Title: Conference Talk

Speakers: Anton Kapustin

Collection: Higher Categorical Tools for Quantum Phases of Matter

Date: March 18, 2024 - 9:15 AM

URL: https://pirsa.org/24030078

Analogies between QFT and lattice systems: thoughts and open questions

Anton Kapustin

Department of Physics, Caltech

March 18, 2024

based in part on work with Nikita Sopenko, IAS, and conversations with Owen Gwilliam, U Mass, and Bas Janssens, TU Delft

Motivation

- There is no satisfactory mathematical formulation of QFT (although there are many semi-satisfactory attempts: Wightman, Haag-Kastler, Segal-Atiyah-Baez-Dolan-Lurie-...).
- But there is a fairly satisfactory theory of quantum statistical mechanics (QSM) of lattice systems

There are many relations between the two subjects. Can we leverage our understanding of lattice systems to shed light on QFT?

Relations between QFT and lattice

- Gapless degrees of freedom of lattice systems are often described by QFT (which typically lacks Lorentz-invariance)
- Low-energy physics of gapped lattice systems is often described by TQFT (but fractons!)
- Gapped lattice systems may have robust gapless edge modes described by a QFT

In the latter case, symmetries of the bulk system often act "anomalously" on the edge QFT.

It is believed that "all" 't Hooft anomalies of the edge QFT can be read off the ground state of the bulk system.

What about gravitational anomalies? I don't know.

Bulk-boundary correspondence

Assume that everything in sight is invariant under a compact Lie group G. For simplicity, I will let $G = U(1)$.

- Zero-temperature Hall conductance σ is a homotopy invariant of the ground state of a $U(1)$-invariant gapped infinite-volume 2d lattice systems.
- A "similar" system on a half-plane has robust gapless edge modes described by a $U(1)$-invariant QFT.
- The action of $U(1)$ on the edge QFT has 't Hooft anomaly: it cannot gauged (i.e. cannot be promoted to an action of the group of smooth maps $\mathbb{R} \to U(1)$)
- The obstruction to gauging is a single real number equal to σ.

Question: can we make the bulk-boundary correspondence "obvious"?

QSM kinematics

The "lattice" $\Lambda \subset \mathbb{R}^d$ is either $\mathbb{Z}^d$ or a more general Delone subset of $\mathbb{R}^d$. To each site $j \in \Lambda$ we attach a f.d. Hilbert space V_j, let $\mathscr{A}_j = End(\mathcal{V}_j)$.

Let $\mathscr{A}$ be the norm-completion of $\otimes_{j\in\Lambda}\mathscr{A}_j$. It is a C^*-algebra. Its elements are quasilocal observables.

Hilbert space appears "later", via the Gelfand-Naimark-Segal construction.

A "state" on $\mathscr{A}$ is a linear function $\omega : \mathscr{A} \to \mathbb{C}$ such that $\omega(a^*a) \geq 0$ for all $a \in \mathscr{A}$.

Given $(\mathscr{A}, \omega)$, the GNS construction produces a Hilbert space $\mathcal{H}$ and a representation $\pi : \mathscr{A} \to B(\mathcal{H})$.

Haag-Kastler approach to QFT

The Haag-Kastler attempt to axiomatize QFT is similar to QSM.

To every open space-time region U they attach a C^*-algebra $\mathscr{A}(U)$. If $U \subseteq V$, then $\mathscr{A}(U)$ is a sub-algebra of $\mathscr{A}(V)$.

In other words, we have a pre-cosheaf of algebras on space-time.

Definition

Let $\mathcal{C}$ be a category and X a topological space. A pre-cosheaf on X with values in $\mathcal{C}$ is an assignment of $\mathcal{F}(U) \in Ob(\mathcal{C})$ to every open $U \subset X$ and a morphism $\rho_{UV} \in Mor_{\mathcal{C}}(\mathcal{F}(U), \mathcal{F}(V))$ to every inclusion of opens $U \subset V$, so that for every three opens $U \subset V \subset W$ we have $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

Equivalently, a pre-cosheaf on X is a functor $\mathcal{F} : Open(X) \to \mathcal{C}$, where $Open(X)$ is a category whose objects are opens in X and morphisms are inclusions.

Pre-cosheaves and pre-sheaves

The morphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is an "extension morphism". In the case of the pre-cosheaf of observables, this is "extension by 1".

Another example of a pre-cosheaf: $U \mapsto \mathcal{F}(U) = C_c^\infty(U, \mathbb{R})$ (compactly-supported smooth functions on U). Here ρ_{UV} is "extension by 0". Could be used to model infinitesimal gauge transformations for $G = U(1)$.

Pre-sheaves are more familiar than pre-cosheaves. A pre-sheaf on X is a functor $Open(X)^{opp} \to \mathcal{C}$.

That is, if $U \subset V$ are opens of X, a pre-sheaf on X with values in $\mathcal{C}$ gives us a "restriction morphism" $\sigma_{UV} : \mathcal{F}(V) \to \mathcal{F}(U)$.

A sheaf is a special kind of pre-sheaf. Dually, one can define a cosheaf as a special kind of pre-cosheaf. More on this later.

Haag-Kastler vs QSM

A similar story in QSM would be to attach to every $U \subset \mathbb{R}^d$ a $*$-algebra $\mathscr{A}_l(U) = \otimes_{j \in (U \cap \Lambda)} \mathscr{A}_j$ or its completion $\mathscr{A}(U)$.

But in QSM it is not particularly natural to consider $\mathscr{A}(U)$, since evolution preserves locality only approximately.

Theorem (Lieb-Robinson bound)

Let $\alpha : \mathbb{R} \times \mathscr{A} \to \mathscr{A}$ be the action of time-translation group on $\mathscr{A}$ generated by a sufficiently nice Hamiltonian. Let $a \in \mathscr{A}_l(U)$. Then $\alpha(t)(a)$ is approximately localized on U for all $t \in \mathbb{R}$.

Say, if the interactions between $j, k \in \Lambda$ decay faster than any power of $d(j, k)$, then the "tails" of $\alpha(t)(a)$, $a \in \mathscr{A}_l(U)$, decay faster than any power of distance to U.

QSM dynamics

Dynamics is specified by a skew-adjoint derivation of $\mathscr{A}$:

$$a \mapsto \mathsf{H}(a) = \sum_{j \in \Lambda} [\mathsf{h}_j, a], \quad \mathsf{h}_j \in \mathscr{A}.$$

H is unbounded and is only defined on a dense domain $\mathscr{D} \subset \mathscr{A}$.

A sufficiently nice H can be exponentiated to a time-evolution $*$-automorphism $\alpha_{\mathsf{H}}(t) = e^{t\mathsf{H}} : \mathscr{A} \to \mathscr{A}$, $t \in \mathbb{R}$.

ω is a gapped ground state for H (or $\alpha_{\mathsf{H}}(t)$) if $\exists \Delta > 0$ such that

$$-i\omega(a^* \mathsf{H}(a)) \geq \Delta(\omega(a^* a) - |\omega(a)|^2), \quad \forall a \in \mathscr{D} \subset \mathscr{A}.$$

In terms of the GNS Hilbert space $\mathcal{H}$ for ω, this means that the vector $|0\rangle$ corresponding to ω is the only state with zero energy and the rest of the energy spectrum lies in $[\Delta, +\infty)$.

Symmetries

A Lie group symmetry in QSM is (roughly) a homomorphism $G \to Aut(\mathscr{A})$. But we also need to demand locality and smoothness.

A generator of a one-parameter symmetry is an unbounded densely-defined derivation $\mathsf{F} : \mathscr{D}_\mathsf{F} \to \mathscr{A}$. Locality means, roughly, that F has the form

$$a \mapsto \mathsf{F}(a) = \sum_j [\mathsf{f}_j, a], \quad \mathsf{f}_j \in \mathscr{A}, \quad a \in \mathscr{D}_\mathsf{F},$$

where f_j is "localized near j".

Technical issues:

- Need to define "localized near j"
- Do local symmetries form a Lie algebra?
- f_j are not uniquely defined

Rigging QSM

- There is a natural $*$-subalgebra $\mathscr{A}_{al} \subset \mathscr{A}$ of well-localized observables ($b \in \mathscr{A}_{al}$ means that b can be approximated by a local observable on a ball of radius r with $O(r^{-\infty})$ accuracy). $\mathscr{A}_{al}$ is a Frèchet algebra and dense in $\mathscr{A}$.
- Work with derivations which preserve $\mathscr{A}_{al}$:

$$b \mapsto \mathsf{H}(b) = \sum_j [\mathsf{h}_j, b],$$

where $\mathsf{h}_j \in \mathscr{A}_{al}$ is well-localized near j and uniformly bounded. Get a Frèchet-Lie algebra of "nice" derivations $\mathfrak{D}_{al}$. They have a common dense domain $\mathscr{D} = \mathscr{A}_{al} \subset \mathscr{A}$.

Let's assume Hamiltonians and generators of other symmetries ("charges") lie in $\mathfrak{D}_{al}$.

Locally generated automorphisms

Since $\mathfrak{D}_{al}$ is a Frèchet space, it makes sense to talk about continuous and smooth functions with values in $\mathfrak{D}_{al}$.

Theorem (Sopenko-AK)

Any F $\in C^0([0,1], \mathfrak{D}_{al})$ *can be "exponentiated" to a family of automorphisms* $\alpha(s) : \mathscr{A}_{al} \to \mathscr{A}_{al}$.

This follows from a version of Lieb-Robinson bounds proved by Nachtergaele, Ogata, and Sims.

Definition

An automorphism $\alpha : \mathscr{A} \to \mathscr{A}$ is called locally-generated (LGA) if it arises from such a family of derivations F. Two states ω, ω' are "in the same phase" if $\omega' = \omega \circ \alpha$ for some LGA α. A state is in a trivial phase if it is related by an LGA to an unentangled pure state ω_0.

Symmetries of systems and states

A symmetry of a system is a homomorphism α from an abstract group G to the group of LGAs. G can be discrete or a Lie group. Here we focus on Lie groups. Then one needs to require smooth dependence on coordinates of G.

For $G = U(1)$, a symmetry is determined by a "charge" $\mathsf{Q} \in \mathfrak{D}_{al}$ such that $\exp(2\pi \mathsf{Q}) = \mathrm{Id}$.

A Hamiltonian $\mathsf{H} \in \mathfrak{D}_{al}$ is G-invariant if $\alpha(g) \circ \mathsf{H} \circ \alpha(g)^{-1} = \mathsf{H}$ for all $g \in G$.

A state is G-invariant if $\omega \circ \alpha(g) = \omega$ for all g. For $G = U(1)$ this implies $\omega(\mathsf{Q}(a)) = 0$ for all $a \in \mathscr{A}_{al}$.

Definition. $\mathsf{F} \in \mathfrak{D}_{al}$ does not excite ω if $\omega(\mathsf{F}(a)) = 0$ for all $a \in \mathscr{A}_{al}$. Such F form a Lie subalgebra $\mathfrak{D}_{al}^{\omega} \subset \mathfrak{D}_{al}$.

Ambiguities in charge densities in field theory and QSM

Field theory: $\mathsf{Q} = \int_{\mathbb{R}^d} \rho$, where ρ is a d-form with values in observables.

Ambiguity: $\rho \mapsto \rho + d\alpha$, α is a $(d-1)$-form.

QSM: $\mathsf{Q} = \sum_{j \in \Lambda} \mathrm{ad}\, \mathsf{q}_j$, where q_j is well-localized near j.

Ambiguity: $\mathsf{q}_j \mapsto \tilde{\mathsf{q}}_j = \mathsf{q}_j + \sum_{k \in \Lambda} \mathsf{p}_{kj}$, where

- $\mathsf{p}_{kj} = -\mathsf{p}_{jk}$
- p_{jk} is well-localized near k and j
- $\|\mathsf{p}_{jk}\| = O\left(|j-k|^{-\infty}\right)$

Gauging the symmetry

Let Q be a generator of $U(1)$ symmetry of a state ω. That is, $Q \in \mathfrak{D}^{\omega}_{al}$ and $\exp(2\pi Q) = \mathrm{Id}$.

Can we promote $U(1)$ to a local symmetry of ω, at least on the infinitesimal level?

Theorem (Kitaev, Sopenko-AK)

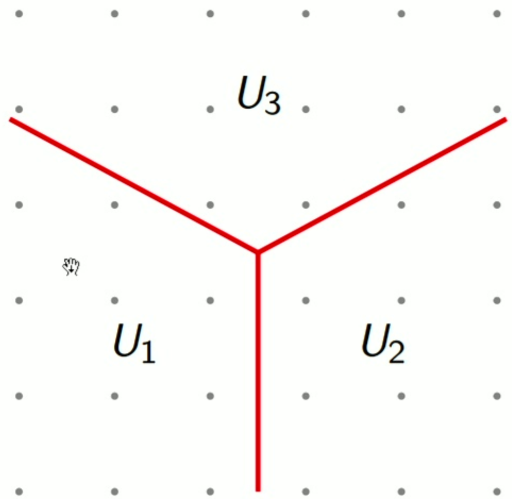
If ω is gapped, one can choose p_{jk}, $j, k \in \Lambda$, so that $\tilde{q}_j$ does not excite ω. Moreover, one can choose $\tilde{q}_j$ to be $U(1)$-invariant: $Q(\tilde{q}_j) = 0$ for all $j \in \Lambda$.

Does it mean every $U(1)$ symmetry of a gapped state can be gauged?

No, still need to ensure $[\tilde{q}_j, \tilde{q}_k] = 0$ for all j, k.

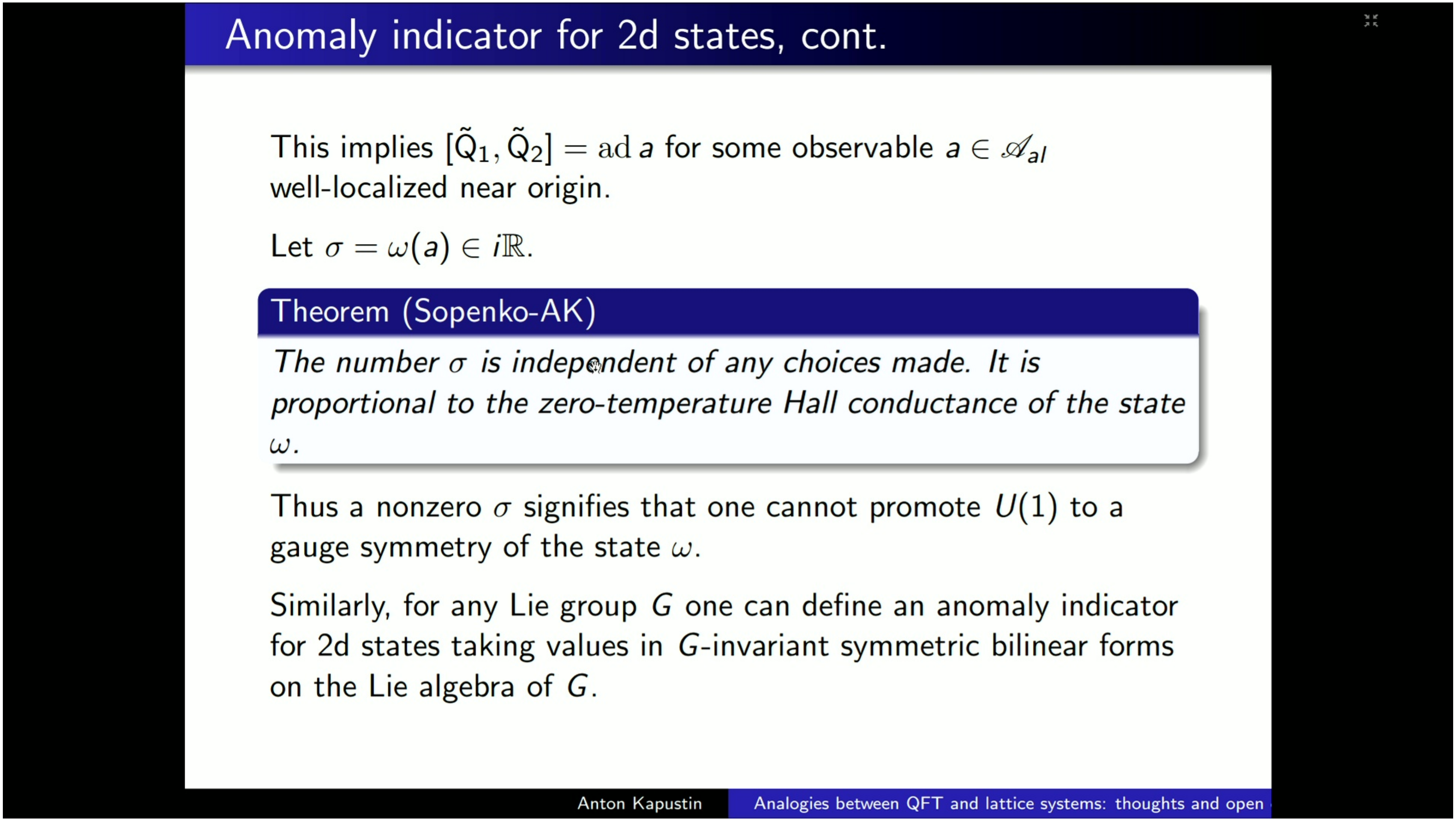
Anomaly indicator for 2d states

Let U_a, $a \in \{1, 2, 3\}$, be a decomposition of $\mathbb{R}^2$ into three cones:



Let $\tilde{Q}_a = \sum_{j \in U_a} \tilde{q}_j$, so that $Q = \tilde{Q}_1 + \tilde{Q}_2 + \tilde{Q}_3$. Then

$$[\tilde{Q}_1, \tilde{Q}_2] = [\tilde{Q}_1, \tilde{Q}_2 - Q] = -[\tilde{Q}_1, \tilde{Q}_3].$$


Anomaly indicator for 2d states, cont.
This implies $[\tilde{Q}_1, \tilde{Q}_2] = \mathrm{ad}\, a$ for some observable $a \in \mathscr{A}_{al}$ well-localized near origin.
Let $\sigma = \omega(a) \in i\mathbb{R}$.
Theorem (Sopenko-AK)
The number σ is independent of any choices made. It is proportional to the zero-temperature Hall conductance of the state ω.
Thus a nonzero σ signifies that one cannot promote $U(1)$ to a gauge symmetry of the state ω.
Similarly, for any Lie group G one can define an anomaly indicator for 2d states taking values in G-invariant symmetric bilinear forms on the Lie algebra of G.
Anton Kapustin
Analogies between QFT and lattice systems: thoughts and open

Some nagging questions

- Are there any other anomaly indicators?
- What is so special about cones?
- What happens if we replace the lattice Λ with some subset of Λ?
- What if we change the metric on $\mathbb{R}^d$?

To answer these questions, need to understand better what we are doing...

Sheaves on a topological space

Let $\mathcal{F}$ be a pre-sheaf of vector spaces on X.

Definition

$\mathcal{F}$ is a sheaf if for any open U and any open cover $\mathfrak{U} = \{U_a\}_{a \in A}$ of U one can relate $\mathcal{F}(U)$ to $\mathcal{F}(U_a)$ via an exact sequence

$$0 \to \mathcal{F}(U) \to \prod_{a \in A} \mathcal{F}(U_a) \to \prod_{a<b} \mathcal{F}(U_a \cap U_b).$$

Meaning: $s \in \mathcal{F}(U)$ is the same as a collection of $s_a \in \mathcal{F}(U_a)$, $a \in A$, which "agree" on all pairwise intersections $U_a \cap U_b$.

Cosheaves on a topological space

Let $\mathcal{F}$ be a pre-cosheaf of vector spaces on X.

Definition

$\mathcal{F}$ is a cosheaf if for any open U and any open cover $\mathfrak{U} = \{U_a\}_{a \in A}$ of U one can relate $\mathcal{F}(U)$ to $\mathcal{F}(U_a)$ via an exact sequence

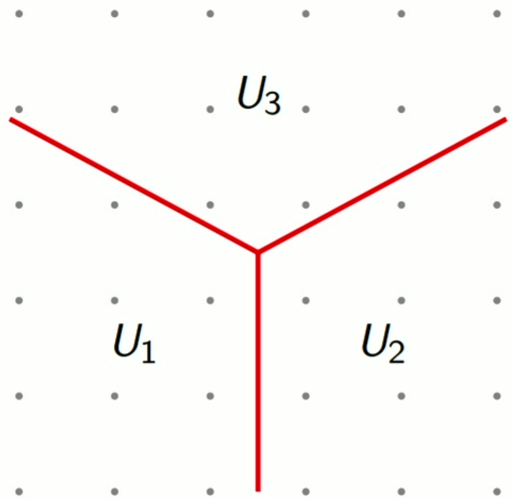
$$\oplus_{a<b} \mathcal{F}(U_a \cap U_b) \to \oplus_{a \in A} \mathcal{F}(U_a) \to \mathcal{F}(U) \to 0.$$

Meaning: $s \in \mathcal{F}(U)$ is the same as a collection of $s_a \in \mathcal{F}(U_a)$ modulo an ambiguity coming from pairwise intersections $U_a \cap U_b$.

(The difference between $\prod$ and $\oplus$ is that when A is infinite, every element of $\oplus_a \mathcal{F}(U_a)$ has only a finite number of nonzero entries).

Anomaly indicator for 2d states

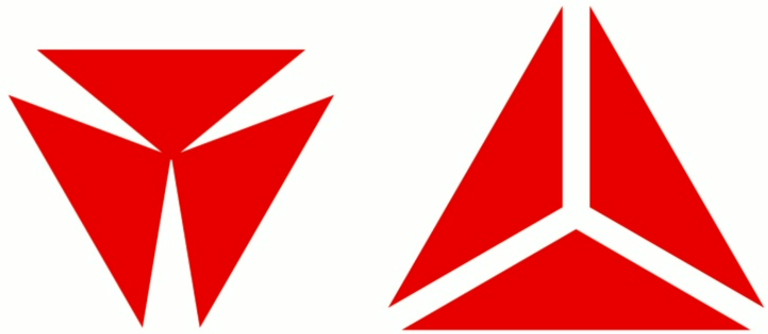
Let U_a, $a \in \{1, 2, 3\}$, be a decomposition of $\mathbb{R}^2$ into three cones:



Let $\tilde{Q}_a = \sum_{j \in U_a} \tilde{q}_j$, so that $Q = \tilde{Q}_1 + \tilde{Q}_2 + \tilde{Q}_3$. Then

$$[\tilde{Q}_1, \tilde{Q}_2] = [\tilde{Q}_1, \tilde{Q}_2 - Q] = -[\tilde{Q}_1, \tilde{Q}_3].$$

Regions that work and regions that don't



The three conical regions on the left do not sufficiently cover $\mathbb{R}^2$.
The three conical regions on the right do.

We need to come up with suitable notions of "generalized union" and "generalized intersection".

Grothendieck topologies in analysis

M. Kashiwara and P. Schapira: spaces of functions with growth conditions can be organized into a sheaf if one uses a "non-classical" notion of topology due to Grothendieck.

This allows one to apply techniques from algebraic geometry to analysis.

Similarly, spaces of functions with decay conditions can be organized into a cosheaf (of vector spaces) in the sense of Grothendieck topology.

In particular, this applies to spaces of derivations approximately localized on regions in $\mathbb{R}^d$.

Grothendieck topologies in a nutshell

- Replace the category $Open(X)$ with some other category $\mathcal{O}$
- Replace intersections of opens with categorical products in $\mathcal{O}$
- Replace unions of opens with coproducts in $\mathcal{O}$
- Define what it means for a $U \in Ob(\mathcal{O})$ to be covered by a collection of objects $U_a \in Ob(\mathcal{O})$, $a \in A$.

A Grothendieck topology on $\mathcal{O}$ is the last item in this list.

A category with a choice of Grothendieck topology is called a site. One can define sheaves and cosheaves on a site in the same way as one defines sheaves and cosheaves on a topological space.

Coherent topology on a pre-ordered set

There is a rather natural choice of Grothendieck topology when $\mathcal{O}$ is a "thin category", i.e. there is at most one morphism between any two $U, V \in Ob(\mathcal{O})$.

If $Mor(U, V)$ is non-empty, we write $U \leq V$. Then $Ob(\mathcal{O})$ becomes a pre-ordered set (proset), just like $Ob(Open(X))$.

A proset is a set Z with a reflexive and transitive relation $\leq$. A proset is exactly the same as a thin category.

In a poset, $U \leq V$ and $V \leq U$ imply $U = V$, but in a proset this is not required. It just means that U and V are isomorphic.

Coproduct becomes "join": $U, V \mapsto U \vee V$. Product becomes "meet": $U, V \mapsto U \wedge V$. Assume they exist.

Definition (Coherent topology on a proset with meets and joins)

$\{U_a, a \in A\}$ covers U iff $U \leq \vee_{b \in A} U_b$ and $U_a \leq U$ for all $a \in A$.

A category of fuzzy subsets

Let U be a subset of a metric space Λ (say, a Delone subset of $\mathbb{R}^d$). For any $r \geq 0$, let U^r be the r-thickening of U.

Definition

We say $U \leq V$ if $\exists r \geq 0$ such that $U \leq V^r$.

Thus all non-empty bounded subsets of Λ become isomorphic objects of the corresponding thin category $\mathcal{O}_\Lambda$.

Definition

Let $\mathfrak{D}^{\omega}_{al}(U)$ be the vector space of "nice" derivations which are approximately localized on U and do not excite ω. This is a pre-cosheaf on $\mathcal{O}_\Lambda$.

Are we done? No, because not every two U, V have a "meet" (generalized intersection).

Semi-linear site

Kashiwara and Schapira deal with a similar issue by allowing only special subsets of $\mathbb{R}^d$: semi-algebraic or sub-analytic.

In our case, it is natural to restrict to semi-linear subsets of $\mathbb{R}^d$.

Definition

A polyhedron is a subset of $\mathbb{R}^d$ defined by a finite number of linear inequalities. A semi-linear subset is a union of a finite number of polyhedra.

Let's use $\|\cdot\|_\infty$ distance on $\mathbb{R}^d$. Then for any semilinear (resp. polyhedral) U the set U^r is also semi-linear (resp. polyhedral).

The proset $\mathcal{O}_\Lambda^{sl} \subset \mathcal{O}_\Lambda$ has finite meets and joins, so it becomes a Grothendieck site.

$\mathfrak{D}_{al}^\omega : U \mapsto \mathfrak{D}_{al}^\omega(U)$ is a cosheaf of vector spaces on this site (and a pre-cosheaf of Lie algebras).

Anomaly indicator again

- The construction of the anomaly indicator depends on a choice of a cover $\{U_a, a \in A\}$ of $U = \Lambda$
- As well as the cosheaf property of $\mathfrak{D}^{\omega}_{al}$
- As well as on certain compatibility between the Lie algebra structure of $\mathfrak{D}^{\omega}_{al}(U)$ and cosheaf structure: $[\mathfrak{D}^{\omega}_{al}(U), \mathfrak{D}^{\omega}_{al}(V)] \subseteq \mathfrak{D}^{\omega}_{al}(U \wedge V)$.
- As well as the fact that every semi-linear set is isomorphic to a cone.

The last item explains what is special about cones, and which cones can be used to construct an anomaly indicator.

Remaining questions

- One still needs to show that different covers give rise to the same anomaly indicator (up to a scalar factor)
- What if we consider a lattice system concentrated on a cone in $\mathbb{R}^2$? Say, on a half-plane. All anomaly indicators are expected to vanish in this case. How do we argue this?

Need to refine the category $\mathcal{O}^{sl}_{\Lambda}$ by keeping more metric information: instead of saying $\exists r \geq 0 \; U \subset V^r$, remember the lower bound of the set of all such r.

Enriched categories

Let $\mathcal{W}$ be a symmetric monoidal category. A category $\mathcal{C}$ enriched in $\mathcal{W}$ has a set of objects, but $Mor(U, V)$ is no longer a set: it is an object of $\mathcal{W}$. Composition uses monoidal structure of $\mathcal{W}$.

Let's take $\mathcal{W}$ to be the ordered monoid $[0, +\infty]$. The monoidal structure is addition, category structure arises from the ordering. The category $\mathcal{O}_\Lambda^{sl}$ can be upgraded to a category enriched in $\mathcal{W}$.

Need to combine this with the coherent topology on $\mathcal{O}_\Lambda^{sl}$, somehow.