Title: Quantum double models and Dijkgraaf-Witten theory with defects

Speakers:

Collection: Higher Categorical Tools for Quantum Phases of Matter

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Abstract: We use 3d defect TQFTs and state sum models with defects to give a gauge theoretical formulation of Kitaev's quantum double model (for a finite group) and (untwisted) Dijkgraaf-Witten TQFT with defects. This leads to a simple description in terms of embedding quivers, groupoids and their representations. Defect Dijkgraaf-Witten TQFTs is then formulated in terms of spans of groupoids and their representations.
This is work in progress with João Faría-Martins, University of Leeds.

Quantum double models and Dijkgraaf-Witten TQFT with defects

Higher categorical tools for quantum phases of matter

Perimeter Institute,

March 19, 2024

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- C. Meusburger: Adv. Maths 429 (2023): 109177,
- J. Faría Martins, C. Meusburger: work in progress

Motivation

- **Turaev-Viro TQFTs and state sums with defects**
  - Turaev-Viro-Barrett-Westbury state sums with defects [C.M 22]
  - defect Turaev-Viro TQFT via orbifoldisation [Carqueville-Müller 23]

⇨ for general categorical data

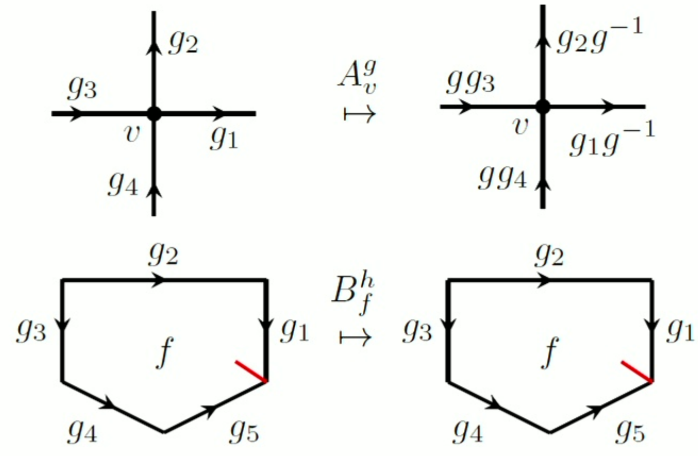
- bulk: spherical fusion categories
- codim 1 defects: bimodule categories with bimodule traces
- codim 2 defects: bimodule functors
- codim 3 defects: bimodule natural transformations

⇨ rather abstract and implicit

- **aim: investigate for untwisted Dijkgraaf-Witten theory**
  - more gauge theoretical and geometric formulation
  - more concrete and efficient model for physics applications
  - 2d part: Kitaev's quantum double model with defects of all codimensions

Kitaev's quantum double model for finite group G

- oriented surface Σ
- embedded graph Γ with $\Sigma \setminus \Gamma = D \amalg \ldots \amalg D$



$\delta_h(g_5 g_4 g_3 g_2^{-1} g_1^{-1})$

- **Hilbert space** $\mathcal{H} = \mathbb{C}[G]^{\otimes E}$
- **vertex** $v \in V$ ⇨ **vertex operator** A_v^g
- **face** $f \in F$ ⇨ **face operator** B_f^h

⇨ **commuting projectors**

$A_v = |G|^{-1} \Sigma_{g \in G} A_v^g$ **gauge invariance at** $v \in V$

$B_f = B_f^e$ **flatness at** $f \in F$

⇨ **Hamiltonian** $H = -\Sigma_{v \in v} A_v - \Sigma_{f \in F} B_f$

⇨ **ground state** $\mathcal{H}_{pr} = (\cap_{v \in V} \mathrm{im}(A_v)) \cap (\cap_{f \in F} \mathrm{im}(B_f))$

topological invariant independent of Γ

$\mathcal{H}_{pr} = \langle \mathrm{Hom}(\pi_1(\Sigma), G)/G \rangle_{\mathbb{C}} = \mathcal{Z}(\Sigma)_{DW}$ **2d part of Dikgraaf-Witten TQFT**

⇨ **excitations** site = pair $s = (v, f)$ of vertex and incident face

⇨ A_v^g , B_f^h form representation of Drinfeld double $D(G)$

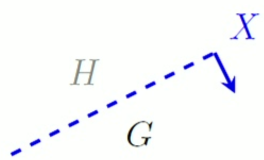
quantum double models with defects - defect data

- oriented surface Σ
- defect graph Γ on Σ

defect data:
- defect data for untwisted Dijkgraaf-Witten theory
- special case of general defect data for Turaev-Viro TQFT

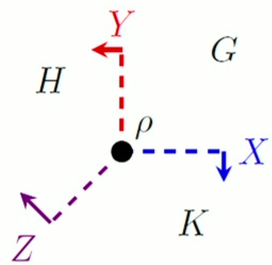
regions $r \in R$ = connected components of $\Sigma \setminus \Gamma$ finite groups $G, H, K \ldots$

edges $e \in E$



finite $G \times H^{op}$ - sets $X, Y, Z, \ldots$

vertices $v \in V$



representations of action groupoids

$$\rho : (X \times Y \times Z)//(G \times H \times K) \to \mathrm{Vect}_{\mathbb{C}}$$

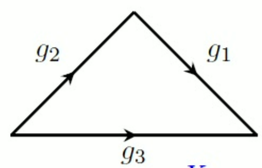
$$(x, y, z) \xrightarrow{(g,h,k)} (k \rhd x \lhd g^{-1}, h \rhd y \lhd g^{-1}, h \rhd z \lhd k^{-1})$$

via Turaev-Viro defect TQFTs and-state sums with defects

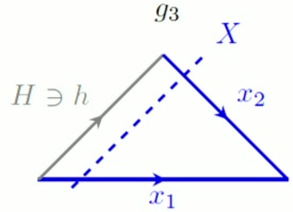
[C.M. 22]
[Carqueville-Müller 23]

- **triangulation of surface** transversal to defects
- **labelling** edges ➩ group elements or elements of action sets

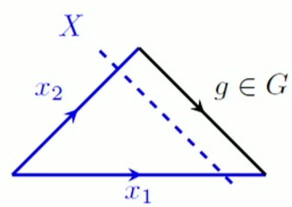
labelled triangle ➩ vector space **flatness - face operator**



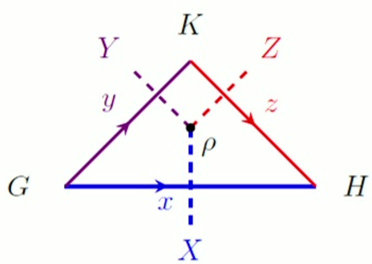
$$\begin{cases} \mathbb{C} & g_3 = g_1 g_2 \\ 0 & g_3 \neq g_1 g_2 \end{cases} \qquad B(\psi) = \delta_{g_3}(g_1 g_2)\, \psi$$



$$\begin{cases} \mathbb{C} & x_1 = x_2 \lhd h \\ 0 & x_1 \neq x_2 \lhd h \end{cases} \qquad B(\psi) = \delta_{x_1}(x_2 \lhd h)\, \psi$$



$$\begin{cases} \mathbb{C} & x_1 = g \rhd x_2 \\ 0 & x_1 \neq g \rhd x_2 \end{cases} \qquad B(\psi) = \delta_{x_1}(g \rhd x_2)\, \psi$$



$$\rho(x, y, z)$$

gauge transformations - vertex operator

$$h \in H \xrightarrow{x} g \in G \quad \mapsto \quad \xrightarrow{g \rhd x \lhd h^{-1}}$$

$$g_2 \in G \xrightarrow{g} g_1 \in G \quad \mapsto \quad \xrightarrow{g_1 g g_2^{-1}}$$

- **questions:**
  - **more conceptual, gauge theoretical description?**
  - **more explicit, suitable for computations?**

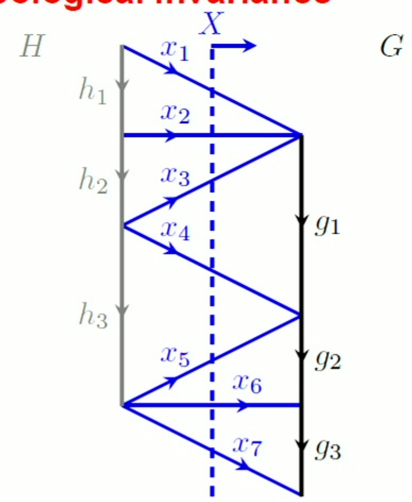
- **no defects:** $\mathcal{H}_{pr} = \mathcal{Z}(\Sigma)_{DW} = \langle \mathrm{Hom}(\pi_1(\Sigma), G)/G \rangle_{\mathbb{C}}$ [Dijkgraaf-Witten, Freed-Quinn,…]
  - G - principal bundles on Σ
  - flat graph gauge connections / graph gauge transformations
  - TQFT: $\partial M = \bar{\Sigma}_1 \amalg \Sigma_2$ ⇨ linear map $\mathcal{Z}(M) : \mathcal{Z}(\Sigma_1) \to \mathcal{Z}(\Sigma_2)$

$$\langle [A_1] \mid M \mid [A_2] \rangle = \left| \frac{\{\text{flat connections } A \text{ on } M : A|_{\Sigma_1} = A_1, A|_{\Sigma_2} = A_2\}}{\text{gauge transformations}} \right|$$

- **with defects ?** $\mathcal{Z}(\Sigma) = ?$
  - TQFT: $\partial M = \bar{\Sigma}_1 \amalg \Sigma_2$ ⇨ linear map $Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$?
  - for domain walls (defect edges) + excitations [Morton '10]
    ⇨ relative G -bundles [Fuchs-Schweigert-Valentino '13]
  - general defects ?

gauge theoretical formulation

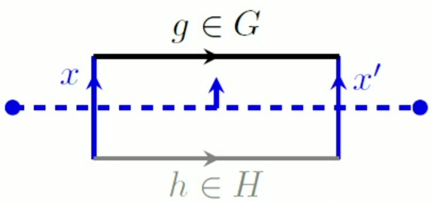
topological invariance



gauge theoretical formulation

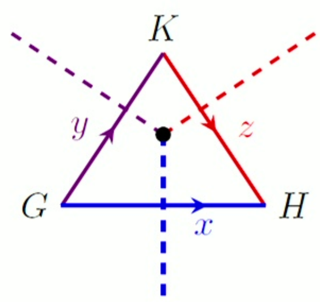
1. thickening of defect graph Γ

- **defect edges ⇨ rectangles**

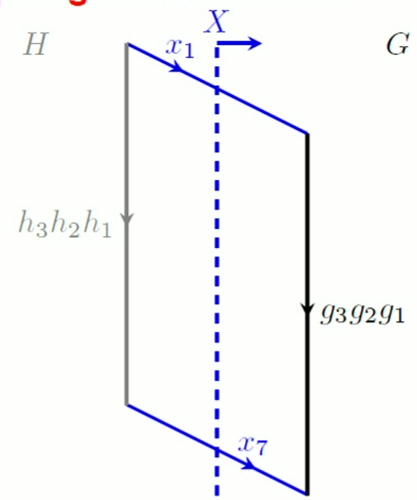


$$x' = g \rhd x \lhd h^{-1}$$

- **defect vertices ⇨ polygons**

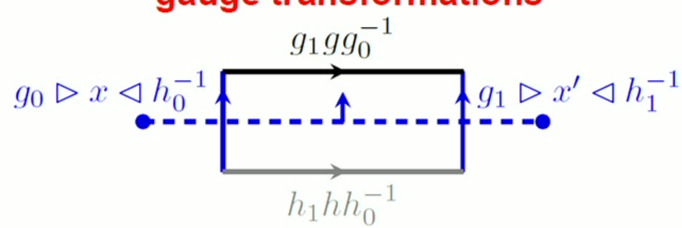


topological invariance



$$x_7 = g_3g_2g_1 \rhd x_1 \lhd (h_3h_2h_1)^{-1}$$

gauge transformations



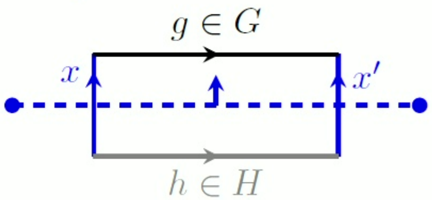
⇨ **edges** of thickening: **gauge fields** - labeled with elements of groups and action sets

⇨ **vertices** of thickening: **gauge transformations** - labeled with group elements

⇨ **faces** of thickening: **defects**

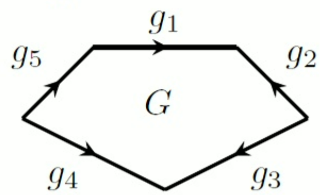
- **special defects**

transparent defect on domain wall $\rho : (X^{\times 2} \times G \times H)//(G^{\times 2} \times H^{\times 2}) \to \mathrm{Vect}_{\mathbb{C}}$



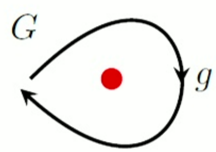
$$\rho(x, x', g, h) = \begin{cases} \mathbb{C} & x' = g \rhd x \lhd h^{-1} \\ 0 & x' \neq g \rhd x \lhd h^{-1} \end{cases}$$

transparent bulk defect $\rho : G^{\times 5}//G^{\times 5} \to \mathrm{Vect}_{\mathbb{C}}$



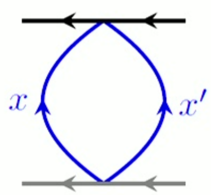
$$\rho(g_1, \ldots, g_5) = \begin{cases} \mathbb{C} & g_5 g_4^{-1} g_3 g_2^{-1} g_1 = 1 \\ 0 & g_5 g_4^{-1} g_3 g_2^{-1} g_1 \neq 1 \end{cases}$$

bulk excitation $\rho : G//G \to \mathrm{Vect}_{\mathbb{C}}$ = representatives of conjugacy classes
+ representations of stabilisers



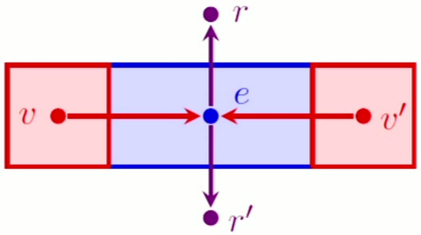
$g_0 \rhd g = g_0 g g_0^{-1}$ = representations of $D(G)$

boundary excitation $\rho : X^{\times 2}//G \times H \to \mathrm{Vect}_{\mathbb{C}}$



= representatives of $G \times H$ - orbits on $X^{\times 2}$
+ representations of stabilisers

2. dual quiver Q ⇨ category $\mathcal{Q}$



- vertices $V_Q = V \dot\cup E \dot\cup R$
- edges $(v,e): v\in V \to e \in E$ for $v \in \bar e$
  $(e,r): e\in E \to r \in R$ for $e \subset \bar r$ ⇨ category $\mathcal{Q}$

⇨ **encodes decomposition of surface and interaction of defect data**

topological content • defect graph in surface ⇨ functor $T: \mathcal{Q} \to \mathrm{Grpd}$

$$r\in R \mapsto \Pi_1(r, V_r) \qquad V_r = \{v\in V \mid v \in \bar r\}$$
$$e\in E \mapsto \Pi_1(e, V_e) \qquad V_e = \{v\in V \mid v \in \bar e\}$$
$$v\in V \mapsto \Pi_1(v, v)$$

$$(v\to e) \mapsto \Pi_1(\iota_{ve}) : \Pi_1(v,v) \to \Pi_1(e, V_e)$$
$$(e\to r) \mapsto \Pi_1(\iota_{er}) : \Pi_1(e,V_e) \to \Pi_1(r, V_r)$$

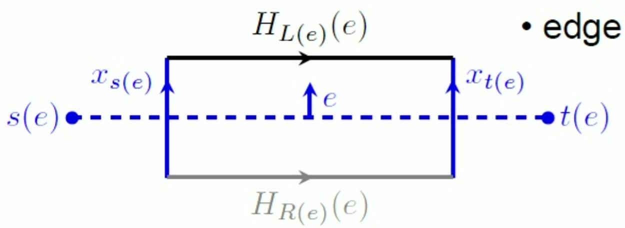
algebraic content • defect data ⇨ functor $D: \mathcal{Q} \to \mathrm{Grpd}$

$$r\in R \mapsto \mathcal{G}_r \qquad \mathcal{G}_r = \bullet//G_r$$
$$e\in E \mapsto \mathcal{G}_e \qquad \mathcal{G}_e = X_e//G_{L(e)} \times G_{r(e)}$$
$$v\in V \mapsto \mathcal{G}_v \qquad \mathcal{G}_v = (\Pi_{v\in e} X_e)//(\Pi_{v\in r} G_r)$$

$$(v\to e) \mapsto \Pi_{ve} : \mathcal{G}_v \to \mathcal{G}_e$$
$$(e\to r) \mapsto \Pi_{er} : \mathcal{G}_e \to \mathcal{G}_r$$

3. gauge fields and gauge transformations

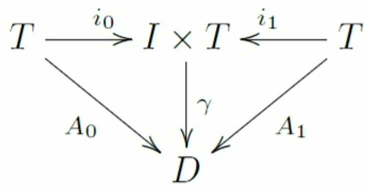
- **gauge field** = natural transformation $A : T \Rightarrow D$
  = assignment • region $r \in R \;\mapsto\;$ functor $H_r : \Pi_1(r, V_r) \to \bullet//G_r$
  • edge $e \in E \;\mapsto\;$ map $f_e : V_e \to X_e$



with flatness $x_{t(e)} = H_{L(e)}(e) \triangleright x_{s(e)} \triangleleft H_{R(e)}(e)^{-1}$

- **gauge transformation** $\gamma : A_0 \Rrightarrow A_1$

interval groupoid $I = \; 0 \xrightarrow{d} 1$

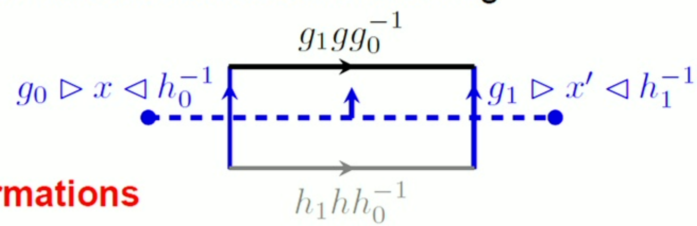


= natural transformation $\gamma : I \times T \Rightarrow D$ with $\gamma i_0 = A_0$ and $\gamma i_1 = A_1$

= assignment of group elements to vertices of thickening

⇨ action on gauge fields

$$H_r(\gamma) \mapsto g_{t(\gamma)} \cdot H_r[\gamma] \cdot g_{s(\gamma)}^{-1}$$



⇨ **groupoid $\mathcal{A}//\mathcal{G}$ of gauge fields and transformations**

4. vector space for a surface

defect data for vertices: representations $\rho_v : (\Pi_{v \in e} X_e)//(\Pi_{v \in r} G_r) \to \mathrm{Vect}_{\mathbb{C}}$

⇨ functor $\rho : \mathcal{A}//\mathcal{G} \to \Pi_{v \in V} \mathcal{G}_v \to \mathrm{Vect}_{\mathbb{C}}$

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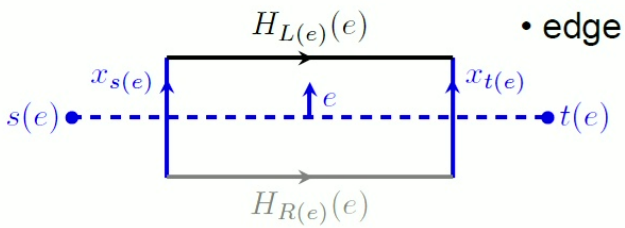
  = assignment
  - region $r \in R \;\mapsto$ functor $H_r : \Pi_1(r, V_r) \to \bullet//G_r$
  - edge $e \in E \;\mapsto$ map $f_e : V_e \to X_e$



with flatness $x_{t(e)} = H_{L(e)}(e) \triangleright x_{s(e)} \triangleleft H_{R(e)}(e)^{-1}$

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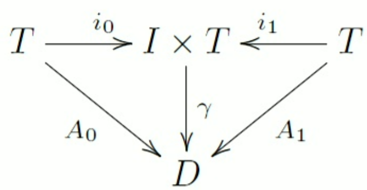
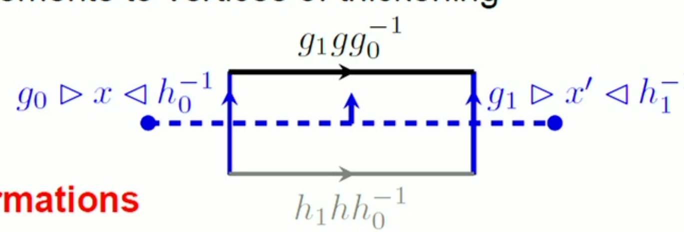


= natural transformation $\gamma : I \times T \Rightarrow D$ with $\gamma i_0 = A_0$ and $\gamma i_1 = A_1$

= assignment of group elements to vertices of thickening

⇨ action on gauge fields

$H_r(\gamma) \mapsto g_{t(\gamma)} \cdot H_r[\gamma] \cdot g_{s(\gamma)}^{-1}$



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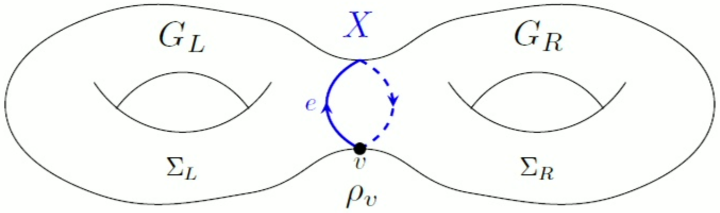
⇨ functor $\rho : \mathcal{A}//\mathcal{G} \to \Pi_{v\in V} \mathcal{G}_v \to \mathrm{Vect}_{\mathbb{C}}$

vector space for surface $\mathcal{Z}(\Sigma) = \lim \rho$

example



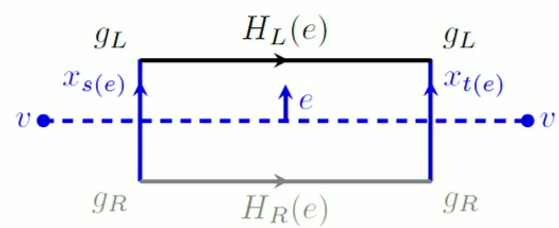
- finite groups G_L, G_R
- finite $G_L \times G_R^{op}$ - set X
- transparent representation

$$\rho_v : X^{\times 2}//(G_L \times G_R) \to \mathrm{Vect}_{\mathbb{C}} \qquad \rho_v(x, x') = \begin{cases} 0 & x \neq x' \\ \mathbb{C} & x = x' \end{cases}$$

gauge fields

$$H_L : \pi_1(\Sigma_L, v) \to G_L \qquad (x_{s(e)}, x_{t(e)})$$
$$H_R : \pi_1(\Sigma_R, v) \to G_R \qquad x_{t(e)} = H_L(e) \triangleright x_{s(e)} \triangleleft H_R(e)^{-1}$$

gauge transformations $(g_L \in G_L, g_R \in G_R)$



$$H_L \mapsto g_L \cdot H_L \cdot g_L^{-1} \qquad x_{s(e)} \mapsto g_L \triangleright x_{s(e)} \triangleleft g_R^{-1}$$
$$H_R \mapsto g_R \cdot H_R \cdot g_R^{-1} \qquad x_{t(e)} \mapsto g_L \triangleright x_{t(e)} \triangleleft g_R^{-1}$$

functor $\rho : \mathcal{A}//\mathcal{G} \to \mathrm{Vect}_{\mathbb{C}} \quad (H_L, H_R, x) \mapsto \begin{cases} \mathbb{C} & H_L(e) \triangleright x \triangleleft H_R(e)^{-1} = x \\ 0 & \text{else} \end{cases}$

vector space

$$\mathcal{Z}(\Sigma) = \lim \rho = \langle \{(H_L, H_R, x) \mid x \in X^{(H_L(e), H_R(e))}\}/\sim \rangle_{\mathbb{C}}$$
$$(H_L, H_R, x) \sim (g_L H_L g_L^{-1}, g_R H_R g_R^{-1}, g_L \triangleright x \triangleleft g_R^{-1})$$

no defect $X = G_L = G_R = G$

example

- graph Γ on Σ with $\Sigma \setminus \Gamma = D \amalg \ldots \amalg D$

region $r \in R$ finite group G_r

edge $e \in E$ finite $G_{L(e)} \times G^{op}_{R(e)}$-set X_e

vertex $v \in V$ representation $\rho_v : (\Pi_{v \in e} X_e)//(\Pi_{v \in r} G_r) \to \mathrm{Vect}_{\mathbb{C}}$

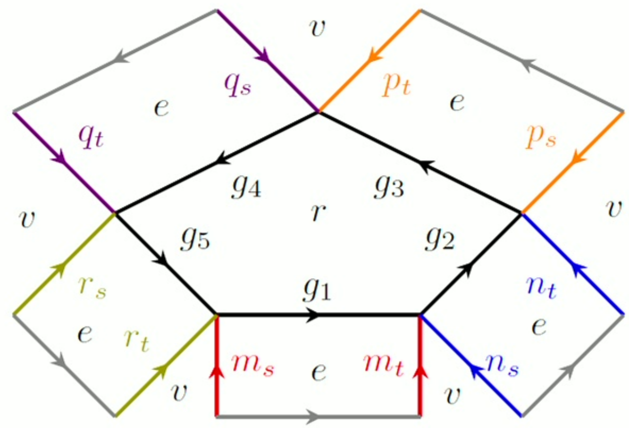
gauge fields and gauge transformations

- gauge transformations at all $v \in r$

  $\Rightarrow g_1 = \ldots = g_5 = 1$

  $\Rightarrow m_s = m_t, \ldots, r_s = r_t$

- residual gauge freedom G_r for $r \in R$



gauge fields $\mathcal{A} = \Pi_{e \in E} X_e$

gauge transformations $\mathcal{G} = \Pi_{r \in R} G_r$

functor $\rho = \otimes_{v \in V} \rho_v : \mathcal{A}//\mathcal{G} \to \mathrm{Vect}_{\mathbb{C}}$

vector space $\mathcal{Z}(\Sigma) = \lim \rho = \bigoplus_{A \in \Pi_0(\mathcal{A}//\mathcal{G})} \rho(A)^{\mathrm{Stab}(A)}$

Dijkgraaf-Witten TQFT with defects

- 3-manifold M with defect stratification $X^M = (\emptyset \subset X_0 \subset X_1 \subset X_2 \subset X_3 = M)$
- boundary stratification $X^{\partial M} = (\emptyset \subset (X_1 \cap \partial M) \subset (X_2 \cap \partial M) \subset \partial M)$

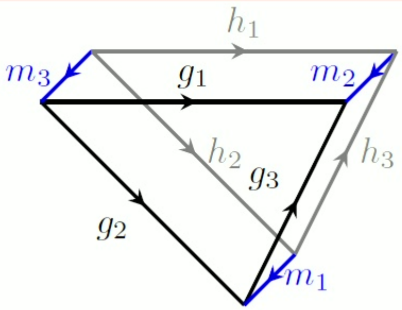
defect data

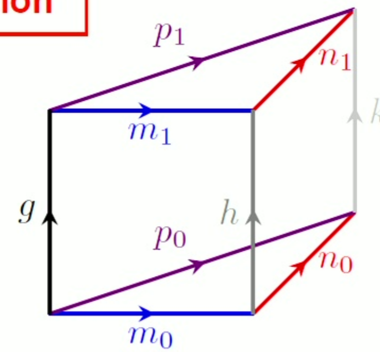
regions	$R^M = \pi_0(X_3 \setminus X_2)$	$r \in R^M$	finite group G_r
defect planes	$P^M = \pi_0(X_2 \setminus X_1)$	$p \in P^M$	finite $G_{L(p)} \times G^{op}_{R(p)}$ - set X_p
defect edges	$E^M = \pi_0(X_1 \setminus X_0)$	$e \in E^M$	representation $\rho_e : \mathcal{G}_e \to \mathrm{Vect}_{\mathbb{C}}$ $\mathcal{G}_e = (\Pi_{e \subset \bar{p}} X_p)//(\Pi_{e \subset \bar{r}} G_r)$
defect vertices	$V^M = X_0$	$v \in V^M$	intertwiner $\tau_v : \rho_v \Rightarrow \mathbb{C}$ $\rho_v = (\otimes_{s(e)=v} \bar{\rho}_e) \otimes (\otimes_{t(e)=v} \rho_e)$ $\mathcal{G}_v = (\Pi_{v \in \bar{p}} X_p)//(\Pi_{v \in \bar{r}} G_r)$

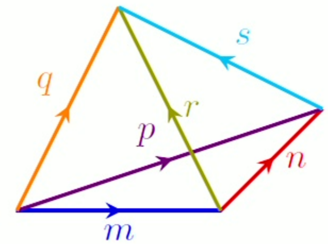
⇨ **defect data for ∂M**

regions	$r \in R^{\partial M}$	finite group G_r
defect edges	$e \in E^{\partial M}$	finite $G_{L(e)} \times G^{op}_{R(e)}$ - set X_e
defect vertices	$v \in V^{\partial M}$	representation $\rho_v : \mathcal{G}_v \to \mathrm{Vect}_{\mathbb{C}}$

1. thickening of defect stratification







thickened defect plane

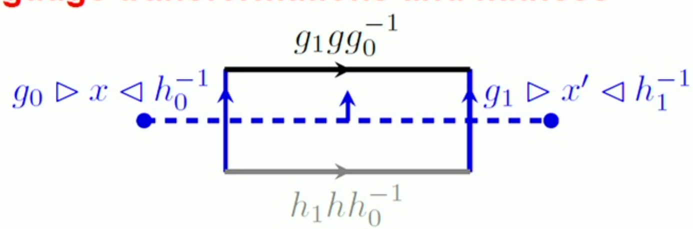
$$g_1 = g_3 g_2$$
$$h_1 = h_3 h_2$$
$$m_2 = g_1 \triangleright m_3 \triangleleft h_1^{-1}$$
$$m_2 = g_3 \triangleright m_1 \triangleleft h_3^{-1}$$
$$m_1 = g_2 \triangleright m_3 \triangleleft h_2^{-1}$$

thickened defect edge

$$m_1 = h \triangleright m_0 \triangleleft g^{-1}$$
$$n_1 = k \triangleright n_0 \triangleleft h^{-1}$$
$$p_1 = k \triangleright p_0 \triangleleft g^{-1}$$

thickened defect vertex

gauge transformations and flatness



⇨ **edges** of thickening: **gauge fields** - labeled with elements of groups and action sets

⇨ **vertices** of thickening: **gauge transformations** - labeled with group elements

⇨ **faces** of thickening: **representations**

2. dual quiver Q^M ⇨ category $\mathcal{Q}^M$

- vertices $V_Q = V^M \dot\cup E^M \dot\cup P^M \dot\cup R^M$
- edges $(v,e) : v \in V^M \to e \in E^M,\ v \in \bar{e}$
  $(e,p) : e \in E^M \to p \in p^M,\ e \subset \bar{p}$
  $(p,r) : p \in P^M \to r \in R^M,\ p \subset \bar{r}$

topological content functor $T^M : \mathcal{Q}^M \to \mathrm{Grpd}$

$$x \in V_Q \mapsto \Pi_1(\bar{x}, V_x)$$
$$(x,y) : x \to y \mapsto \Pi_1(\iota_{xy}) : \Pi_1(\bar{x}, V_x) \to \Pi_1(\bar{y}, V_y)$$

algebraic content functor $D^M : \mathcal{Q}^M \to \mathrm{Grpd}$

$$x \in V_Q \mapsto \mathcal{G}_x$$
$$(x,y) : x \to y \mapsto P_{xy} : \mathcal{G}_x \to \mathcal{G}_y$$

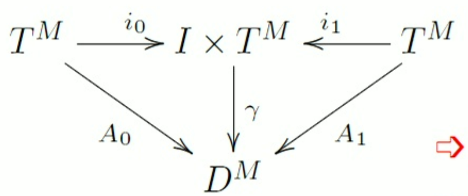
3. gauge fields and gauge transformations

- **gauge field** = natural transformation $A : T^M \Rightarrow D^M$

$r \in R^M \mapsto$ functor $H_r : \Pi_1(\bar{r}, V_r) \to G_r$

$p \in P^M \mapsto$ map $f_p : V_p \to M_p$ with flatness condition

- **gauge transformation** $\gamma : A_0 \Rrightarrow A_1$ = natural transformation $\gamma : I \times T^M \Rightarrow D^M$
  with $\gamma i_0 = A_0$ and $\gamma i_1 = A_1$



= assignment group elements - vertices of thickening

⇨ **groupoid** $\mathcal{A}^M//\mathcal{G}^M$ **of gauge fields and transformations**

⇨ **functor** $P_\partial : \mathcal{A}^M//\mathcal{G}^M \to \mathcal{A}^{\partial M}//\mathcal{G}^{\partial M}$

4. defect Dikgraaf-Witten TQFT

- 3-manifold M with defect stratification and boundary $\partial M = \bar{\Sigma}_1 \times \Sigma_2$

- **"classical" defect data**

  ⇨ groupoid $\mathcal{A}^M//\mathcal{G}^M$ of gauge fields and transformations

  ⇨ projection functors $P_1 : \mathcal{A}^M//\mathcal{G}^M \to \mathcal{A}^{\Sigma_1}//\mathcal{G}^{\Sigma_1}$

  $P_2 : \mathcal{A}^M//\mathcal{G}^M \to \mathcal{A}^{\Sigma_2}//\mathcal{G}^{\Sigma_2}$

  ⇨ fibrant span of groupoids

$$\mathcal{A}^{\Sigma_1}//\mathcal{G}^{\Sigma_1} \xleftarrow{P_1} \mathcal{A}^M//\mathcal{G}^M \xrightarrow{P_2} \mathcal{A}^{\Sigma_2}//\mathcal{G}^{\Sigma_2}$$

- **"quantum" defect data**

  defect data at boundary vertices ⇨ representations $\rho_1 : \mathcal{A}^{\Sigma_1}//\mathcal{G}^{\Sigma_1} \to \mathrm{Vect}_{\mathbb{C}}$

  $\rho_2 : \mathcal{A}^{\Sigma_2}//\mathcal{G}^{\Sigma_2} \to \mathrm{Vect}_{\mathbb{C}}$

  defect data at internal edges, vertices ⇨ intertwiner $\sigma : \rho_1 P_1 \Rightarrow \rho_2 P_2$

  ⇨ compatible with compositions of spans

  ⇨ linear map $\mathcal{Z}(M) : \lim \rho_1 \to \lim \rho_2$ induced by $\sigma_{A_1,A_2} = |\mathcal{G}^{\Sigma_2}||\mathcal{G}^M|^{-1} \sum_{A \in P_\partial^{-1}(A_1,A_2)} \sigma_A$

  $= \mathcal{Z}(\Sigma_1) \qquad = \mathcal{Z}(\Sigma_2)$

Outlook and Conclusions

- **summary**
  - gauge theoretical description of quantum double models with defects of all codimensions
  - gauge theoretical description of Dijkgraaf-Witten theory with defects of all codimensions
  - reproduces results of [Morton '10] [Fuchs-Schweigert-Valentino '13] in untwisted case, codim 1 defects
  - concrete, easy computations

- **To Dos**
  - compute for more complicated 3d examples
  - applications in knot theory? invariants of embedded manifolds?
  - inclusion of cocycles (twisted defect Dijkgraaf-Witten TQFT)
  - extension to quantum double models with defects for Hopf algebras
  - extension to defect Turaev-Viro TQFT with spherical fusion categories
  - generalisation to homotopy TQFT à la Quinn