

Title: Cosmology Lecture

Speakers: Kendrick Smith

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BACKGROUND

$(\phi(t), a(t))$

FLUCTUATIONS

EXACT EOMS

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

$$H^2 = \frac{1}{3M_{pl}^2} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right)$$

SLOW ROLL APPROX, I.E.

$$\epsilon = \frac{M_{pl}^2}{2} \left(\frac{V'}{V} \right)^2 \quad \eta = M_{pl}^2 \left(\frac{V''}{V} \right)$$

$$\Rightarrow \dot{\phi} \approx -\frac{V'}{3H}$$

$$H^2 \approx \frac{V}{3M_{pl}^2}$$

FLUCTUATIONS

$$\underbrace{S(\vec{k}, t)}_{1 \text{ DOF}}$$

$$\underbrace{\chi_{ij}(\vec{k}, t)}_{2 \text{ DOFs}}$$

$$\underbrace{\chi_{ij} = k_i \chi_{ij} = 0}_{\text{CONSTRAINTS}}$$

$$\chi_{.j} = \chi_{j.}$$

E.G. IF $\vec{k} = k \hat{z}$

$$\text{CONSTRAINTS} \Rightarrow \chi_{ij}(\vec{k}) = \begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

FLUCTUATIONS

$$\xi(\vec{k}, t)$$

1 DOF

$$\delta_{ij}(\vec{k}, t)$$

2 DOFs

$$\delta_{ii} = k_i \delta_{ij} = 0$$

CONSTRAINTS

$$\delta_{ij} = \delta_{ji}$$

E.G. IF $\vec{k} = k \hat{z}$

$$\text{CONSTRAINTS} \Rightarrow \delta_{ij}(\vec{k}) = \begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

MUKHANOV-SASAKI ACTION:

$$S = \frac{1}{2} \int dt d^3x \frac{\dot{\phi}^2 a^3}{H^2} \left[\dot{\xi}^2 - \frac{1}{a^2} (\partial_i \xi)(\partial_i \xi) \right]$$

$$+ \frac{M_{pl}^2}{8} \int dt d^3x a^3 \left[\dot{\delta}_{ij} \dot{\delta}_{ij} - \frac{1}{a^2} (\partial_\mu \delta_{ij})(\partial_\mu \delta_{ij}) \right]$$

$$\delta_{ij} = \lambda_{ij} \delta_{ij} = 0$$

CONSTRAINTS

$$\gamma_{ij} = \gamma_{ji}$$

$$\text{CONSTRAINTS} \Rightarrow \gamma_{ij}(\vec{k}) = \begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\partial_i \psi)(\partial_i \psi)$$

$$\delta_{ij} - \frac{1}{a^2} (\partial_u \delta_{ij})(\partial_u \delta_{ij})$$

$$\int D\psi e^{iS/\hbar}$$

TIME DEPENDENT HARMONIC OSCILLATOR

SETUP: $\hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{2} \omega(t)^2 \hat{x}^2$

$\omega(t) \rightarrow \omega_0$ AT $t \rightarrow \infty$

STARTS IN $|\psi(t=0)\rangle$

GOAL: COMPUTE "VARIANCE" $\langle \psi(t) | \hat{x}^2 | \psi(t) \rangle = \frac{1}{2\omega(t)}$

"
 $|\psi(t)\rangle^2$

$\ddot{u}(t) = -\omega(t)^2 u(t)$

SCHRODINGER PICTURE: $\frac{d}{dt} |\psi(t)\rangle = -i H_S(t) |\psi(t)\rangle$

EVOLUTION OPERATOR $|\psi(t)\rangle = \underbrace{U(t, t_0)}_{= e^{-iH(t-t_0)} \text{ IF } H \text{ IS}} |\psi(t_0)\rangle$
 $= T \exp\left(-i \int_{t_0}^t H(t') dt'\right)$

HEISENBERG PICTURE $\mathcal{O}_H(t) \equiv U(t, t_0)^\dagger \mathcal{O}_S(t) U(t, t_0)$

EQUIVALENCE

$$\langle \psi | \mathcal{O} | \psi \rangle = \langle \psi(t) | \mathcal{O}_S(t) | \psi(t) \rangle$$
$$= \langle \psi(t_0) | \mathcal{O}_H(t) | \psi(t_0) \rangle$$

TIME

PROPERTIES

$$(AB)_H = A_H B_H \quad [A, B]_H = [A_H, B_H]$$

$$[\hat{X}_H(t), \hat{P}_H(t)] = [\hat{X}_S, \hat{P}_S]_H = (i)_H = i$$

"CANONICAL
COMMUTATOR"

HEISENBERG EOM

$$\frac{d\hat{O}_H(t)}{dt} = i \underbrace{[\hat{H}_H(t), \hat{O}_H(t)]}_{= [\hat{H}_S(t), \hat{O}_S(t)]_H} + \left(\frac{d\hat{O}_S}{dt} \right)_H$$

AT EARLY TIMES

EQUIVALENCE

$$\langle \psi | \mathcal{O} | \psi \rangle = \langle \psi(t) | \mathcal{O}_S(t) | \psi(t) \rangle$$
$$= \langle \psi(t_0) | \mathcal{O}_H(t) | \psi(t_0) \rangle$$

INDEPENDENT OF TIME

$$\frac{d}{dt} U(t, t_0) = -iH U(t, t_0)$$

$$= \frac{1}{2} (a_0 + a_0^\dagger)$$

BACK TO SHLO

AT $t = t_0$

$$\hat{X}_H = \hat{X}_S = \frac{1}{(2\omega_0)^{1/2}} (a_0 + a_0^\dagger)$$
$$\hat{P}_H = \hat{P}_S = -i \left(\frac{\omega_0}{2} \right)^{1/2} (a_0 - a_0^\dagger)$$

$$[a_0, a_0^\dagger] = 1$$

HE

HEISENBERG EOM

$$\frac{d\hat{x}_H}{dt} = i \left[\hat{H}_H, \hat{x}_H \right]$$

$$= i \left[\frac{1}{2} \hat{p}_H^2 + \frac{\omega(H)^2}{2} \hat{x}_H, \hat{x}_H \right]$$

$$= \frac{i}{2} \left(\hat{p}_H [\hat{p}_H, \hat{x}_H] + [\hat{p}_H, \hat{x}_H] \hat{p}_H \right)$$

$$= \hat{p}_H \quad \text{SINCE } [\hat{p}_H, \hat{x}_H] = -i$$

SAME AS CLASSICAL EOM

HEISENBERG EOM

$$\begin{aligned}\frac{d\hat{X}_H}{dt} &= i \left[\hat{H}_H, \hat{X}_H \right] \\ &= i \left[\frac{1}{2} \hat{P}_H^2 + \frac{\omega(t)^2}{2} \hat{X}_H^2, \hat{X}_H \right] \\ &= \frac{i}{2} \left(\hat{P}_H [\hat{P}_H, \hat{X}_H] + [\hat{P}_H, \hat{X}_H] \hat{P}_H \right) \\ &= \hat{P}_H \quad \text{SINCE } [\hat{P}_H, \hat{X}_H] = -i\end{aligned}$$

WE AS CLASSICAL EOM

SIMILARLY

$$\frac{d\hat{P}_H}{dt} = -\omega(t)^2 \hat{X}_H$$

SAME

LINEAR EOMS $\Rightarrow \hat{X}_H(t), \hat{P}_H(t)$ MUST BE LINEAR COMBS
OF a_0, a_0^+ AT ALL TIMES

$$\begin{pmatrix} \hat{X}_H(t) \\ \hat{P}_H(t) \end{pmatrix} = \begin{pmatrix} u(t) & u(t)^* \\ \dot{u}(t) & \dot{u}(t)^* \end{pmatrix} \begin{pmatrix} a_0 \\ t \\ a_0 \end{pmatrix}$$

LINEAR COMBS
 a_0^+ AT ALL TIMES

$$\frac{d\hat{p}_H}{dt} = -\omega(t)^2 \hat{x}_H$$
$$\Rightarrow \ddot{u} = -\omega(t)^2 u$$

AT EARLY TIMES

$$u \rightarrow \frac{1}{(2\omega_0)^{1/2}}$$
$$\dot{u} \rightarrow -i \left(\frac{\omega_0}{2}\right)^{1/2}$$

$$\Rightarrow u(t) \rightarrow \frac{1}{(2\omega_0)^{1/2}} e^{-i\omega_0 t}$$

AT EARLY TIMES

GOAL. COMPUTE $\langle \psi | x^2 | \psi \rangle = \langle 0 | \hat{x}_H(t)^2 | 0 \rangle$

$$= \langle 0 | (u(t) a_0 + u(t)^* a_0^\dagger) (u(t) a_0 + u(t)^* a_0^\dagger) | 0 \rangle$$

$$= u(t) u(t)^* \langle 0 | a_0 a_0 | 0 \rangle$$

$$= |u(t)|^2$$

$$[\hat{x}_H(t), \hat{p}_H(t)] = [u(t) a_0 + u(t)^* a_0^\dagger, \dot{u}(t) a_0 + \dot{u}(t)^* a_0^\dagger]$$

$$= \text{Det} \begin{pmatrix} u(t) & u(t)^* \\ \dot{u}(t) & \dot{u}(t)^* \end{pmatrix} = W(t) \text{ WRONSKIAN DETERMINANT}$$

0 >

$$\frac{dw}{dt} = 0$$

FOLLOWS FROM EOMs $\ddot{u} = -\omega(t)^2 u$

$$W(t_0) = i$$

FOLLOWS SINCE

$$u(t) \rightarrow \frac{1}{(2\omega_0)^{1/2}} e^{i\omega_0 t}$$

AT EARLY

QFT EXAMPLE

MASSLESS SCALAR FIELD $\psi(x, t)$
ON A NONDYNAMICAL DE SITTER SPACETIME

"DE SITTER" \Rightarrow FLRW WITH $H = \text{CONST.}$ AND $K = 0$

$$ds^2 = -dt^2 + a(t)^2 dx^2$$

$$a(t) = e^{Ht} \quad -\infty < t < \infty$$

$$= a(t)^2 [-d\tau^2 + dx^2]$$

QFT EXAMPLE → MASSLESS SCALAR FIELD $\phi(x, t)$
ON A NONDYNAMICAL DE SITTER SPACETIME

"DE SITTER" \Rightarrow FLRW WITH $H = \text{CONST.}$ AND $K = 0$

$$ds^2 = -dt^2 + a(t)^2 dx^2$$

$$= a(\tau)^2 [-d\tau^2 + dx^2]$$

$$a(t) = e^{Ht} \quad -\infty < t < \infty$$

$$a(\tau) = -\frac{1}{H\tau} \quad -\infty < \tau < 0$$

FIELD $\psi(x, t)$

DE SITTER SPACETIME

CONST. AND $K=0$

$$Ht \quad -\infty < t < \infty$$

$$\frac{1}{H\tau} \quad -\infty < \tau < 0$$

$$d\tau = \frac{dt}{a(t)} = \frac{dt}{e^{Ht}} \quad \text{DE SITTER}$$

$$\Rightarrow \tau = -\frac{1}{H} e^{-Ht} = -\frac{1}{aH}$$

$$\Rightarrow a = -\frac{1}{H\tau}$$

$$\begin{aligned}
 S &= \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) \right) & g_{\mu\nu} &= \begin{pmatrix} -a(\tau)^2 & \\ & a(\tau)^2 \delta_{ij} \end{pmatrix} \\
 &= \frac{1}{2} \int d\tau d^3x a(\tau)^2 \left[(\partial_\tau \phi)^2 - (\partial_i \phi)^2 \right] & & \text{FLRW} \\
 &= \frac{1}{2} \int d\tau \frac{d^3k}{(2\pi)^3} \frac{1}{(H\tau)^2} \left[(\partial_\tau \phi_k) (\partial_\tau \phi_k^*) - k^2 \phi_k \phi_k^* \right] & & \text{DE SITTER}
 \end{aligned}$$

HEISEN

$\frac{dx_H}{dt}$

JANE