Title: Twisted Tools for (Untwisted) Quantum Field Theory

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Collection: Higher Categorical Tools for Quantum Phases of Matter

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Twisted Tools for (Untwisted) Quantum Field Theory

Higher Categorical Tools for Quantum Phases of Matter

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with Davide Gaiotto, and Jingxiang Wu
and earlier works with Kasia Budzik,
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Outlines and Punchlines 1/2

- Discuss ideas from formal deformation theory in QFT
  - Familiar example is Ocneanu rigidity of fusion categories
- QFTs have "higher" **multilinear k-ary operations** ("brackets")

$$\{-, -, \ldots, -\} \tag{1}$$

  - Control: **deformations**, (generalized) **OPEs**, and **anomalies**
  - ∞-algebras, factorization algebras, and operads
- Familiar to high energy physicists and mathematicians who have studied twisted SQFTs (descent relations)
  - Not limited to twisted scenarios
- Can go very far in the case of (mixed) Holomorphic and/or Topological (HT) theories

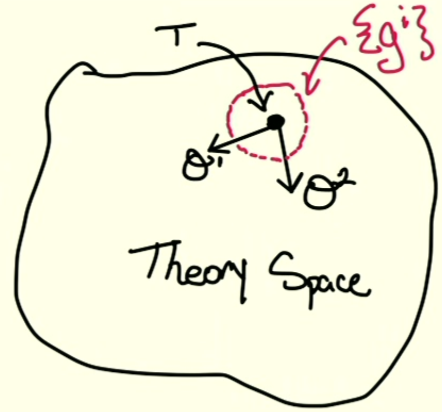
Outlines and Punchlines 2/2

I. Reminder on beta-function and BRST-symmetry
II. Introduce the eta-function and higher brackets
III. Example of computing the eta-function
IV. Generalizations to other scenarios
V. Quick survey of some results in holomorphic-topological scenarios

Three Punchlines

1. η-vector exists and is computable
2. η-vector contains anomalies, OPEs, and more
3. Non-renormalization theorem for HT theories

Deformations of QFTs



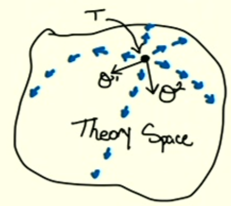
- Given a QFT T, it can be deformed by turning on interactions

$$S_T + \sum_i g^i \int_{\mathbb{R}^d} \mathcal{O}^i(x) d^d x \qquad (2)$$

  - g^i are coordinates on theory space
  - Work perturbatively in couplings g^i
- Defines a **formal pointed neighbourhood** $\mathcal{D}[T]$ of T, consisting of all effective QFTs obtained by perturbative deformation of T
  - Pointed because there is a distinguished point, called T.
  - Formal because we only consider deformations in an infinitesimal nbd of T (we are not at finite coupling).
  - Think of formal/infinitesimal as synonym for "perturbative"

The Beta Function

- Generic QFT (point) is not scale invariant
  - Scale transformation on T is traded for a change of the couplings e.g. integrating out DoF modifies couplings
- We encode infinitesimal scale transformations in vector field on theory space, the **beta function**



$$\boldsymbol{\beta} = \sum_i \beta^i(g) \frac{\partial}{\partial g^i} \,. \tag{3}$$

  - Perturb around (typically free) scale-invariant theory, $\boldsymbol{\beta} = 0$
  - Deformations of T preserving scale invariance are zeroes of $\boldsymbol{\beta}$
- The coefficients $\beta^i(g)$ are power series in g

$$\beta^i(g) = \underbrace{(d - \Delta_i)}_{\text{Classical}} g^i + O(g^2) \tag{4}$$

  - Tune relevant terms to 0 and study $\boldsymbol{\beta}$ as a measure of scale generated by "quantum effects"

The BV-BRST Formalism 1/2

- Quantization of non-abelian gauge theories is hard: formulated redundantly in exchange for other properties

$$Z = \int [DAD\bar{\psi}D\psi]e^{-S[A,\bar{\psi},\psi]} =: \int [D\Phi]e^{-S[\Phi]} , \qquad (5)$$

  - Introduce a gauge fixing procedure and Fadeev-Poppov ghosts b and c

$$Z = \int [D\Phi DB_A db_A dc^\alpha]e^{-S[\Phi]+iB_A F^A[\Phi]-b_A c^\alpha \delta_\alpha F^A[\Phi]} \qquad (6)$$

- Gauge fixed action still has residual nilpotent odd global symmetry involving fields and ghosts, called **BRST symmetry**.

$$\delta_{\mathrm{BRST}}\Phi = -i\epsilon c^\alpha \delta_\alpha \Phi , \qquad \delta_{\mathrm{BRST}} B_A = 0 ,$$
$$\delta_{\mathrm{BRST}} c^\alpha = \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma , \qquad \delta_{\mathrm{BRST}} b_A = \epsilon B_A , \qquad (7)$$

  - Physical theory can be identified with Q_{BRST}-cohomology

The BV-BRST Formalism 2/2

- Focus on theories defined in BV-BRST formalism:
  - T is embedded in a bigger ambient theory $\widetilde{T}$ with ghosts, anti-ghosts, anti-fields, etc.
  - Grassmann odd nilpotent symmetry Q_{BRST}
  - Observables in T are recovered from $\widetilde{T}$ by taking Q_{BRST} coho

$$\begin{aligned} \mathrm{Ops}_T &= (\mathrm{Ops}_{\widetilde{T}}, Q_{\mathrm{BRST}}) \\ \mathrm{Int}_T &= (\mathrm{Ops}_{\widetilde{T}}[dx], \mathrm{d} + Q_{\mathrm{BRST}}) \end{aligned} \tag{8}$$

- i.e. we will work in BV formalism
  - Essential to quantizing p-form gauge theories, theories which only close on-shell, field-dependent structure constants, or theories with other complicated constraints
  - Not restricted to such complicated theories either

The Eta Function

- Can compute analog of β **for any type of transformation**.
  - Ex. Non-relativistic scale transformations $(t, x) \mapsto (\lambda^z t, x)$
  - Ex. Anomalous axial transformation on θ angle in gauge theory
- Consider $T \hookrightarrow (\widetilde{T}, Q_{\mathrm{BRST}})$ described in a BRST formalism in terms of ambient $\widetilde{T}$
  - To deform T, we deform $\widetilde{T}$ without breaking BRST symmetry
  - Consider deformations of $\tilde{T}$ with Grassmann odd couplings, non-trivial ghost number, etc. This is formal pointed dg-supermanifold $\mathcal{D}[\widetilde{T}]$.
- BRST symmetry will be encoded in a vector field

$$\boldsymbol{\eta} = \sum_i \eta^i(g) \frac{\partial}{\partial g^i} \,. \tag{9}$$

  - Linear term tells us if adding an interaction $\mathcal{I}$ explicitly/classically violates BRST symmetry
  - Higher order terms do so “quantum mechanically”

HIGHER ALGEBRA

- Since $Q^2 = 0$, the eta function $\boldsymbol{\eta}^2 = 0$.
  - Wess-Zumino consistency condition for BRST symmetry
  - Gves **quadratic constraints** on coefficient functions $\eta^i(g)$

$$\eta^i(g) = \sum_{n>0} \frac{1}{n!} \sum_{j_1 \cdots j_n} \eta^i_{j_1 \cdots j_n} g^{j_1} \cdots g^{j_n}\,, \tag{10}$$

- Define the following multilinear operation $\mathrm{Int}^{\otimes n} \to \mathrm{Int}$

$$\{g^{j_1}\mathcal{I}_{j_1}, \cdots, g^{j_n}\mathcal{I}_{j_n}\} = \eta^i_{j_1 \cdots j_n} g^{j_1} \cdots g^{j_n} \mathcal{I}_i\,. \tag{11}$$

  - The BRST variation becomes

$$\boldsymbol{\eta}\mathcal{I} = \{\mathcal{I}\} + \frac{1}{2!}\{\mathcal{I},\mathcal{I}\} + \frac{1}{3!}\{\mathcal{I},\mathcal{I},\mathcal{I}\} + \dots \tag{12}$$

$\boldsymbol{\eta}^2 = 0 \quad \Leftrightarrow$ The coefficients $\eta^i_{j_1 \cdots j_n}$ and brackets $\{\cdot, \dots, \cdot\}$ define an $L_\infty[1]$**-algebra** structure on In.

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Basic Eta-Function Calculation - General 1/2

- η is not just abstract fun, it is computable fun
- Correlation functions of T deformed by $\mathcal{I}$ are correlation functions of T with additional insertions

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_{T+\mathcal{I}} = \left\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)e^{g\int d^dx\,\mathcal{I}}\right\rangle_T \quad (13)$$

  - At **linear order**, the BRST anomaly generated by $\mathcal{I}_i$ is just

$$\int_{\mathbb{R}^d}[Q,\mathcal{I}_i]\,. \quad (14)$$

  - Write

$$[Q,\mathcal{I}_i] = \sum_j \eta_i^j\,\mathcal{I}_j + d\mathcal{J}_i \quad (15)$$

- In perturbation theory, higher order terms

$$O(g^n) \sim \int_{\mathbb{R}^{dn}} \mathcal{I}(y_1)\cdots\mathcal{I}(y_n) \quad (16)$$

  - Need **regularization** to avoid UV divergences from colliding $\mathcal{I}$

Basic Eta-Function Calculation - General 2/2

- E.g. **at** $O(g^2)$ we can regularize the deformation to

$$\int_{\mathbb{R}^{2d}} f_\epsilon^{(2)}(x_1, x_2)\mathcal{I}(x_1)\mathcal{I}(x_2) \tag{17}$$

- Now we can compute

$$[Q, \int_{\mathbb{R}^{2d}} f_\epsilon^{(2)}(x_1, x_2)\ \mathcal{I}(x_1)\mathcal{I}(x_2)]\ |_{d\mathcal{J}} \tag{18}$$

$$= -\int_{\mathbb{R}^{2d}} df_\epsilon^{(2)}(x_1, x_2)(\mathcal{I}(x_1)\mathcal{J}(x_2) + \mathcal{J}(x_1)\mathcal{I}(x_2)) \tag{19}$$

- With a sharp cutoff (**point-splitting**) this becomes

$$\{\mathcal{I}, \mathcal{I}\}(x_2) \overset{\text{Sharp Cutoff}}{=} \int_{|x_{12}|=\epsilon} \mathcal{I}(x_1)\mathcal{J}(x_2) + \mathcal{J}(x_1)\mathcal{I}(x_2) \tag{20}$$

  - Anomaly appears in point-splitting regularization because total derivative terms give a boundary contribution.

Basic Eta-Function Calculation - Concrete 1/2

- 2d S_{Matter} with G global symmetry and G gauge theory

$$S_T = -\frac{1}{4}\int d^2x\, F_{\mu\nu}F^{\mu\nu} + S_{\text{Matter}} \tag{21}$$

Ex. Free fermions with vector current $J^\mu_a = \bar{\psi}\gamma^\mu t_a \psi$.

- Study the **interaction** $\mathcal{I} = A_\mu J^\mu$

- Add ghost and auxiliary fields $T \hookrightarrow (\widetilde{T}, Q)$
  - BRST transformation of $\mathcal{I}$ gives:

$$\delta_{\text{BRST}}(A_\mu J^\mu) = (\epsilon D_\mu c)J^\mu + A_\mu(i\epsilon g c J^\mu) = \epsilon\partial_\mu c J^\mu\,, \tag{22}$$

  - See $\mathcal{I}$ is BRST-closed up to total derivative $\mathcal{J} = cJ$
  - Term can potentially cause BRST anomaly

Basic Eta-Function Calculation - Concrete 2/2

- The two-bracket receives a contribution from the 2d JJ OPE:

$$\{\mathcal{I},\mathcal{I}\}(x_2) = \int_{|x_{12}|=\epsilon} :AJ:(x_1)\ :cJ:(x_2) + :cJ:(x_1)\ :AJ:(x_2)$$
$$= \oint_{S^1_{x_2}} (\ :A(x_1)c(x_2): + :c(x_1)A(x_2):\ )\ \langle J(x_1)J(x_2)\rangle$$
$$= \#\ \ :c\, dA:\ (x_2)\,. \tag{23}$$

  - We use $JJ \sim |x_{12}|^{-2}$, taylor expanded, and integrated by parts
  - $\#$ denotes combinatorial and rep-theoretic factors

- Recover well-known **1-loop anomaly for G-gauge theory**

$$\{A_\mu J^\mu, A_\nu J^\nu\} = \#\, cF_{12}\,. \tag{24}$$

  - In $2k$-dim, you recover anomaly from $(k+1)$-ary bracket

Recap and Further Connections

- Recap:
  - Deformations are integral to our understanding of QFT
  - Working in a BV-BRST formalism, we can introduce η that tracks violation of BRST symmetry due to interactions
  - η defines an L_∞-algebra on interactions and Maurer-Cartan equation is solved by well-defined deformations
  - η contains useful information like anomalies
- Descent operations in twisted theories
  - Higher brackets appear by colliding/integrating descendants
  - E.g. OPE and secondary product of cohomological TFTs is a 2-ary bracket
  - [Bomans, Wu] compute higher-central charges (a and c) of 4d SUSY gauge theories from brackets

GENERALIZATIONS

- Can consider **position dependent interactions**: causes momentum-inflow p^i at each vertex
  - L_∞-brackets extended to $\otimes_i \mathrm{In}_{p^{(i)}} \to \mathrm{In}_{\sum_i p^{(i)}}$
  - Momentum-coloured operad

$$\{\mathcal{I}_{i_1}\ {}_{p^{(1)}}\ \mathcal{I}_{i_2}\ {}_{p^{(2)}}\ \dots\ {}_{p^{(n-1)}}\ \mathcal{I}_{i_n}\} \tag{25}$$

- Distinguished subcase: holomorphic theories with holomorphic momentum λ recovers λ-brackets and higher n-Lie or homotopy conformal algebras
- **Auxiliary and defect systems**: the brackets of $T \times T_{\mathrm{probe}}$ extract information about T

  Ex. 't Hooft anomaly of S_{Matter} is apparent in the non-trivial bracket when coupled to G-gauge theory T_{probe}

  Ex. If T is topological QM, brackets of T recover Moyal commutator. Brackets with an auxiliary fermion recovers full Moyal-star product. 1d-topological defect brackets have A_∞.

Holomorphic-Topological Theories

Holomorphic-Topological Theories

- "**Holomorphic-Topological**" means flat spacetime has structure of $\mathbb{C}^H \times \mathbb{R}^T$ with coords $(x^{\mathbb{C}}, \bar{x}^{\mathbb{C}}, x^{\mathbb{R}})$
  - Anti-holomorphic translations in $\mathbb{C}^H$ and translations in $\mathbb{R}^T$ are gauge symmetries (Q_{BRST}-exact)
  - Interested in theories with action

$$\int_{\mathbb{C}^H \times \mathbb{R}^T} \left[(\Phi, \mathrm{d}\,\Phi) + \mathcal{I}(\Phi)\right]\, d^H x^{\mathbb{C}} \,. \tag{26}$$

  - Φ is a "superfield," and $dx^{\mathbb{R}}$ and $d\bar{x}^{\mathbb{C}}$ are "superspace coordinates"
- Appears in holomorphic-topological twists of SUSY theories.
  - Take cohomology of nilpotent supercharge Q_{SUSY}, i.e. "twist"
  - Translations are Q-exact; (cohomologically) HT sub-theory

$$\{Q, \bar{Q}_{\dot{\alpha}}\} = \partial_{\bar{z}^{\dot{\alpha}}} \tag{27}$$

- Can be used to prove a general non-renormalization theorem for theories with $T \geq 2$.

Holomorphic-Topological Integrals

- In such theories, we will be interested in brackets of the form

$$\{\mathcal{O}_1 \,{}_{\lambda_1} \cdots {}_{\lambda_{n-1}}\, \mathcal{O}_n\} \quad (28)$$

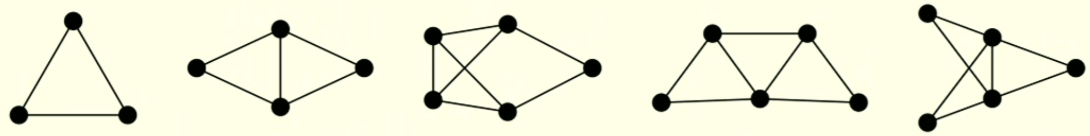
- The **Feynman integrals** that contribute will take the form:

$$I_\Gamma(\lambda; z) = \int_{\mathcal{M}^{|\Gamma_0|-1}} \left[\prod_{v\in\Gamma_0}^{v\neq v_*} \mathrm{dVol}_v\, e^{\lambda_v \cdot x_v^{\mathbb{C}}}\right] \mathrm{d}\left[\prod_{e\in\Gamma_1} P_\epsilon(x_{e(0)} - x_{e(1)} + z_e)\right]$$

- Let's count the form degree of the integrand:
  - (Regulated) propagator P_ϵ is a $(0, H+T-1)$ form
  - $(H+T)\times(|\Gamma_0|-1)$ integration variables: one for each vertex of graph, and throw one vertex away by translation symmetry.
  - $(|\Gamma_1|-1)$ regulated propagators and one $(0, H+T)$-form cut propagator $\mathrm{d}\, P_\epsilon(x)$

Holomorphic-Topological Feynman Diagrams

- Non-vanishing Feynman diagrams are **Laman graphs** and generalizations



- Feynman diagrams and integrals satisfy infinite collections of geometric identities relating to the configuration space of almost-Laman graphs
  - Feynman integrals satisfy infinite collection of **quadratic identities**
  - Identities imply (higher)-associativity of the accompanying brackets in a diagram-by-diagram way
  - Can **bootstrap** all Feynman integrals from these identities?
- Can use these Laman graphs to prove a completely combinatorial/diagrammatic proof of **perturbative non-renormalization** of $T \geq 2$ theories.

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Recap

I. Reviewed basic examples of deformations in QFT (β-function)
II. Introduced **η-function**; interactions have L_∞-algebra structure, tracks violation of BRST symmetry by interactions
III. η-function contains familiar data like anomaly data, and briefly discussed the relation to twisted SQFTs
IV. Introduced holomorphic-topological theories, and showed brackets are very strongly constrained (**Laman graphs**)
V. Laman graphs prove associativity relations, and no perturbative corrections when $T \geq 2$ topological directions.

Three Punchlines

1. η-vector exists and is computable
2. η-vector contains anomalies, OPEs, and more
3. Non-renormalization theorem for HT theories