Title: Cosmology Lecture

Speakers: Kendrick Smith

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BACKGROUND $(\phi(t), a(t))$

EXACT EOMS $\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$

$$H^2 = \frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

SLOW ROLL APPROX, I.E. $\varepsilon = \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2$ $\eta = M_{pl}^2\left(\frac{V''}{V}\right)$

$$\Rightarrow \quad \dot{\phi} \approx -\frac{V'}{3H} \quad H^2 \approx \frac{V}{3M_{pl}^2}$$

FLUCTUATIONS

FLUCTUATIONS. $\underbrace{\zeta(\vec{k},t)}_{1\ \text{DOF}}$ $\underbrace{\gamma_{ij}(\vec{k},t)}_{2\ \text{DOFs}}$ $\underbrace{\gamma_{ii} = k_i \gamma_{ij} = 0}_{\text{CONSTRAINTS}}$ $\gamma_{ij} = \gamma_{ji}$

E.G. IF $\vec{k} = k\hat{z}$ CONSTRAINTS $\Rightarrow$ $\gamma_{ij}(\vec{k}) = \begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ 0 & 0 & 0 \end{pmatrix}$

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MUKHANOV-SASAKI ACTION:

$$S = \frac{1}{2}\int dt\, d^3x\, \frac{\dot{\phi}^2 a^3}{H^2}\left[\dot{\zeta}^2 - \frac{1}{a^2}(\partial_i \zeta)(\partial_i \zeta)\right]$$

$$+ \frac{M_{pl}^2}{8}\int dt\, d^3x\, a^3\left[\dot{\gamma}_{ij}\dot{\gamma}_{ij} - \frac{1}{a^2}(\partial_k \gamma_{ij})(\partial_k \gamma_{ij})\right]$$

$$\underbrace{\gamma_{ii} = k_i \gamma_{ij} = 0}_{\text{CONSTRAINTS}} \qquad \gamma_{ij} = \gamma_{ji}$$

$$\text{AINTS} \Rightarrow \gamma_{ij}(\vec{k}) = \begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\partial_i s)(\partial_i s) \Big]$$

$$\dot{\gamma}_{ij} - \frac{1}{a^2}(\partial_k \gamma_{ij})(\partial_k \gamma_{ij}) \Big]$$

$$\int D\phi \, e^{iS/\hbar}$$

TIME DEPENDENT HARMONIC OSCILLATOR

SETUP: $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega(t)^2\hat{x}^2$

$\omega(t) \to \omega_0$ AT E

STARTS IN $|\psi($

GOAL: COMPUTE "VARIANCE"

$$\langle \psi(t) | \hat{x}^2 | \psi(t) \rangle = \frac{1}{2\omega}$$

$$= |u(t)|^2$$

$\ddot{u}(t) = -$

$u(t) -$

SCHRODINGER PICTURE: $\frac{d}{dt}|\psi(t)\rangle = -i H_s(t) \, |\psi(t)\rangle$

EVOLUTION OPERATOR $|\psi(t)\rangle = \underbrace{U(t,t_0)} \, |\psi(t_0)\rangle$

$= e^{-iH(t-t_0)}$ IF H IS

$= T \exp\left(-i \int_{t_0}^{t} H(t') \, dt'\right)$

HEISENBERG PICTURE $\mathcal{O}_H(t) \equiv U(t,t_0)^\dagger \, \mathcal{O}_s(t) \, U(t,t_0)$

TIME

EQUIVALENCE

$$\langle \psi | \mathcal{O} | \psi \rangle = \langle \psi(t) | \mathcal{O}_S(t) | \psi(t) \rangle$$

$$= \langle \psi(t_0) | \mathcal{O}_H(t) | \psi(t_0) \rangle$$

PROPERTIES

$$(AB)_H = A_H B_H \qquad [A, B]_H = [A_H, B_H]$$

$$[\hat{x}_H(t), \hat{p}_H(t)] = [\hat{x}_S, \hat{p}_S]_H = (i)_H = i$$

"CANONICAL COMMUTATOR"

HEISENBERG EOM

$$\frac{d\hat{O}_H(t)}{dt} = i\underbrace{[\hat{H}_H(t), \hat{O}_H(t)]}_{= [H_S(t), O_S(t)]_H} + \left(\frac{d\hat{O}_S}{dt}\right)_H$$

AT EARLY TIMES

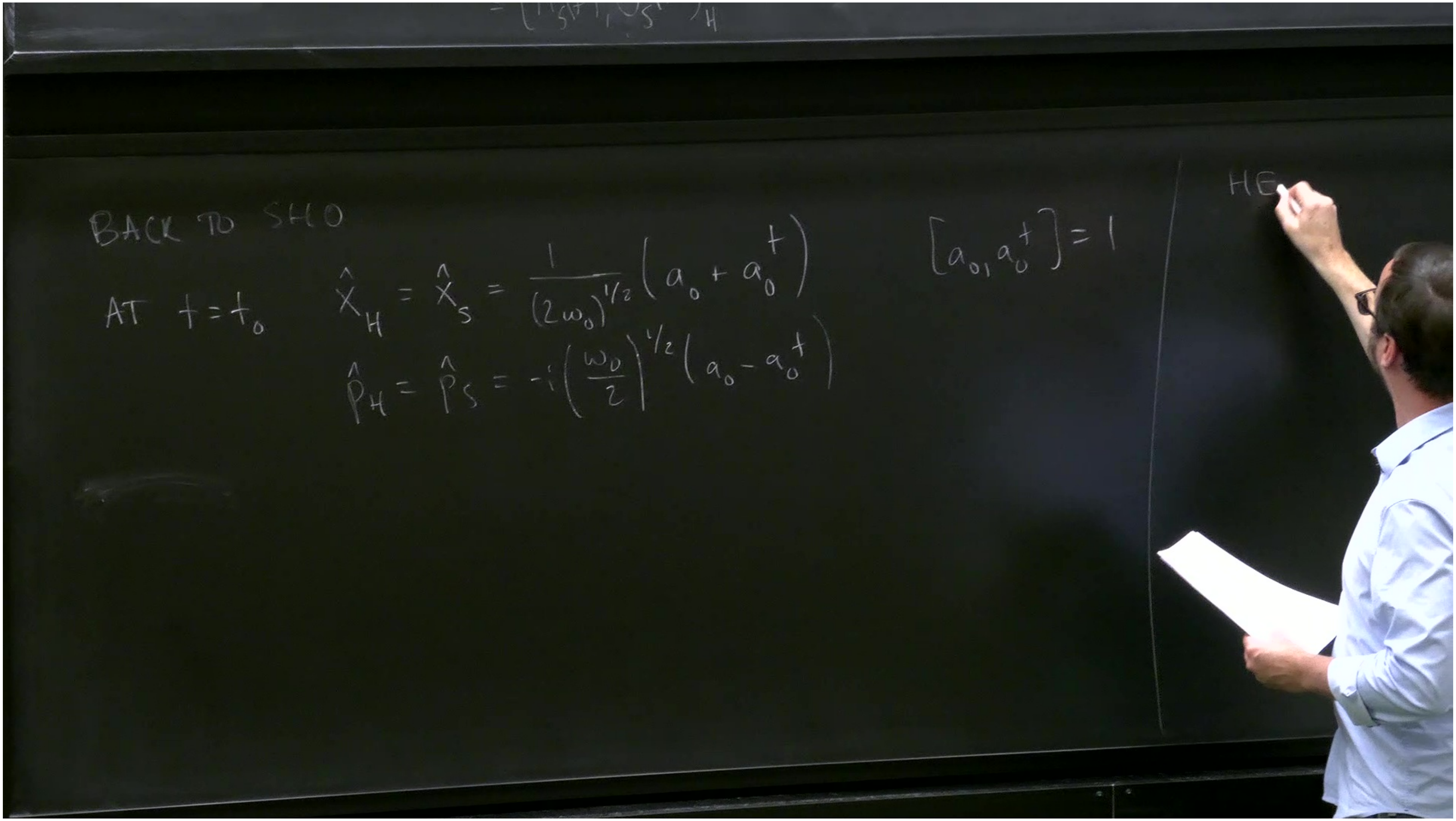
EQUIVALENCE

$$\langle \psi | \mathcal{O} | \psi \rangle = \langle \psi(t) | \mathcal{O}_S(t) | \psi(t) \rangle$$

$$= \langle \psi(t_0) | \mathcal{O}_H(t) | \psi(t_0) \rangle$$

NDEPENDENT OF TIME

$$\frac{d}{dt} U(t,t_0) = -iH\, U(t,t_0)$$


BACK TO SHO
AT $t = t_0$ $\hat{X}_H = \hat{X}_S = \frac{1}{(2\omega_0)^{1/2}}(a_0 + a_0^\dagger)$
$\hat{P}_H = \hat{P}_S = -i\left(\frac{\omega_0}{2}\right)^{1/2}(a_0 - a_0^\dagger)$
$[a_0, a_0^\dagger] = 1$
HE

HEISENBERG EOM

$$\frac{d\hat{x}_H}{dt} = i\left[\hat{H}_H, \hat{x}_H\right]$$

$$= i\left[\frac{1}{2}\hat{p}_H^2 + \underbrace{\cancel{\frac{\omega(t)^2}{2}\hat{x}_H^2}, \hat{x}_H}_{0}\right]$$

$$= \frac{i}{2}\left(\hat{p}_H\left[\hat{p}_H, \hat{x}_H\right] + \left[\hat{p}_H, \hat{x}_H\right]\hat{p}_H\right)$$

SINCE $\left[\hat{p}_H, \hat{x}_H\right] = -i$

$$= \hat{p}_H$$

SAME AS CLASSICAL EOM

ENBERG EOM

$$\frac{d\hat{X}_H}{dt} = i\left[\hat{H}_H, \hat{X}_H\right]$$

$$= i\left[\frac{1}{2}\hat{p}_H^2 + \cancel{\frac{\omega(t)^2}{2}\hat{X}_H^2}, \hat{X}_H\right]$$

$$0$$

$$= \frac{i}{2}\left(\hat{p}_H\left[\hat{p}_H, \hat{X}_H\right] + \left[\hat{p}_H, \hat{X}_H\right]\hat{p}_H\right)$$

SINCE $\left[\hat{p}_H, \hat{X}_H\right] = -i$

$$= \hat{p}_H$$

NE AS CLASSICAL EOM

SIMILARLY

$$\frac{d\hat{p}_H}{dt} = -\omega(t)^2\,\hat{X}_H$$

SAME

LINEAR EOMs $\Rightarrow$ $\hat{x}_H(t)$, $\hat{p}_H(t)$ MUST BE LINEAR COMBS OF a_0, $a_0^\dagger$ AT ALL TIMES

$$\begin{pmatrix} \hat{x}_H(t) \\ \hat{p}_H(t) \end{pmatrix} = \begin{pmatrix} u(t), & u(t)^* \\ \dot{u}(t) & \dot{u}(t)^* \end{pmatrix} \begin{pmatrix} a_0 \\ a_0^\dagger \end{pmatrix}$$

E LINEAR COMBS

$, a_0^\dagger$ AT ALL TIMES

$$\frac{d\hat{p}_H}{dt} = -\omega(t)^2 \hat{x}_H$$

$$\Rightarrow \boxed{\ddot{u} = -\omega(t)^2 u}$$

AT EARLY TIMES

$$u \to \frac{1}{(2\omega_0)^{1/2}}$$

$$\dot{u} \to -i\left(\frac{\omega_0}{2}\right)^{1/2}$$

$$\Rightarrow \boxed{u(t) \to \frac{1}{(2\omega_0)^{1/2}} e^{-i\omega_0 t}}$$ AT EARLY TIMES

GOAL. COMPUTE $\langle \psi | x^2 | \psi \rangle = \langle 0 | \hat{x}_H(t)^2 | 0 \rangle$

$= \langle 0 | \left(u(t)\, a_0 + u(t)^* a_0^\dagger \right) \left(u(t)\, a_0 + \dots \right.$

$= u(t)\, u(t)^* \langle 0 | a_0 a_0^\dagger | 0 \rangle$

$= |u(t)|^2$

i

$\shortparallel$

$[\hat{x}_H(t), \hat{p}_H(t)] = [u(t)\, a_0 + u(t)^* a_0^\dagger ,\ \dot{u}(t)\, a_0 + \dot{u}(t)^* a_0^\dagger]$

$= \mathrm{Det} \begin{pmatrix} u(t) & u(t)^* \\ \dot{u}(t) & \dot{u}(t)^* \end{pmatrix} = W(t)$ "WRONSKIAN DETERMINANT"

$0\rangle$

$$\frac{dW}{dt} = 0$$ FOLLOWS FROM EOMs $\ddot{u} = -\omega(t)^2 u$

$W(t_0) = i$ FOLLOWS SINCE $u(t) \to \frac{1}{(2\omega_0)^{1/2}} e^{-i\omega_0 t}$

AT EARLY

QFT EXAMPLE: MASSLESS SCALAR FIELD $\phi(x,t)$ ON A NONDYNAMICAL DE SITTER SPACETIME

"DE SITTER" $\Rightarrow$ FLRW WITH $H =$ CONST. AND $K = 0$

$a(t) = e^{Ht} \qquad -\infty < t < \infty$

$$ds^2 = -dt^2 + a(t)^2\, dx^2$$

$$= a(\tau)^2 \left[-d\tau^2 + dx^2 \right]$$

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$$= a(\tau)^2\left[-d\tau^2 + dx^2\right] \qquad a(\tau) = -\frac{1}{H\tau} \qquad -\infty < \tau < 0$$

FIELD $\phi(x,t)$

DE SITTER SPACETIME

NST. AND $K=0$

$Ht \qquad -\infty < t < \infty$

$\frac{1}{H\tau} \qquad -\infty < \tau < 0$

$$d\tau = \frac{dt}{a(t)} = \frac{dt}{e^{Ht}} \qquad \text{DE SITTER}$$

$$\Rightarrow \quad \tau = -\frac{1}{H} e^{-Ht} = -\frac{1}{aH}$$

$$\Rightarrow \quad a = -\frac{1}{H\tau}$$

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) \right)$$

$$g_{\mu\nu} = \begin{pmatrix} -a(\tau)^2 & \\ & a(\tau)^2 \delta_{ij} \end{pmatrix}$$

HEISEN

$$\frac{d\hat{x}_H}{dt}$$

FLRW

$$= \frac{1}{2} \int d\tau \, d^3x \; a(\tau)^2 \left[(\partial_\tau \phi)^2 - (\partial_i \phi)^2 \right]$$

DE SITTER

$$= \frac{1}{2} \int d\tau \frac{d^3k}{(2\pi)^3} \frac{1}{(H\tau)^2} \left((\partial_\tau \phi_k)(\partial_\tau \phi_k^*) - k^2 \phi_k \phi_k^* \right)$$

SAME