Title: Bosonization and anomalies of 3d fermionic topological orders

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Abstract: here are two notions of a symmetry of a group G on a 3d topological order (TO): an "algebraic" symmetry, where G acts by automorphisms on the tensor category defining the (TO), and a "field-theoretic" symmetry, where the TFT corresponding to the TO is extended to manifolds with a principal G-bundle. The "field-theoretic" notion is stronger than the "algebraic" one, and the obstruction is sometimes referred to as the anomaly of the TO. The goal of this talk is to discuss a project joint with Weicheng Ye and Matthew Yu on computing these anomalies for fermionic TOs/spin TFTs: we develop a general framework employing Gaiotto-Kapustin's bosonic shadow construction. I will discuss both the mathematical conjectures our framework rests on as well as its use in
examples. The Smith long exact sequence appears in our computations.

Bosonization and anomalies of 3d fermionic topological orders

Arun Debray — March 22, 2024

arXiv:2312.13341 joint with Weicheng Ye and Matthew Yu

Symmetries of 3d topological orders

- Many three-dimensional oriented topological field theories (3d TFTs) Z are described by the data of a **modular tensor category** (MTC) C
- Two kinds of symmetry by a group G:
  - "Algebraic": $G \to \mathrm{Aut}(\mathsf{C})$
  - "Field-theoretic": the TFT Z is promoted to a TFT of 3-manifolds equipped with a principal G-bundle
- These notions are **not equivalent**: a field-theoretic symmetry is stronger than an algebraic one
- The **anomaly** of an algebraic symmetry is the obstruction to promoting it to a field-theoretic symmetry

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Anomalies as invertible field theories

- One way to think about (at least some) anomalies is as the data of an **invertible field theory** (IFT) in one dimension higher
- Freed-Hopkins show that this theory is determined by its partition functions, and that the partition functions are bordism invariants (see also Freed-Hopkins-Teleman, Yonekura, Rovi-Schoembauer, Kreck-Stolz-Teichner)
- Upshot: to calculate the anomaly of a G-action on an MTC, it suffices to calculate it on some small list of closed 4-manifolds
- These formulas for the anomaly on those manifolds are called **anomaly indicators**

What's new (and what isn't)

- Finding anomaly indicators for G-symmetries is settled, to an extent: for common choices of G, complete invariants for the anomaly of a G-action have been written down, and there are also tools to attack the problem for general G (Barkeshli-Bonderson-Cheng-Jian-Walker, Bulmash-Barkeshli, Barkeshli-Bonderson-Cheng-Wang, Wang-Levin, Lapa-Levin, Wang-Lin-Levin, Kobayashi-Barkeshli, Ye-Zou, ...)
- Things are very different in the **fermionic TO** case, corresponding to **3d spin TFT**: the constructions one would want to use do not exist yet, and the bordism computations are harder. (Tata-Kobayashi-Bulmash-Barkeshli)

What's new (and what isn't)

- We study anomaly indicators for group actions on super MTCs
- Given a group action on a super MTC, we produce a 4d TFT which is an invariant of the group action as well as a procedure for evaluating the TFT on a 4-manifold given a Kirby diagram
- We provide a conjecture implying that our TFT is the anomaly of the group action
- We study this in several examples, showing our method agrees with prior work by other methods

The story in the bosonic case

- Pick a group G and a homomorphism $a\colon G \to \{\pm 1\}$
- Have G act on a unitary MTC C
  - If $a(g) = 1$, g acts unitarily on C
  - If $a(g) = -1$, g acts antiunitarily on C

The story in the bosonic case

- We would like to extend the TFT Z_{C} to manifolds with a **(BG, a)-twisted orientation**: a principal G-bundle $P \to M$ and an identification of $a(P)$ with the orientation bundle of M
- The obstruction to doing so is a 4d invertible TFT $\alpha_{G,\mathsf{C}}$; its partition function is a bordism invariant $\Omega_4^{\mathrm{SO}}(BG, a) \to \mathbb{C}^\times$

Example: $\mathbb{Z}_2^T$

- Consider $G = \mathbb{Z}/2$ and a is the isomorphism, as studied by Wang-Levin, Barkeshli-Cheng, ...
- A $(B\mathbb{Z}/2, a)$-twisted orientation is the data of the orientation bundle, so no data at all
- $\Omega_4^{\mathrm{O}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by $\mathbb{RP}^4$ and $\mathbb{CP}^2$

$$\alpha(\mathbb{RP}^4) = \frac{1}{\mathcal{D}} \sum_{\text{anyons } b\colon\ \rho(b) = b} (\pm 1)\mathrm{qdim}(b) e^{i\theta_b},$$

where $\rho\colon \mathbb{Z}/2 \to \mathrm{Aut}(\mathsf{C})$ is the action and the sign is essentially whether ρ acts on b as 1 or -1.

$$\alpha(\mathbb{CP}^2) = \frac{1}{\mathcal{D}} \sum_{\text{anyons } b} (\mathrm{qdim}(b))^2 e^{i\theta_b}.$$

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Where do these formulas come from?

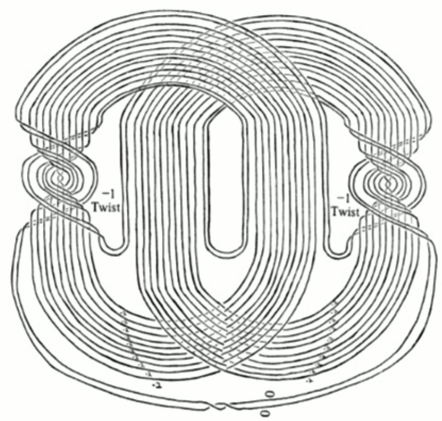
- There are general procedures for building anomaly indicator formulas using **triangulations** (Crane-Yetter, Bulmash-Barkeshli) or **Kirby diagrams** (Ye-Zou) for the generating 4-manifolds
- Upshot: "less complicated" generators lead to tractable formulas
- Given (G, a), how complicated can generators be? **In practice seems not too bad**

What changes in the fermionic case

- We're interested in porting this story over to **super MTCs**: braided fusion categories with Müger center sVect
- If C is a super MTC, $\mathbb{Z}/2$ acts by "$(-1)^F$": tensoring with the odd line in sVect
- So a G-action on a super MTC comes with the data $a\colon G \to \mathbb{Z}/2$ from before, together with an extension $1 \to \mathbb{Z}/2 \to \widetilde{G} \to G \to 1$ expressing how the symmetries mix with $(-1)^F$
- This data is classified by $b \in H^2(BG; \mathbb{Z}/2)$. (G, a, b) is called a **fermionic group** (Benson, Stolz, Stehouwer)
- The spacetime structure corresponding to the fermionic group (G, a, b) is a **(BG, a, b)-twisted spin structure**: a principal bundle $P \to M$ and identifications $w_1(M) = a(P)$ and $w_2(M) + w_1^2(M) = b(P)$ (B.L. Wang)

Things are more difficult in the fermionic case

- If $G = 1$, look at $\Omega_4^{\mathrm{Spin}} = \mathbb{Z}$, generated by the K3 surface and detected by the **A-roof genus** (aka **index of the Dirac operator**)
- The K3 surface has very complicated topology! Here's its Kirby diagram (source: Harer-Kas-Kirby)



- This is simply too complicated to turn into an anomaly indicator perimeter_3_22.pdf

Things are more difficult in the fermionic case

- A problem for more general (G, a, b): **spin Crane-Yetter theory doesn't exist yet** and it looks like it will be a significant undertaking
- This is needed in the bosonic case to construct the 4d invertible TFT α
- For a systematic approach, we must find a different way forward

Bosonization and fermionization

- If the spin structure is the problem, maybe we can make it go away...
- This is made real with the **bosonic shadow** construction of Gaiotto-Kapustin
- We use a generalization due to Tata-Kobayashi-Bulmash-Barkeshli for twisted spin structures

Bosonization and fermionization

- Let F_{Spin} be the 3d oriented TFT given by summing the trivial theory over spin structures, and $F_{\mathbb{Z}/2}$ be the analogous construction for $\mathbb{Z}/2$-bundles
- There is an $(F_{\mathrm{Spin}}, F_{\mathbb{Z}/2})$-defect z_c: tensoring with z_c turns 2d spin theories (boundaries for F_{Spin}) to 2d $\mathrm{SO} \times \mathbb{Z}/2$ theories (boundaries for F_{SO}) and vice versa
- Going from spin to $\mathrm{SO} \times \mathbb{Z}/2$ is called **bosonization**, and backwards is **fermionization**
- Concretely, tensoring with z_c amounts to: stack your theory with the theory $\alpha(\Sigma, \mathfrak{s}, P) \mapsto \mathrm{Arf}(\mathfrak{s} + P)$ and sum over spin structures (or, going the other way, sum over $\mathbb{Z}/2$-bundles)
- $z_c \otimes_{F_{\mathrm{Spin}}} z_c$ and $z_c \otimes_{F_{\mathbb{Z}/2}} z_c$ are Euler theories, so nearly trivial: this means that bosonization and fermionization are essentially inverses

Bosonization and fermionization

- In higher dimensions, this isn't quite possible, but Gaiotto-Kapustin modified it to make it work
- F_{Spin} is the same, but $F_{\mathbb{Z}/2}$ is replaced with a Dijkgraaf-Witten-type theory with fields classes $x \in H^{n-2}(-;\mathbb{Z}/2)$ and action $\int \mathrm{Sq}^2(x)$
- So bosonization and fermionization exchange spin theories with **anomalous** oriented theories
- These two operations are still essentially inverses
- Tata-Kobayashi-Bulmash-Barkeshli generalize this to (BG, a, b)-twisted spin TFTs $\longleftrightarrow$ (Bg, a)-twisted orientated TFTs with a $\mathbb{Z}/2$ coh class as before and a specified anomaly
- Note: far from symmetric monoidal, no guarantee this preserves invertibility

Bosonization and fermionization

- This suggests a heuristic approach to finding anomaly indicators:
- Begin with a 3d theory with a G-symmetry, and bosonize
- Calculate the anomaly TFT of the bosonized theory using prior work in the bosonic case
- Use Kirby diagrams at this stage to write down formulas on a set of generators!
- Fermionize the anomaly

Bosonization and fermionization

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What we did

First, the part that is settled:

Theorem (D.-Ye-Yu '23)

- *The construction sketched above, given a super MTC* C *and a fermionic group* (G, a, b) *acting on it, produces a 4d TFT* α *on* (BG, a, b)*-twisted spin manifolds*
- *Partition functions can be computed from a Kirby diagram and a small amount of extra data representing the twisted spin structure*

What we did

Next, the part that is still conjecture:

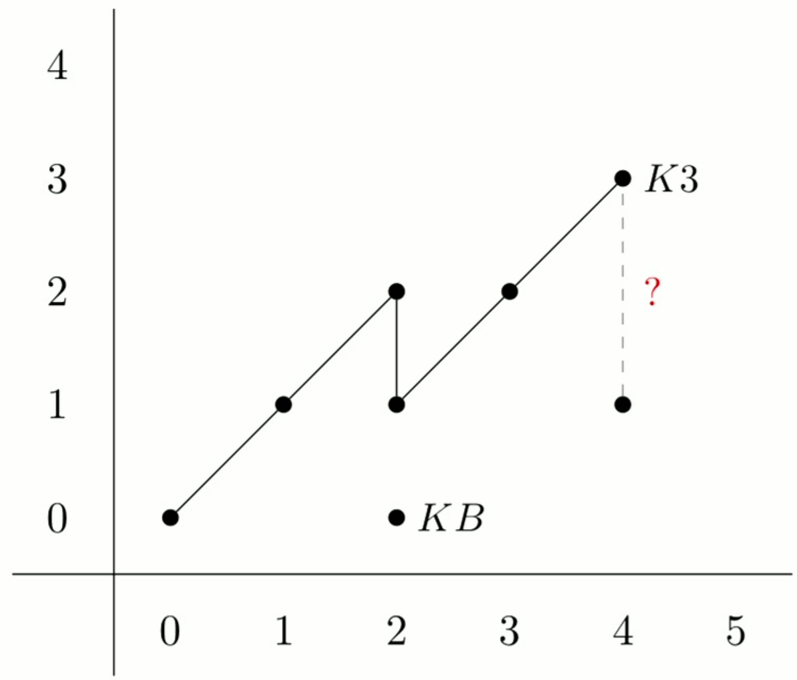
- **Is α invertible?** Bosonization/fermionization are not symmetric monoidal and do not preserve invertibility
- **Does α even have anything to do with the anomaly?**
- We provide a conjecture which implies these two; roughly, it says that **the bosonization of the anomaly is the anomaly of the bosonization**

Evidence for the conjecture

- Reproduces formulas obtained in work of Lapa-Levin, Kobayashi-Barkeshli, and Ning-Mao-Li-Wang for G the **tenfold way fermionic groups** of Freed-Hopkins
- New calculations ($\mathbb{Z}/4$, $a \neq 0$, $b = 0$) match computations by Delmastro-Gomis
- Calculations agree with those obtained by the **anomaly cascade method** of Bulmash-Barkeshli

The $\mathbb{Z}/4$ example

- Setting: $\mathbb{Z}/4$ **does not mix** with $(-1)^F$; the generator acts **antiunitarily**. Examples of this symmetry on super MTCs studied by Delmastro-Gomis
- The spacetime symmetry type $(B\mathbb{Z}/4, a, 0)$-twisted spin structure is called **epin structure** by Wan-Wang-Zheng
- Ω_4^{EPin} has been studied here and there since 1997 (Botvinnik-Gilkey, Barrera-Yanez, Wan-Wang-Zheng) but its structure was an open question
  - Botvinnik-Gilkey: Ω_4^{EPin} is $\mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$, and the **Adams spectral sequence** doesn't disambiguate between these options
  - Barrera-Yanez: **eta invariants** also don't disambiguate
  - The **Atiyah-Hirzebruch spectral sequence** also doesn't help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Does $\mathbb{Z}/4$ vs $\mathbb{Z}/2 \times \mathbb{Z}/2$ matter?

- You can look for a set of generators without knowing the exact isomorphism type of the group. However...
- Botvinnik-Gilkey's argument implies: $\Omega_4^{\mathrm{EPin}} = \mathbb{Z}/2 \times \mathbb{Z}/2$ if and only if one of the generators is a K3 surface
- So we really hope to get $\mathbb{Z}/4$; otherwise a complete set of anomaly indicators would be intractable!

Good news!

Theorem (D.-Ye-Yu '23)

$\Omega_4^{\mathrm{EPin}} \cong \mathbb{Z}/4$. *We may take as a generator a Klein bottle bundle* $\mathcal{M}$ *over* S^2

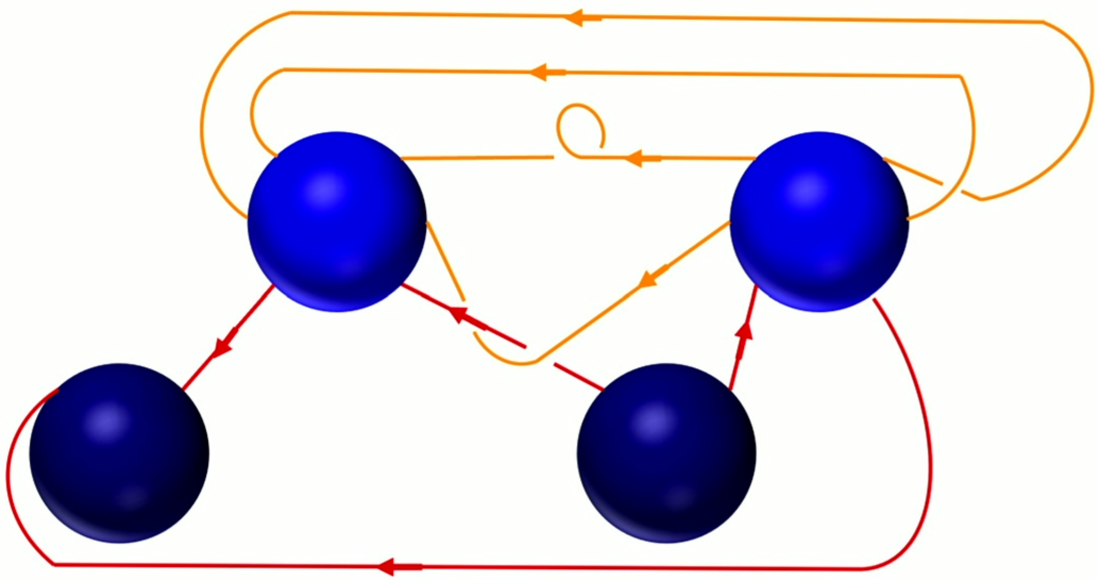
How to define $\mathcal{M}$:

- KB is the quotient of $\mathbb{C}$ by the group generated by $z \mapsto \overline{z} + 1/2$ and $z \mapsto z + i$
- The $\mathbb{R}$-action $z \mapsto z + t$ on $\mathbb{C}$ descends to an $\mathbb{R}/\mathbb{Z}$-action on the Klein bottle
- $\mathcal{M} := S^3 \times_{S^1} KB \to S^2$
- Twisted spin structure essentially induced from Klein bottle

Smith long exact sequence

- For determining both $\mathbb{Z}/4$ and the generator, we used the **Smith long exact sequence** of bordism groups (D.-Devalapurkar-Krulewski-Liu-Pacheco-Tallaj-Thorngren)
- Unrelated(?) to physics applications motivating that project, we used the LES to pass information from simpler symmetry types to EPin
- Interesting note: for $\mathbb{Z}/2^k$, $k > 2$, the same extension problem arises, but the bordism group is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$

Kirby Diagram for $\mathcal{M}$



Evaluating the anomaly indicator

- Plug the super MTC data into the formulas:
- The $\mathbb{Z}/4$-action on $\mathrm{U}(1)_5$ has trivial anomaly
- On $\mathrm{U}(1)_2 \times \mathrm{U}(1)_{-1}$, the anomaly is $2 \in \mathbb{Z}/4$
- On $\mathrm{SO}(3)_3$, the anomaly is $3 \in \mathbb{Z}/4$

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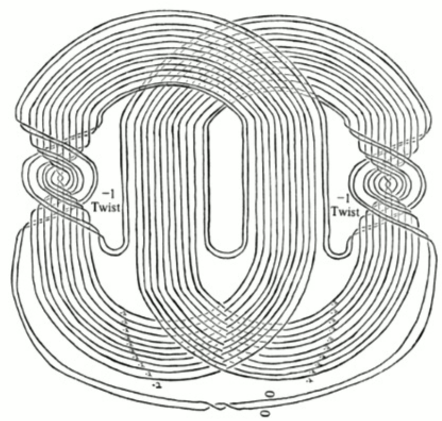

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