{A}$ be an orbifold datum and $[(\Omega, S)\colon (M, s) \to (M', s')] \in \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{A})$ a dual-triangulated bordism. Then

$$\Psi_s^{s'}(\Omega) := \left[Z(M,s) \xrightarrow{Z(\Omega,S)} Z(M',s') \right]$$

depends only on the boundary dual-triangulations s, s' and not the interior of the stratification S.

$\Psi_s^s(T^2) =$ 

cf. condensation monad! [Gaiotto–Johnson-Freyd'19]

[Carqueville–Runkel–Schaumann'17]

Generalised orbifold TQFT

Definition and Proposition.

The generalised orbifold TQFT is a symmetric monoidal functor

$$Z^{\mathrm{orb}\,\mathbb{A}} \colon \mathrm{Bord}_n \longrightarrow \mathrm{Vect}_{\Bbbk}\,,$$

defined

- on objects:

$$Z^{\mathrm{orb}\,\mathbb{A}}(M) := \mathrm{colim}\{\Psi_s^{s'}(M \times [0,1]) \colon Z(M,s) \to Z(M,s')\}$$

- on morphisms:

$$Z^{\mathrm{orb}\,\mathbb{A}}(M \xrightarrow{\Omega} M') := \Big[Z^{\mathrm{orb}\,\mathbb{A}}(M) \hookrightarrow Z(M,s) \xrightarrow{\Psi_s^{s'}(\Omega)} Z(M',s') \twoheadrightarrow Z^{\mathrm{orb}\,\mathbb{A}}(M) \Big]$$

Trivial defect TQFTs

Some ideas and terminology:

- *Trivial n-dim. TQFT (vacuum)*:

$$Z\colon \text{Bord}_n \longrightarrow \text{Vect}_{\Bbbk}\,, \quad Z(M) \longmapsto \Bbbk\,, \quad Z(M \xrightarrow{\Omega} M') \longmapsto \text{id}_{\Bbbk}\,.$$

- Slogan: d-dim. defect in the trivial n-dim. TQFT $\Leftrightarrow$ d-dim. TQFT
- *Trivial n-dim. defect TQFT*

$$Z_n^{\text{triv}}\colon \text{Bord}_n^{\text{def}}(\mathbb{D}^{nd}) \longrightarrow \text{Vect}_{\Bbbk}$$

can be defined iteratively as

i) having only the trivial bulk theory;
ii) lower dimensional defects are obtained as generalised orbifolds of lower dimensional trivial defect TQFTs.

Because of i) the trivial defect TQFTs are ironically quite non-trivial!

[Carqueville–Müller'23]

Trivial defect TQFTs

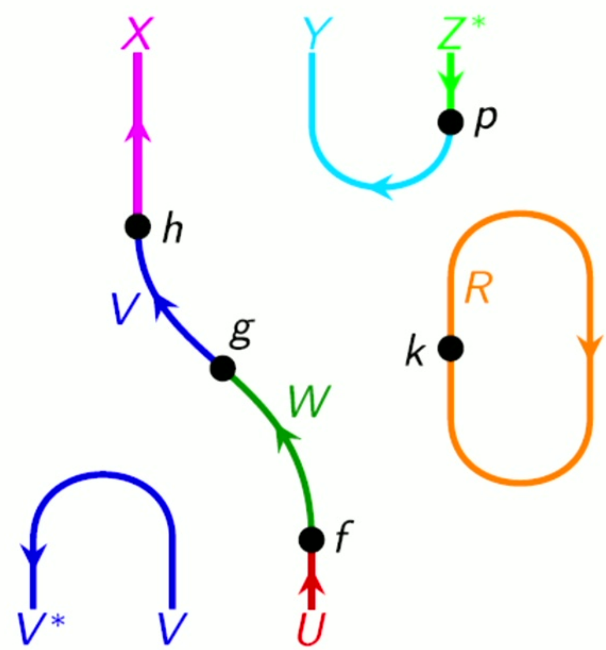
The general guideline to obtain the defect datum of Z_n^{triv}:
take $\mathbb{D}^{nd} = \mathbb{D}^{\mathcal{C}_n}$ for an n-category built as follows:

- $\mathcal{C}_1 := \mathrm{vect}_{\Bbbk}$ (or any other symm. mon. 1-category)
- $\mathcal{C}_2 := B\mathcal{C}_1$ (delooping - 2-category with a single object)
- $\mathcal{C}_2^{\mathrm{orb}}$ - 2-category associated to the *orbifold completion* of Z_2^{triv}
  - objects: (Euler completed) orbifold data in Z_2^{triv};
  - 1-morphisms: line defects between generalised orbifold bulk theories (implemented i.t.o. Z_2^{triv});
  - 2-morphisms: dito for point defects

  ... $\simeq \mathrm{Frob}_{\mathcal{C}_1}^{\mathrm{ss}}$ (symm. sep. Frobenius algebras in $\mathcal{C}_1$, bimodules, bimod. hom.'s)
- $\mathcal{C}_3 := B(\mathcal{C}_2^{\mathrm{orb}})$
- $\mathcal{C}_4 := B\left((B\mathcal{C}_2^{\mathrm{orb}})^{\mathrm{orb}} \right)$
- ...

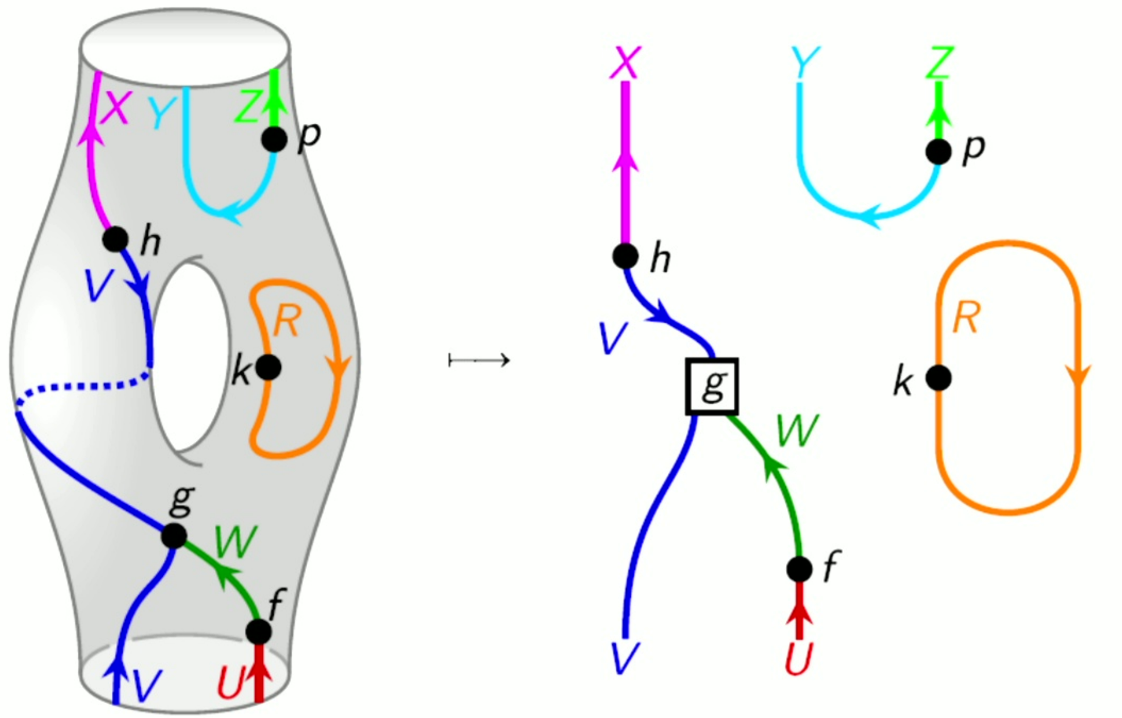
Examples

Bordisms in $\mathrm{Bord}_1^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}_1})$ are just string diagrams in $\mathcal{C}_1 = \mathrm{vect}_{\Bbbk}$
- to obtain $Z_1^{\mathrm{triv}}\colon \mathrm{Bord}_1^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}_1}) \to \mathrm{Vect}_{\Bbbk}$ can just evaluate:

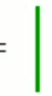
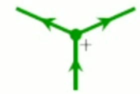
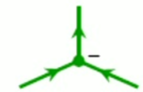


Examples

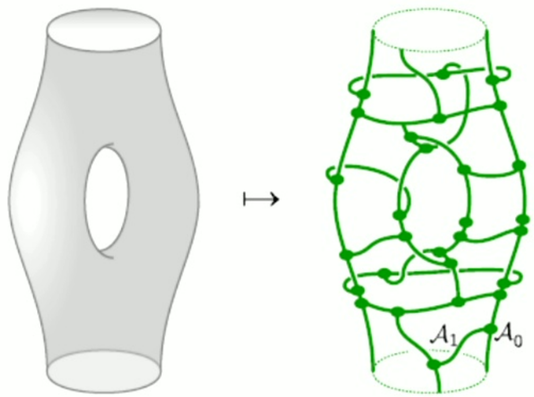
$Z_2^{\text{triv}}\colon \text{Bord}_2^{\text{def}}(\mathbb{D}^{\mathcal{C}_2}) \to \text{Vect}_{\Bbbk}$ is just the forgetful functor $\text{Bord}_2^{\text{def}}(\mathbb{D}^{\mathcal{C}_2}) \to \text{vect}_{\Bbbk}$:

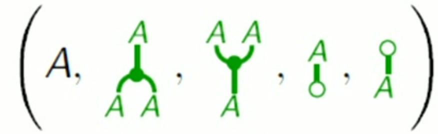


Examples

Orbifold data in Z_2^{triv}: labels $\mathcal{A}_2 = \star$, $\mathcal{A}_1 =$ , $\mathcal{A}_0^+ =$ , $\mathcal{A}_0^- =$ 

such that dual Pachner moves 

hold upon evaluation 

Proposition. Symmetric Δ-seperable Frobenius algebra 

yields an orbifold datum in Z_2^{triv}.

Examples

Orbifold completion:

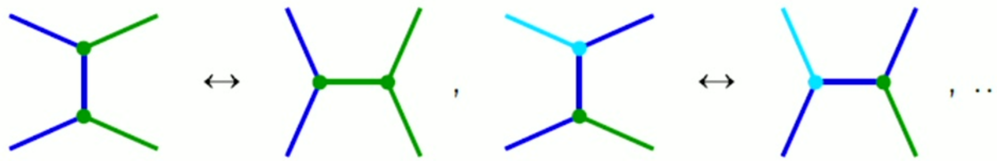
need to add line/point defects and Euler insertions to gen. orbifold theories

- line defect datum - similar to orbifold datum, but add a coloured line:

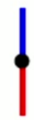
  labels for

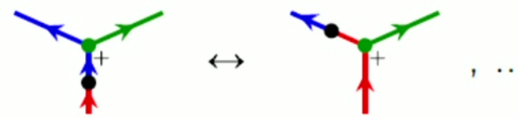
  such that

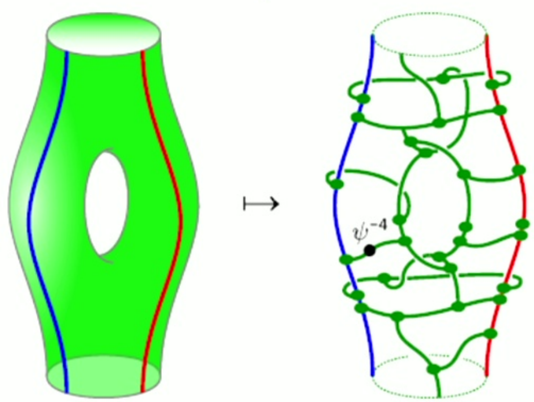
- point defect datum: label for

  such that

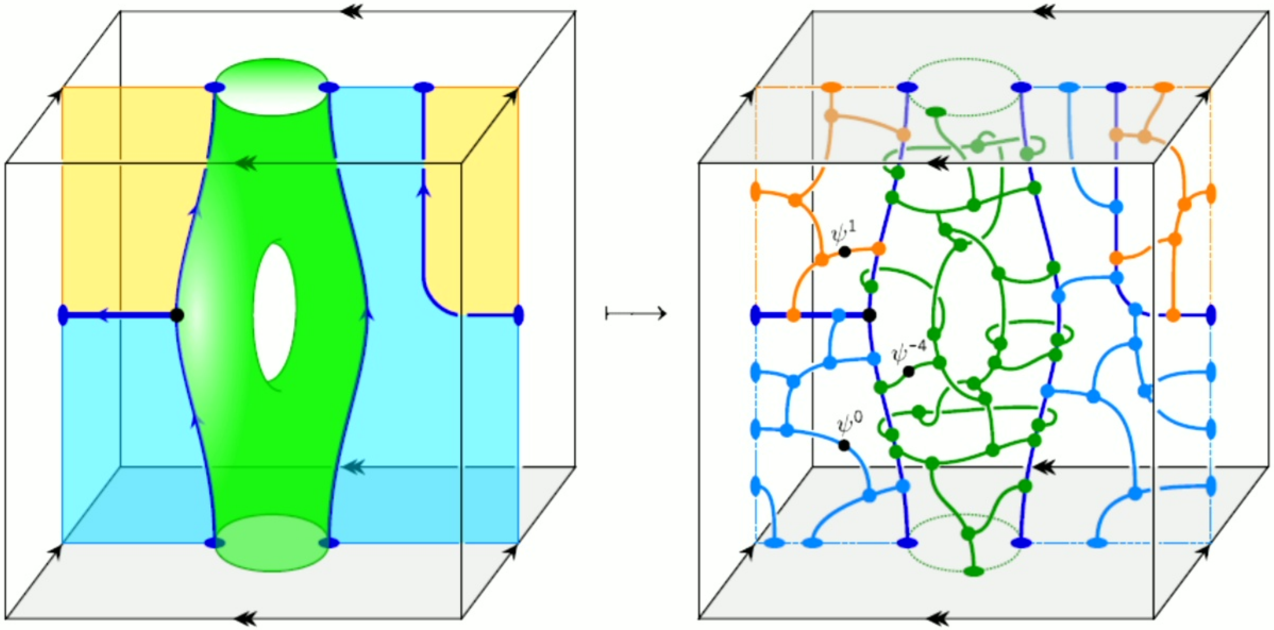
  

Evaluation:



Examples

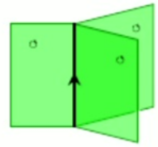
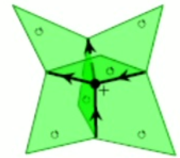
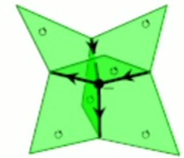
Z_3^{triv}: $\mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}_3}) \to \mathrm{Vect}_{\Bbbk}$ - "ribbonise", forget topology and evaluate:



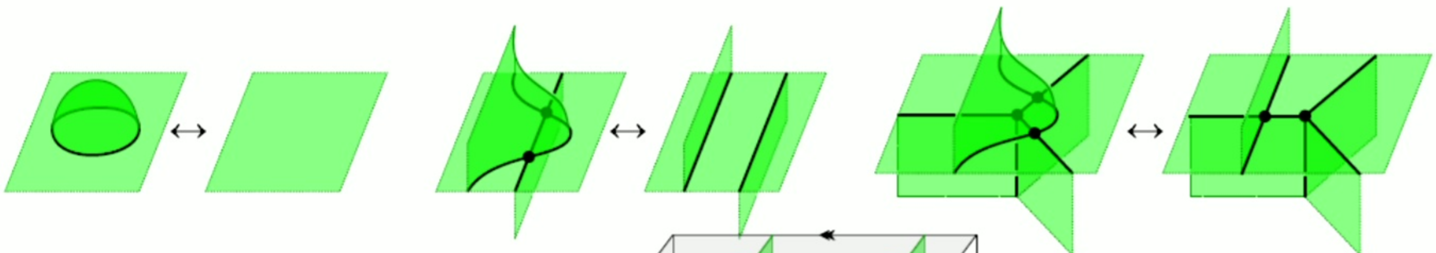
[Carqueville–Runkel–Schaumann'17]
[Carqueville–M–Runkel–Schaumann–Scherl'21]

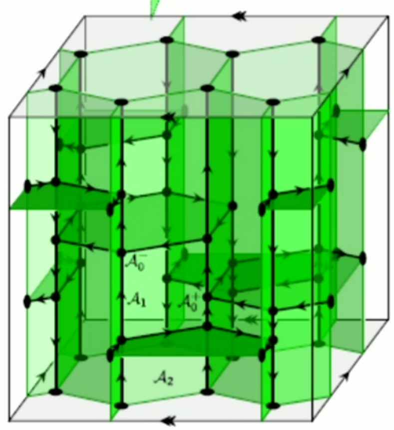
Examples

Orbifold data in Z_3^{triv}:

labels $\mathcal{A}_3 = \star$, $\mathcal{A}_2 =$  , $\mathcal{A}_1 =$  , $\mathcal{A}_0^+ =$  , $\mathcal{A}_0^- =$  ,

such that the dual Pachner moves hold.
Special orbifold data - stronger BLT moves hold instead:

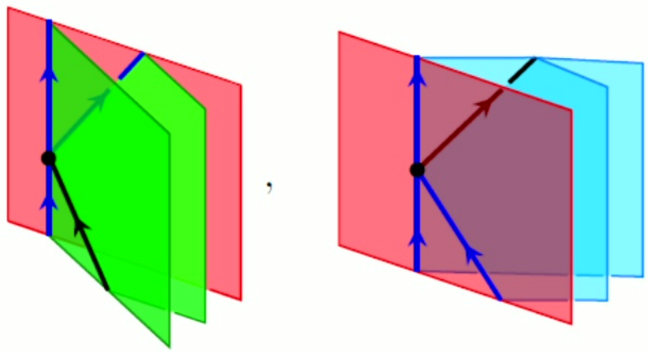


Evaluation: e.g. $T^2 \longmapsto$ 

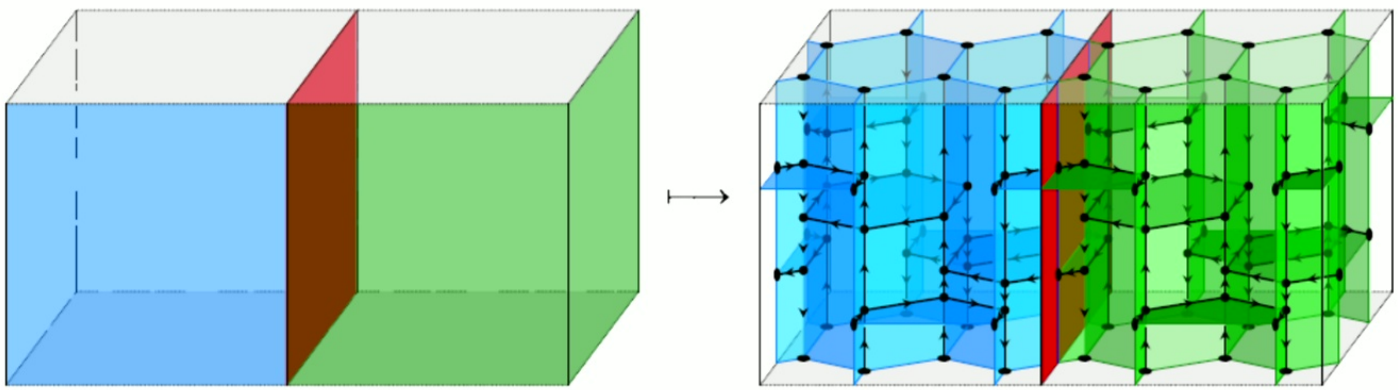
Examples

[Fuchs–Schweigert–Valentino'13]
[Koppen–M–Runkel–Schweigert'21]
[Meusburger'22]
[Carqueville–Müller'23]

Orbifold completion:
like in 2d: add colours to dual triangulations and dual Pachner moves, e.g.



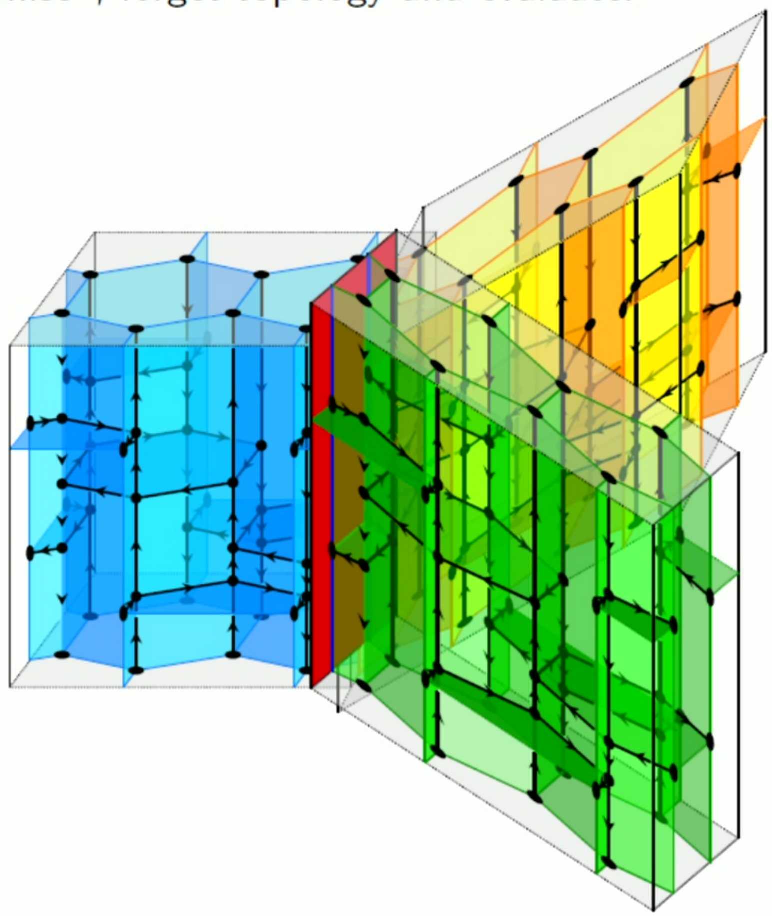
Evaluation:



Examples

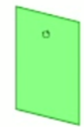
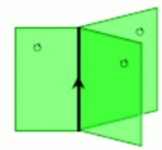
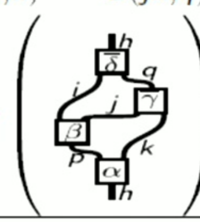
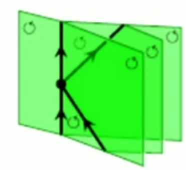
$Z_4^{\mathrm{triv}}\colon\ \mathrm{Bord}_4^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}_4}) \to \mathrm{Vect}_{\Bbbk}$

- "foamify", "ribbonise", forget topology and evaluate:



3d state-sum models

Turaev–Viro–Barrett–Westbury TQFT is a generalised orbifold of Z_3^{triv}:

$\mathcal{S}$ – spherical fusion cat.	$\mathbb{A}^{\mathcal{S}}$ – orb. datum
(simple) objects i	$\mathcal{A}_2 = \bigoplus_i \big(\underbrace{\mathrm{End}_{\mathcal{C}}(i)}_{\cong \Bbbk}, \mathrm{tr}_{\mathcal{C}} \big)$ 
tensor product $i \otimes j$	$\mathcal{A}_1 = \bigoplus_{ijk} \mathcal{S}(i \otimes j, k)$ 
associator $\alpha \colon (ij)k \xrightarrow{\sim} i(jk)$	$\mathcal{A}_0^{\pm} = \bigoplus_{ijkh} [\underbrace{\mathcal{S}((ij)k, h)}_{\mathcal{S}(ij,p) \otimes_{\mathcal{S}(p,p)} \mathcal{S}(pk,h)} \leftrightarrow \underbrace{\mathcal{S}(i(jk), h)}_{\mathcal{S}(jk,q) \otimes_{\mathcal{S}(q,q)} \mathcal{S}(iq,h)}]$ – get from 6j-symbols $\mathrm{tr}_{\mathcal{S}}$  
Euler completion factors	guess

Theorem (Carqueville–Runkel–Schaumann'18).
$\mathbb{A}^{\mathcal{S}} = (\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_0^{\pm}, \dots)$ is an orbifold datum in Z_3^{triv}, and $(Z_3^{\mathrm{triv}})^{\mathrm{orb}\,\mathbb{A}^{\mathcal{S}}} \cong Z_{\mathcal{S}}^{\mathrm{TVBW}}$.

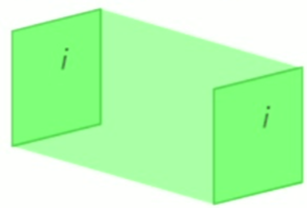
4d state-sum models

1) [Découppet'20] (unoriented version)
2) [Carqueville–Müller'23]

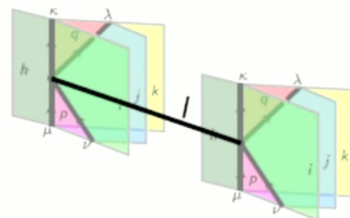
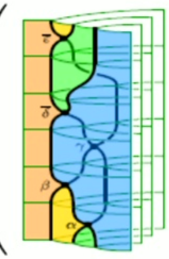
In 3d: $\mathbb{A}^{\mathcal{S}}$ was obtained by mapping $(\mathcal{S}, \otimes, \alpha)$ via Eilenberg–Watts equivalence:

$$\text{EW}: \left\{ \begin{matrix} \text{semisimple cat.'s w/ trace,} \\ \text{lin. functors, \quad nat. transf.'s} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{symm. sep. Frobenius algberas,} \\ \text{bimodules, \quad bimodule maps} \end{matrix} \right\}$$

In 4d proceed similarly: $\{\text{s.si. 2-cat.'s w/ tr.,...}\} \xrightarrow{\sim^{1)}} \{\text{multifus. cat.'s w/ tr.,...}\} \xrightarrow{2)} \mathcal{C}_3$

$\mathcal{S}$ - spherical (pre)fusion 2-cat.	$\mathbb{A}^{\mathcal{S}}$
(simple) objects i	$\mathcal{A}_3 = \boxplus_i \text{End}_{\mathcal{S}}(i)$ – sum of 3d orb. data 
tensor product $i \boxtimes j$	$\mathcal{A}_2 = \boxplus_{ijk} \mathcal{S}(i \boxtimes j, k)$ – sum of domain walls 
...	...

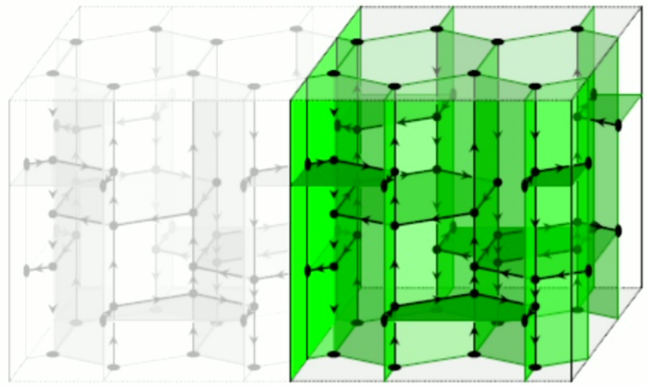
4d state-sum models

...	...
associator $\alpha\colon \underbrace{(ij)k \to i(jk)}_{\text{1-morphisms}}$	$\mathcal{A}_1 = \boxplus_{ijkh}\Big[\underbrace{\mathcal{S}((ij)k,h)}_{\mathcal{S}(ij,p)\boxtimes_{\mathcal{S}(p,p)}\mathcal{S}(pk,h)} \to \underbrace{\mathcal{S}(i(jk),h)}_{\mathcal{S}(jk,q)\boxtimes_{\mathcal{S}(q,q)}\mathcal{S}(iq,h)}\Big]$ 
pentagonator $$\begin{array}{ccccc} & (i(jk))l & \longrightarrow & i((jk))l) & \\ \nearrow & & & & \searrow \\ ((ij)k)l & & \Downarrow\pi & & i(j(kl)) \\ & \searrow & & \nearrow & \\ & & (ij)(kl) & & \end{array}$$	$\mathcal{A}_0^{\pm}$ – get from 10j symbols $\mathrm{Tr}_{\mathcal{S}}\Bigg($  $\Bigg)$
Euler completion factors	guess

Theorem (Carqueville–M–Müller).
$\mathbb{A}^{\mathcal{S}} = (\mathcal{A}_3, \mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_{\pm}, \dots)$ is an orbifold datum in Z_4^{triv}.
The orb. TQFT yields Douglas–Reutter invariants: $(Z_4^{\text{triv}})^{\text{orb}\,\mathbb{A}^{\mathcal{S}}}(\varnothing \xrightarrow{\Omega} \varnothing) = Z_{\mathcal{S}}^{\text{DR}}(\Omega)$.

Boundary theories

Generalised orbifold TQFT has a natural domain wall with the trivial bulk theory:

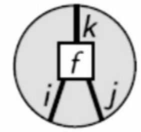


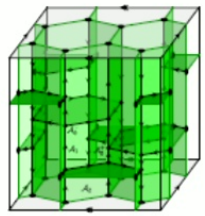
Specialising to $\mathcal{S} = B\mathcal{C}$ for a modular fusion category $\mathcal{C}$, can (re)prove:

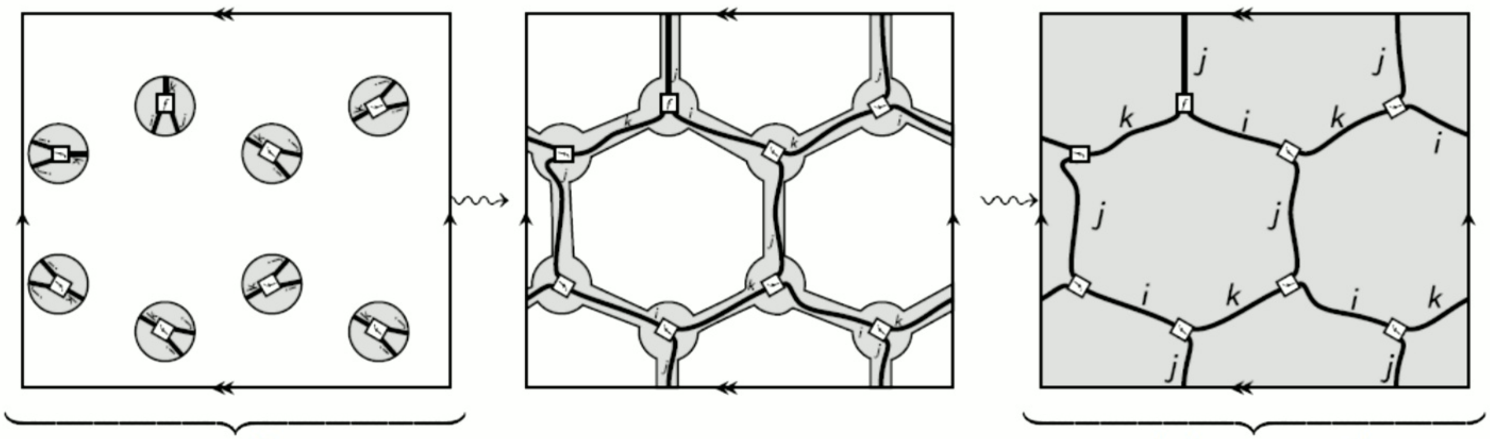
Reshetikhin–Turaev TQFT is a boundary theory of Crane–Yetter 4d TQFT [Tham'21]

$$\begin{array}{ccc} \mathrm{Bord}_4^{\mathrm{def}}(B\mathcal{C})\big|_{1w} & \xrightarrow{\;Z^{\mathrm{DR}}_{B\mathcal{C}}\;} & \mathrm{Vect}_{\Bbbk} \\ \downarrow \sigma & \nearrow Z^{\mathrm{RT}}_{\mathcal{C}} & \\ \widehat{\mathrm{Bord}_3} & & \end{array}$$

State spaces

Recall 3d orb. datum for TVBW: $\mathcal{A}_1 = \bigoplus_{ijk} \mathcal{S}(i \otimes j, k) \cong \Bigg\{$  $\Bigg\}$.

State-space $\mathrm{im}(Z_3^{\mathrm{triv}}$ $\Bigg($  $\Bigg))$ is the common image of edge/face projectors:

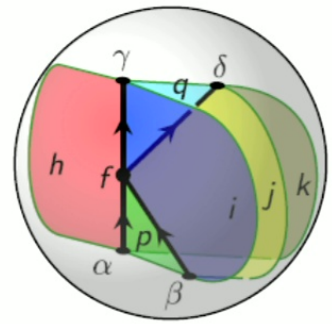


$\mathcal{A}_1^{\otimes N}$

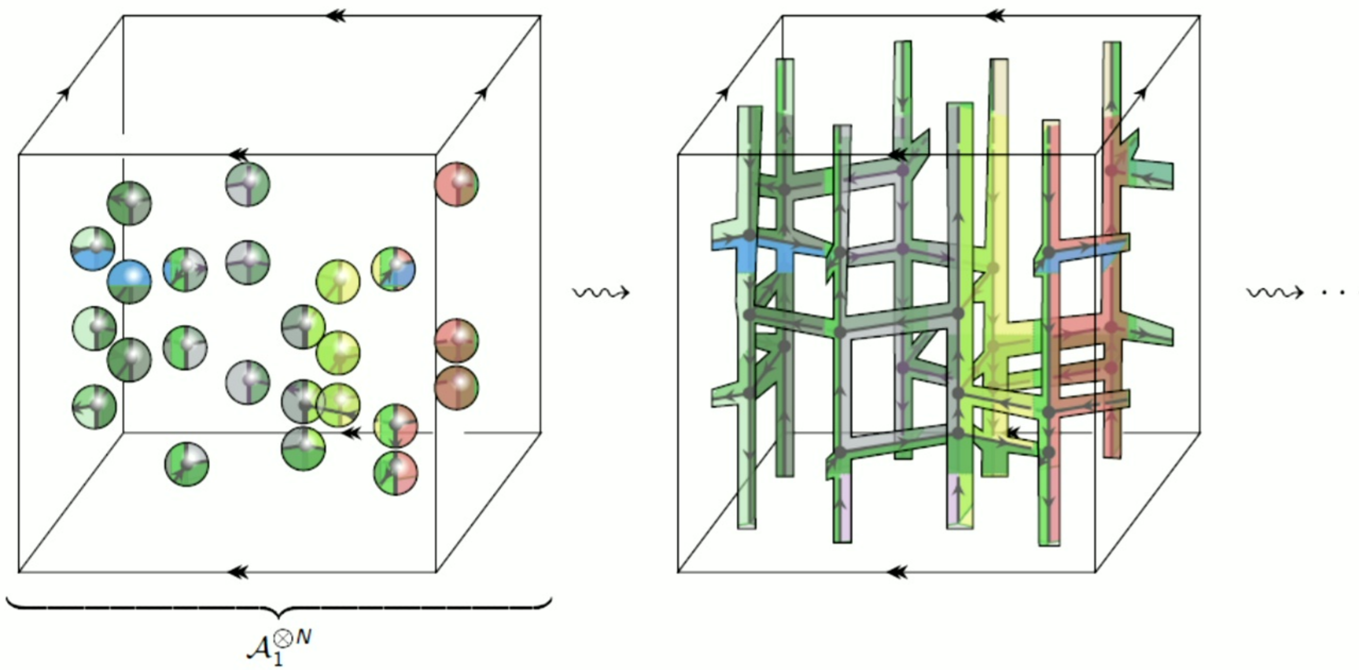
'string-net soup'

[Kirillov'11]

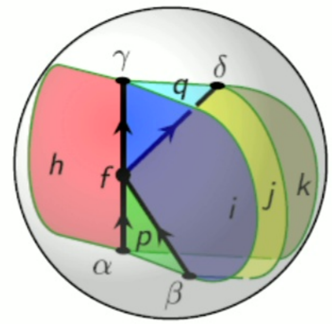
State spaces

Recall 4d orb. datum for DR: $\mathcal{A}_1 = \Bigg\{$  $\Bigg\}$.

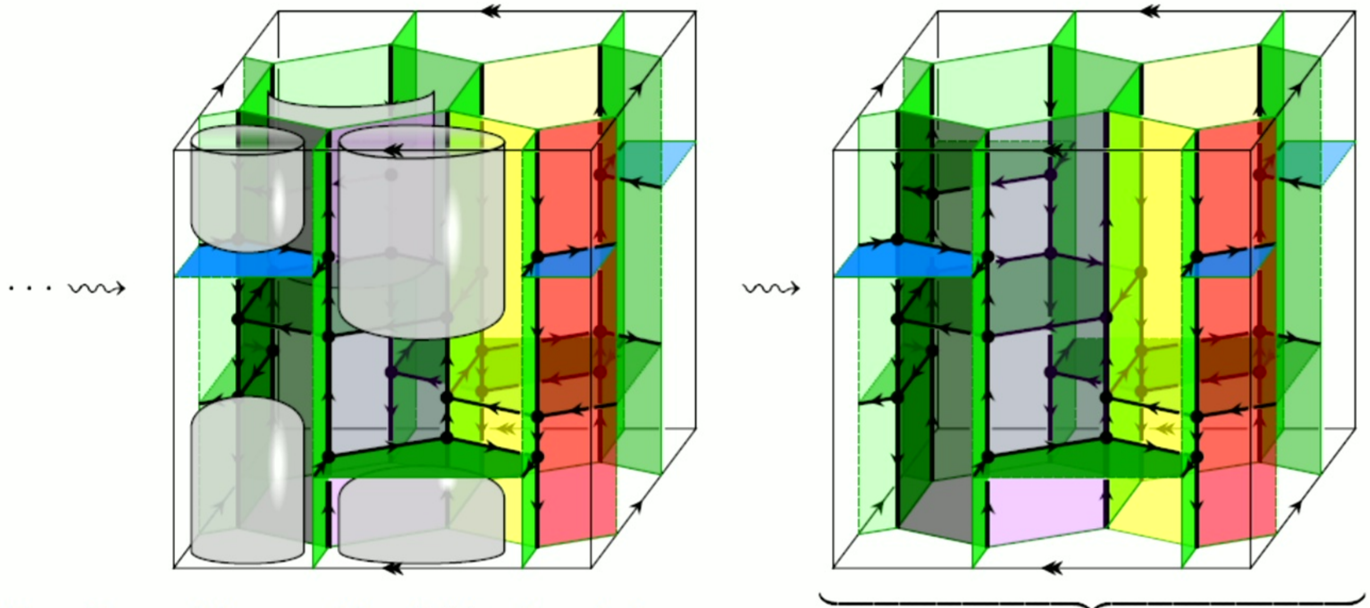
State-spaces should be similar:



State spaces

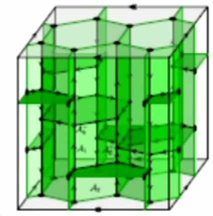
Recall 4d orb. datum for DR: $\mathcal{A}_1 = \left\{ \right.$  $\left. \right\}$.

State-spaces should be similar:



For Crane–Yetter this yields the skein space.

Lattice models

From the edge/face/volume/... projectors in $\mathrm{im}(Z_n^{\mathrm{triv}}$  $)$

one can get a commuting projector Hamiltonian acting on $\mathcal{A}_1^{\otimes N}$:

- 3d TVBW orbifold datum $\rightsquigarrow$ Levin–Wen models
- 4d DR orbifold datum for $\mathcal{S} = B\mathcal{C} \rightsquigarrow$ Walker–Wang models
  for arbitrary $\mathcal{S} \rightsquigarrow$??

Can also apply this for other defect TQFTs to get 'internal' lattice models

- For 3d Reshetikhin–Turaev defect TQFT [Carqueville–Runkel–Schaumann'17]
  orbifold data yield 'internal Levin–Wen models' [M–Runkel–Voß'23]

These are instances of lattice models obtained from
(oriented versions of) condensation monads in the higher categories of defects.
[Gaiotto–Johnson-Freyd'19]

Thank you!