Title: Quantum Information Lecture

Speakers: Eduardo Martin-Martinez

Collection: Quantum Information 2023/24

Date: March 22, 2024 - 9:00 AM

URL: https://pirsa.org/24030060

9.4 Quantum Logic

Now that we played with classical gates, let us move on to quantum ones.

9.4.1 Single qubit gates

We consider single qubits in the basis:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{114}$$

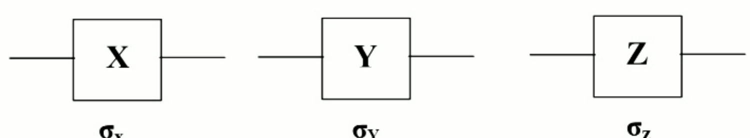
Then, an arbitrary state can be expressed as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{115}$$

Pauli gates The Pauli gates (X,Y,Z) correspond to the three Pauli matrices $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ acting on single qubits.

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{116}$$

and are represented in circuits as shown in figure 3. Notice that the X gate, is a NOT gate in the sense that it flips the components of the vector state.



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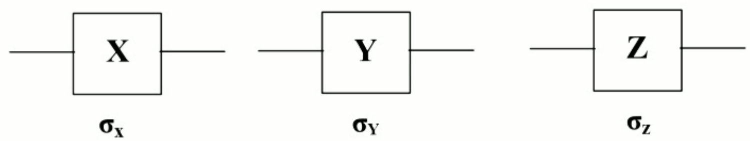
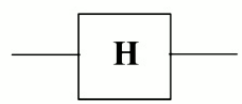


Figure 3: Pauli gates

Hadamard Another elementary gate is the Hadamard gate. The corresponding operator is:

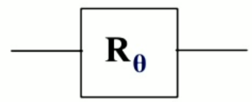
Figure 4: Hadamard gate



$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{117}$$

Phase rotation gate Phase rotation gate acts by adding a phase to the second component. The operator for the phase rotation is:

Figure 5: Phase Rotation gate.



$$\hat{R}_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \tag{120}$$

9.4.2 Classical bits and measurements

Measurements in the computational basis are represented as shown in figure 6. The double line corresponds to a classical bit, in this case it is storing the measurement outcome. In the qubit after the measurement is discontinued, this means that the measurement is destructive.

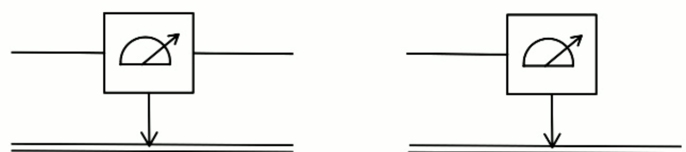


Figure 6: Measurement representation in quantum circuits.

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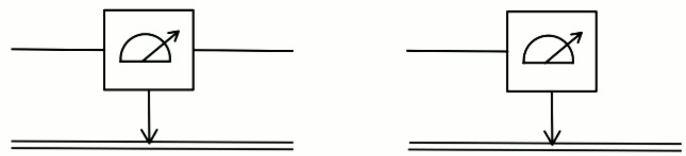

Figure 6: Measurement representation in quantum circuits.

Classically-controlled NOT A classically-controlled NOT is a quantum NOT gate, i.e. a X-Pauli gate, that is controlled by a classical bit (see figure 7). So the classical information controls the quantum operation.

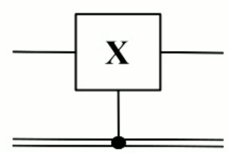

Figure 7: Classically-controlled NOT gate.

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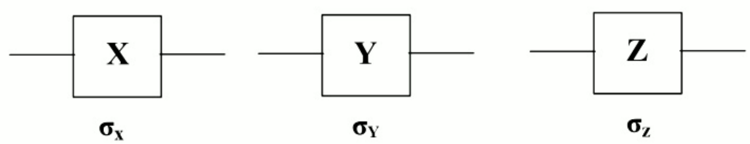

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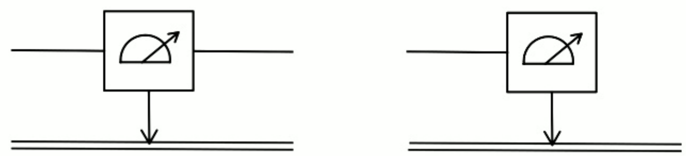

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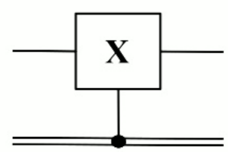

Figure 7: Classically-controlled NOT gate.

9.4.3 Two qubit gates

For two qubits we work in a basis where:

$$|00\rangle \to |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |01\rangle \to |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle \to |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |11\rangle \to |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

CNOT The CNOT gate is a two qubit gate that maps:

$$\text{CNOT}\,|a,\, b\rangle \to |a,\, a \oplus b\rangle. \tag{121}$$

Acting with CNOT on the basis vectors yields:

$$\begin{aligned} |00\rangle &\to |00\rangle & |01\rangle &\to |01\rangle \\ |10\rangle &\to |11\rangle & |11\rangle &\to |10\rangle \end{aligned}$$

and is represented in circuits as shown in the following figure.

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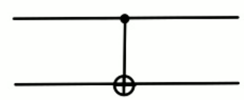


Figure 8: CNOT gate

In the chosen basis, the matrix representation of the CNOT gate is the following operator:

$$\text{CNOT} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \sigma_x \end{pmatrix} \tag{122}$$

TOFFOLI The quantum version of the TOFFOLLI gate acts again as a

Figure 9: TOFFOLI gate

SWAP The SWAP gate swaps the elements of the vector states.

$$a = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \rightarrow b = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \tag{125}$$

$$b = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \rightarrow a = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \tag{126}$$

In circuits we represent the SWAP gate as in figure 10,

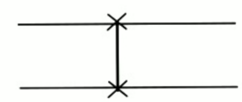


Figure 10: SWAP gate

while the corresponding matrix operator is:

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{127}$$

The "control" qubit in the quantum case. In the classical case, it is

Gates.nb

```
In[107]:= (*Pauli Gates*)
```

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \; Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

```
(*Hadamard and Phase Rotation*)
```

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \quad R[\theta\_] := \begin{pmatrix} 1 & 0 \\ 0 & \text{Exp}[i\,\theta] \end{pmatrix};$$

```
(*CNOT AND SWAP*)
```

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \; \text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

```
(*Toffoli*)
```

$$\text{TOF} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

(125)

(126)

(127)

...se, it is ...t in the quantum case too? Let's take for example the circuit in figure 11. If we apply a Hadamard gate in all qubits and change basis for the qubits, we can show

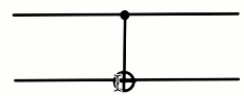
CNOT The CNOT gate is a two qubit gate that maps:

$$\text{CNOT}\,|a,\,b\rangle \to |a,\,a \oplus b\rangle. \tag{121}$$

Acting with CNOT on the basis vectors yields:

$$\begin{array}{ll} |00\rangle \to |00\rangle & |01\rangle \to |01\rangle \\ |10\rangle \to |11\rangle & |11\rangle \to |10\rangle \end{array}$$

and is represented in circuits as shown in the following figure.



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$$\text{TOFFOLI}\,|a,\,b,\,c\rangle \to |a,\,b,\,c \oplus a \cdot b\rangle \tag{123}$$

The matrix representation of the TOFFOLI gate is [14]:

```
2]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
13]:= MatrixForm[KroneckerProduct[H, H].CNOT.KroneckerProduct[H, H]]

3]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
14]:= MatrixForm[CNOT.SWAP]

4]//MatrixForm=
```

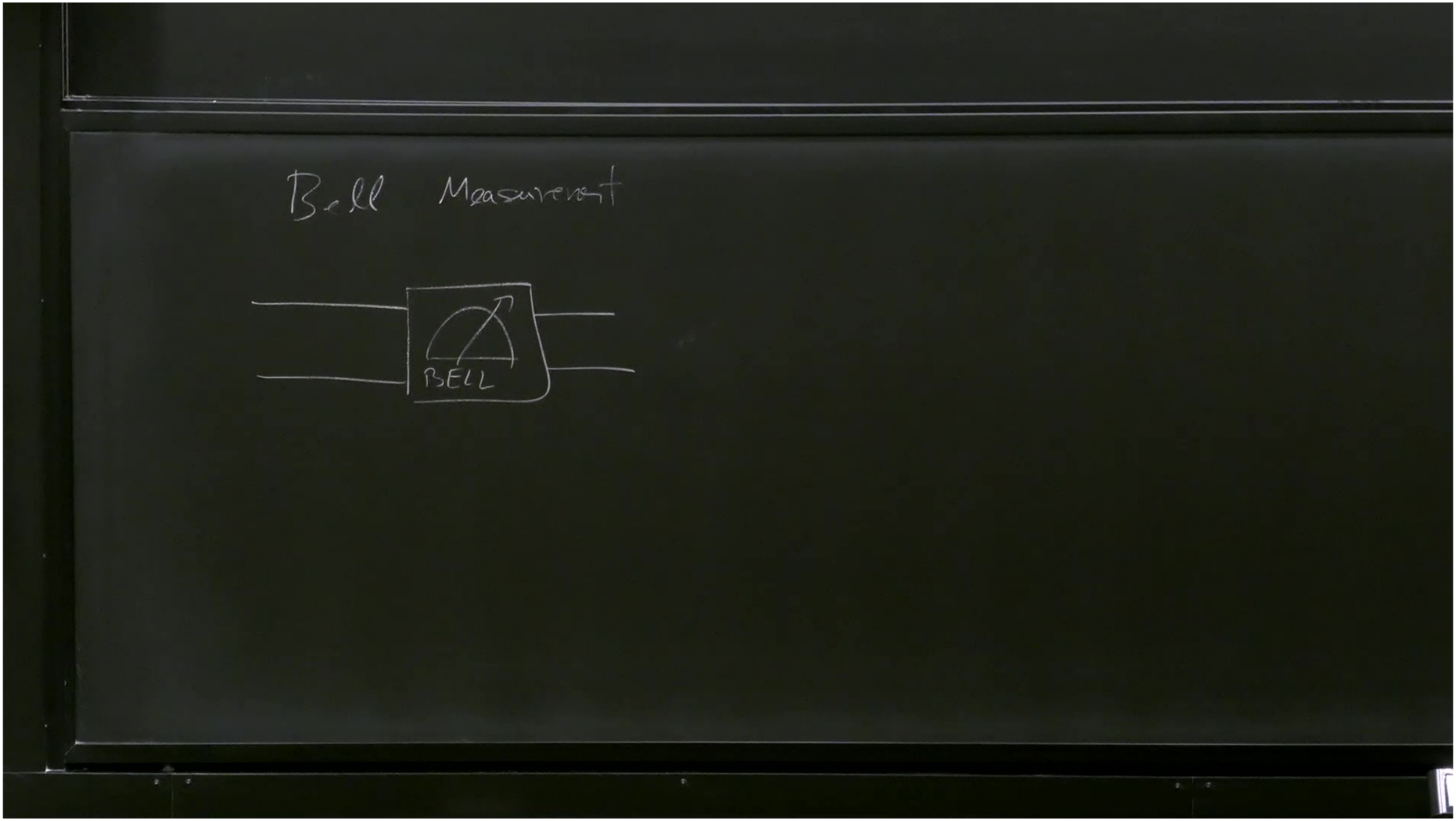
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

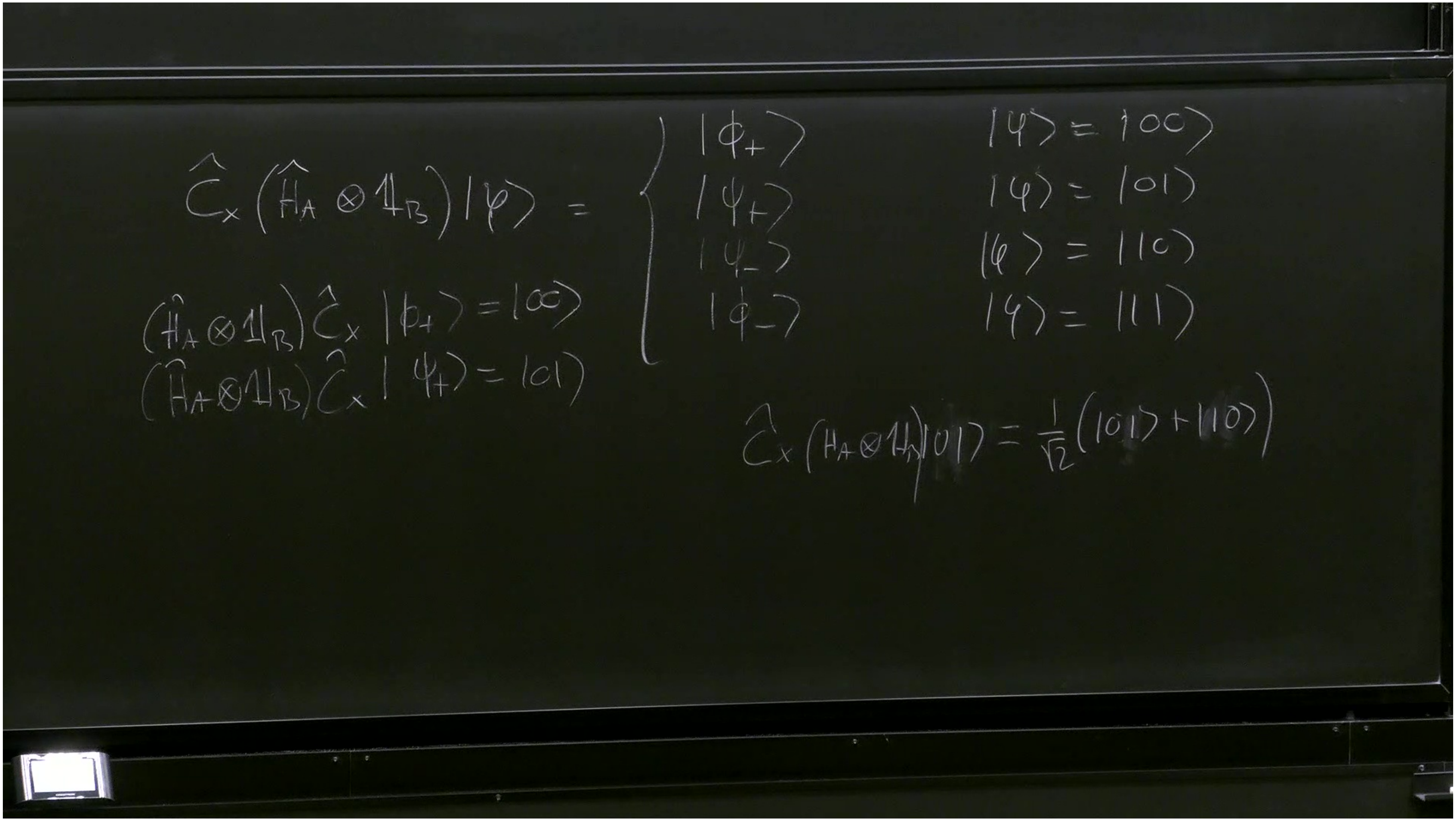
uming a matrix | Use as a *list of lists* instead

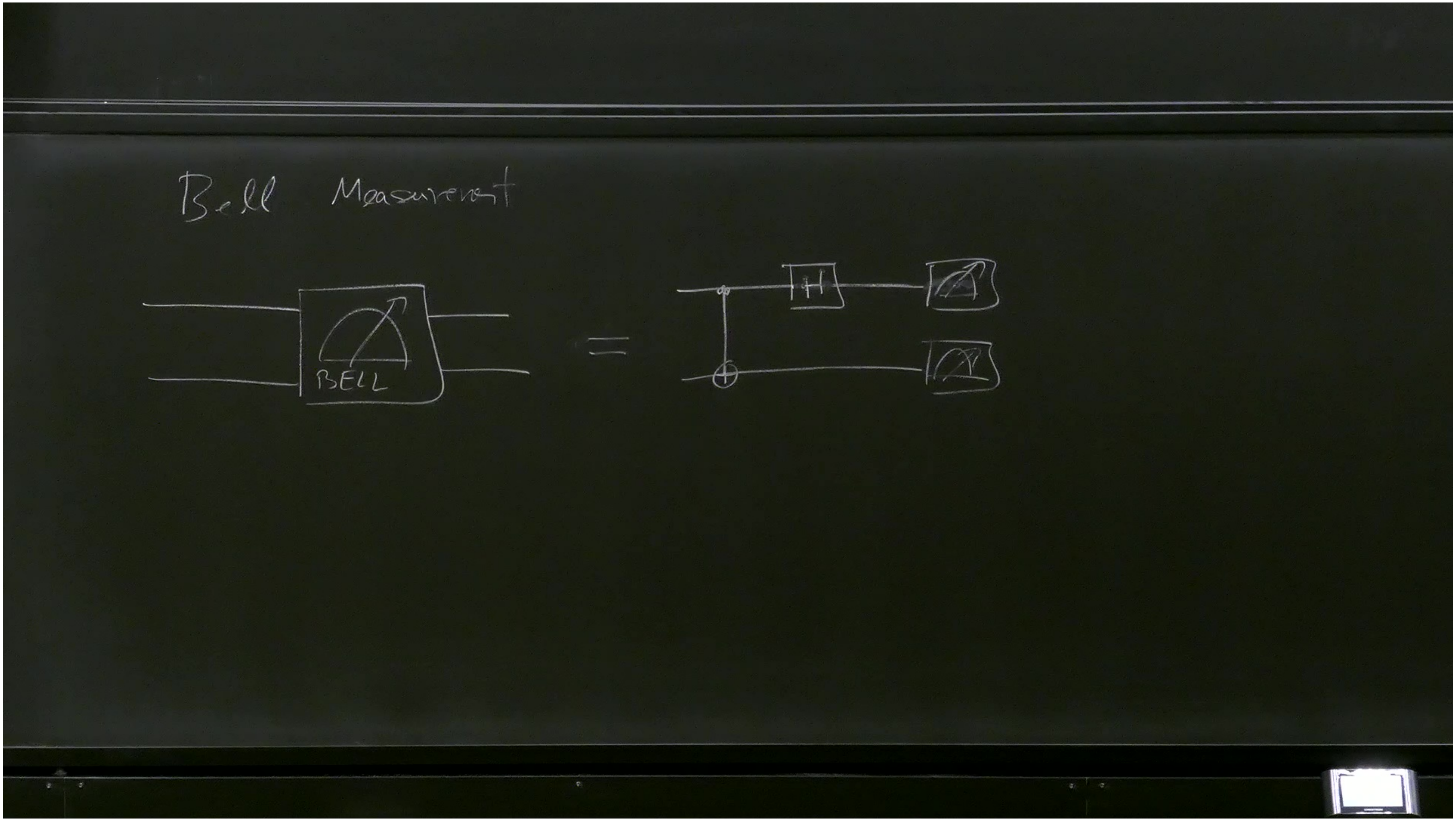
ay plot | display as | determinant | inverse | more...

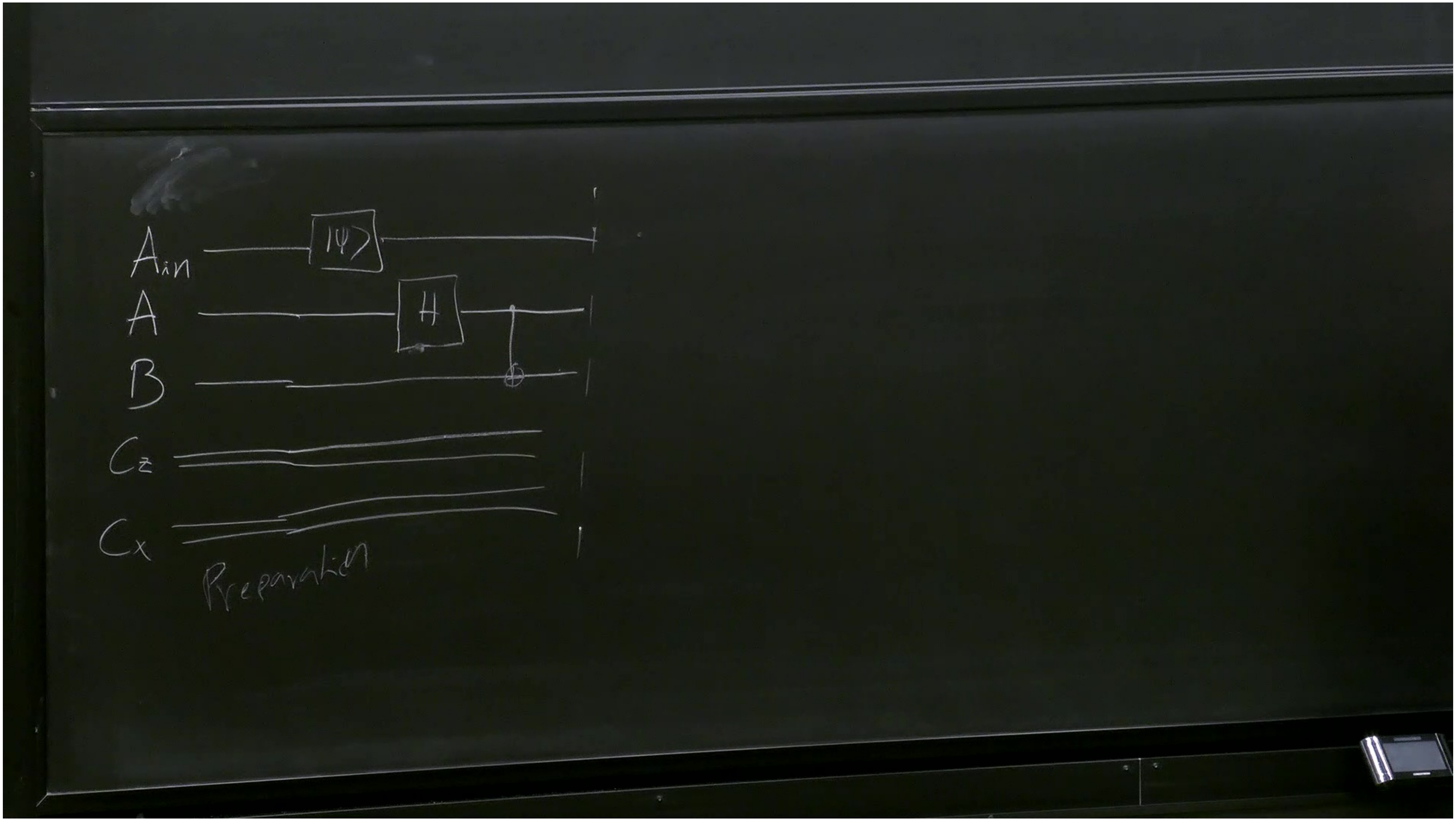
ate per-
the con-

10 Lecture 10

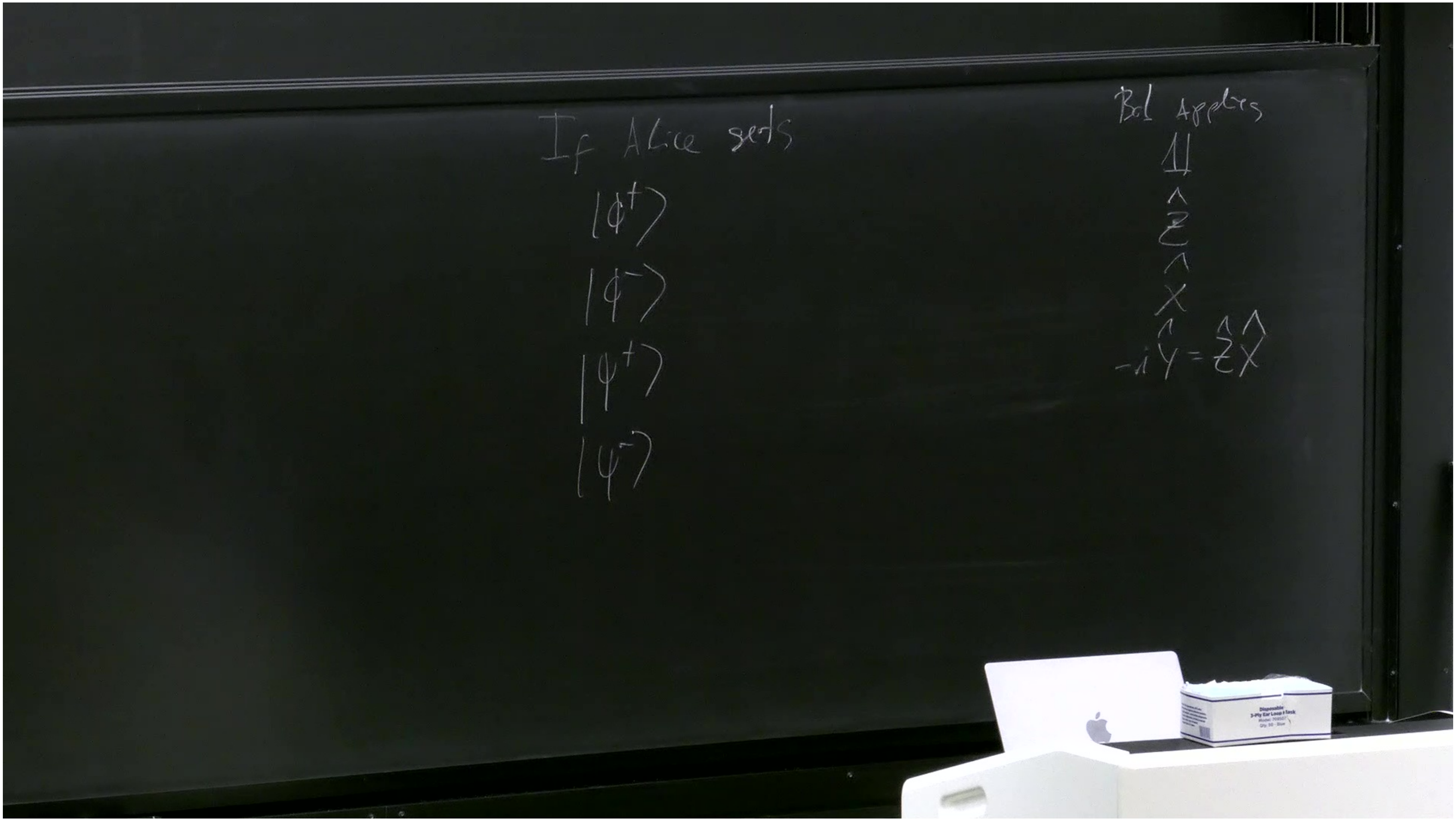

Bell Measurement
BELL

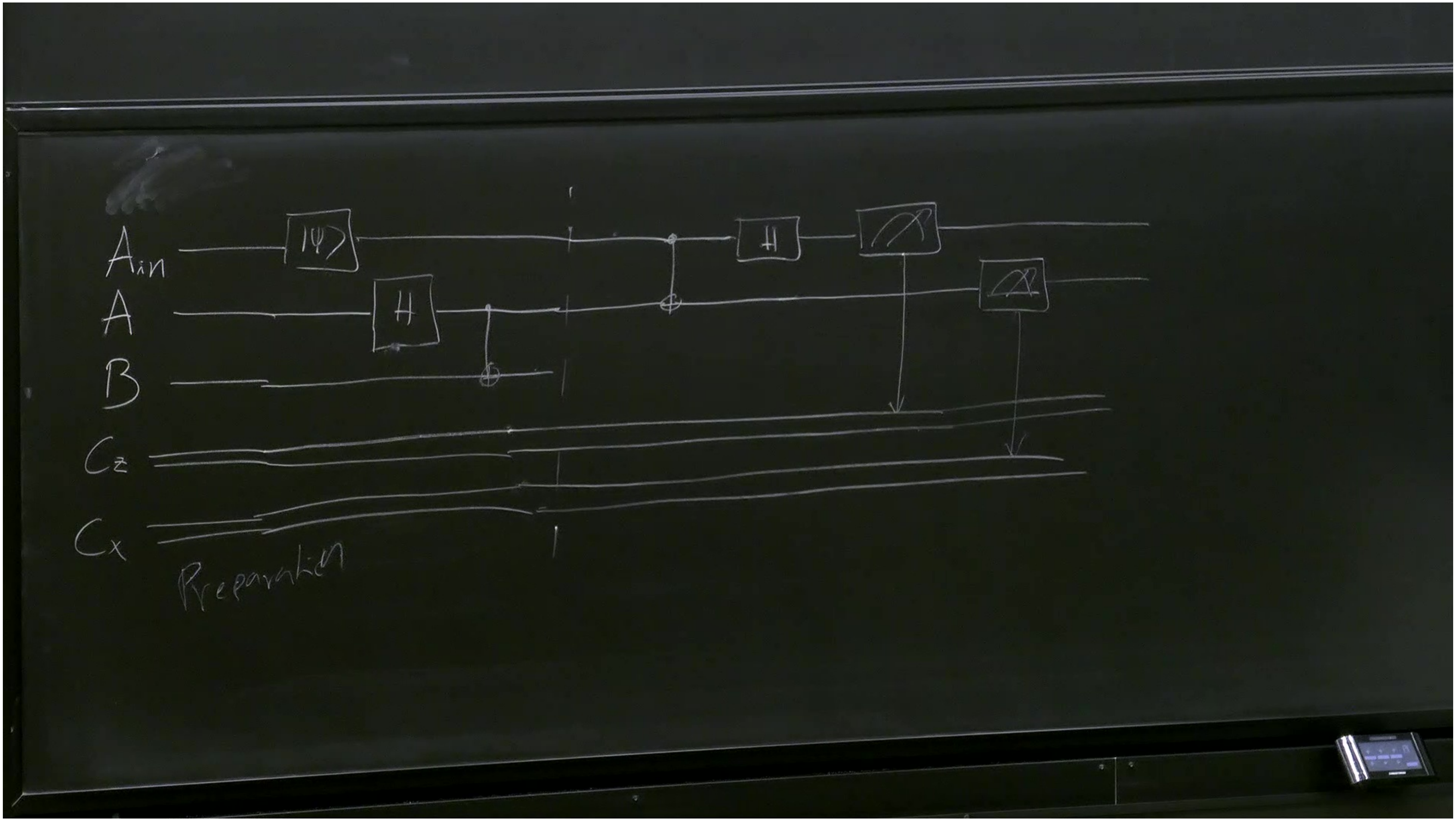

$\hat{C}_x(\hat{H}_A \otimes 1_B)|\varphi\rangle = \begin{cases} |\Phi_+\rangle \\ |\Psi_+\rangle \\ |\Psi_-\rangle \\ |\Phi_-\rangle \end{cases}$
$|\varphi\rangle = |00\rangle$
$|\varphi\rangle = |01\rangle$
$|\varphi\rangle = |10\rangle$
$|\varphi\rangle = |11\rangle$
$(\hat{H}_A \otimes 1_B)\hat{C}_x|\Phi_+\rangle = |00\rangle$
$(\hat{H}_A \otimes 1_B)\hat{C}_x|\Psi_+\rangle = |01\rangle$
$\hat{C}_x(H_A \otimes 1_B)|01\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$


Bell Measurement
BELL
=
H

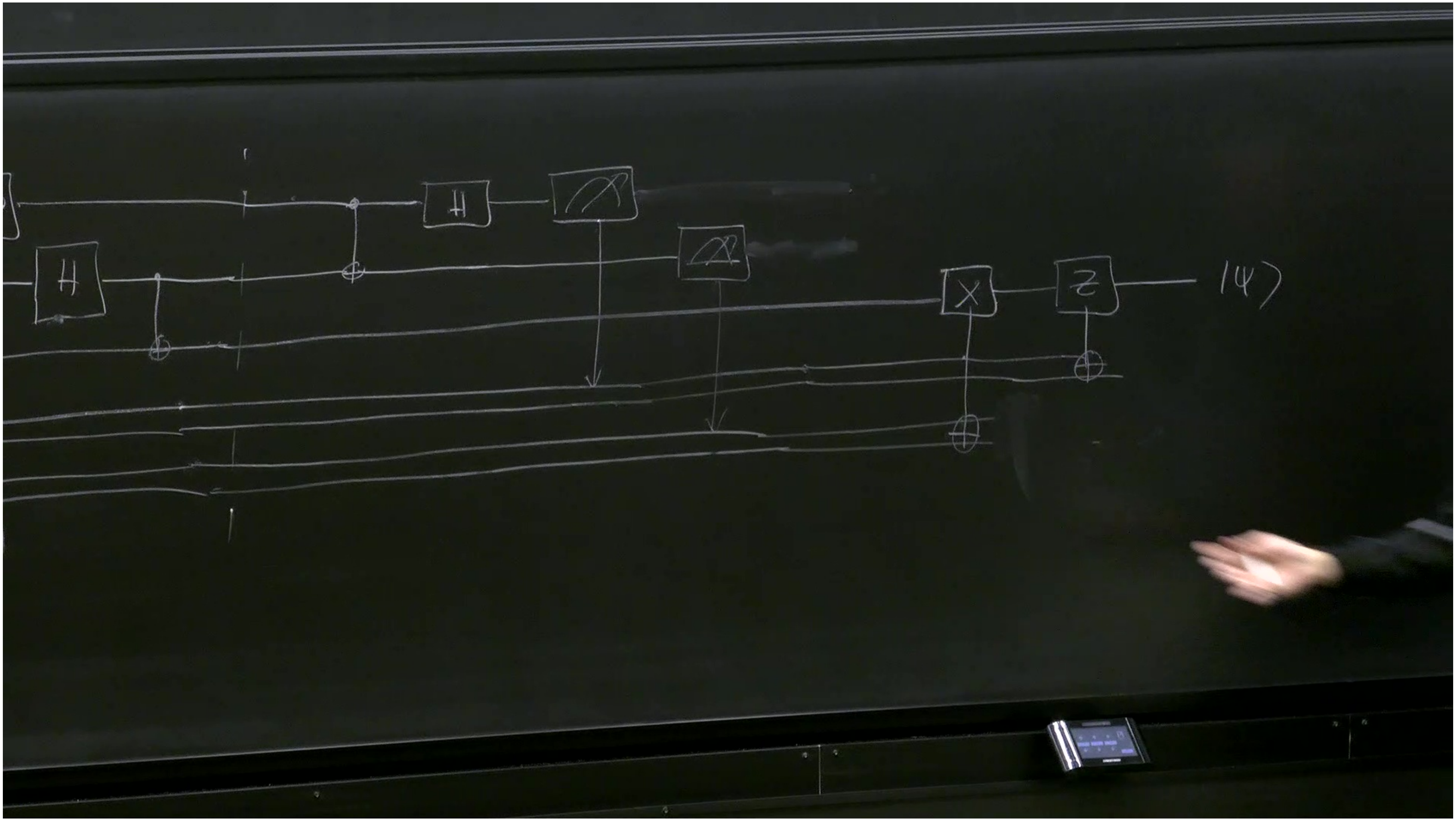


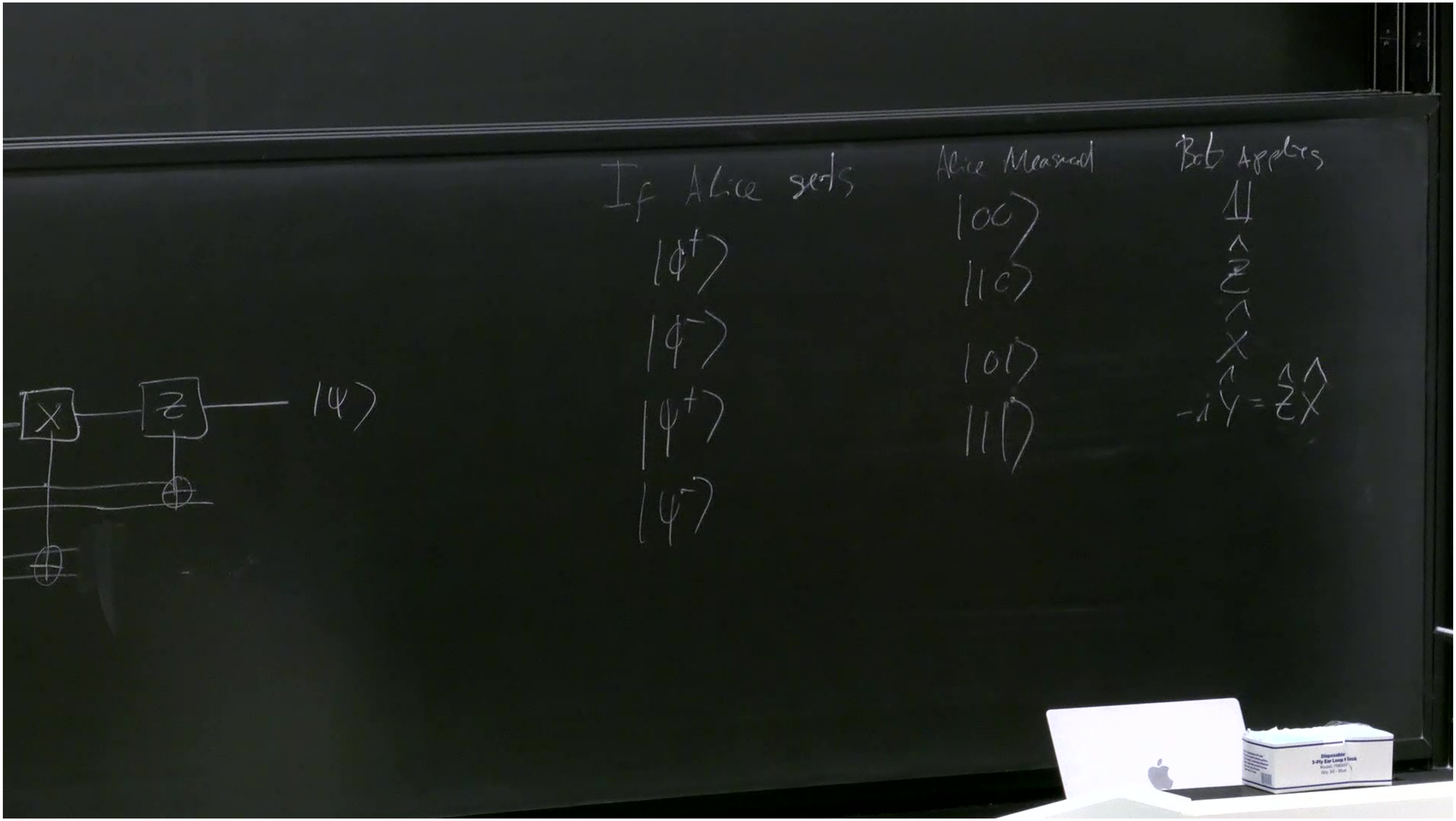
Ain
A
B
Cz
Cx
|ψ>
H
Preparation


If Alice sends
$|\phi^+\rangle$
$|\phi^-\rangle$
$|\psi^+\rangle$
$|\psi^-\rangle$
Bob Applies
$\mathbb{1}$
$\hat{Z}$
$\hat{X}$
$-i\hat{Y} = \hat{Z}\hat{X}$

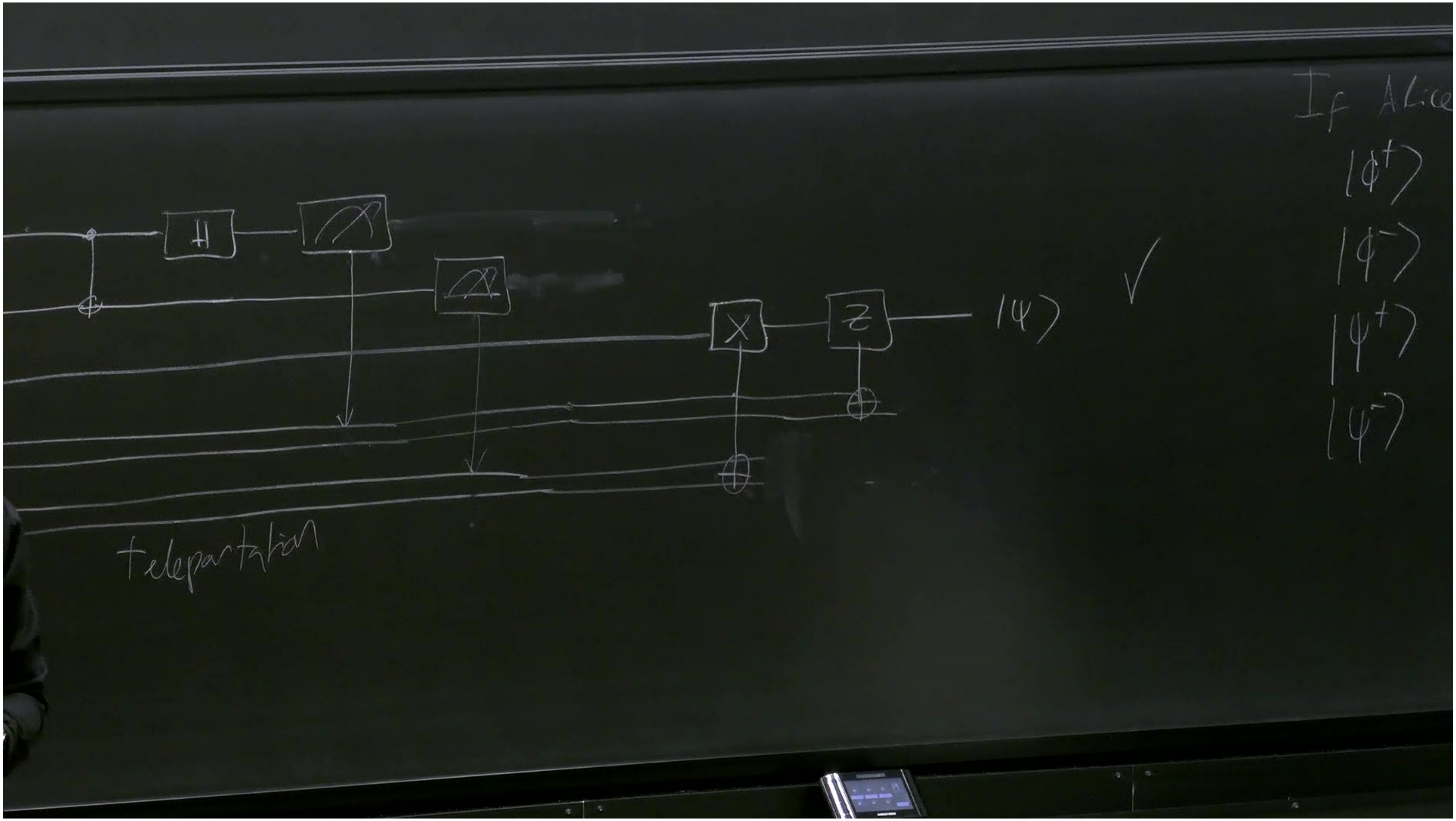


A_{in}
A
B
C_z
C_x
|ψ⟩
H
H
Preparation


H
H
X
Z
|ψ⟩



X
Z
|ψ⟩
If Alice sends
|φ+⟩
|φ−⟩
|ψ+⟩
|ψ−⟩
Alice Measured
|00⟩
|10⟩
|01⟩
|11⟩
Bob Applies
1
Ẑ
X̂
−iŶ = ẐX̂



If Alice
H
X
Z
$|\psi\rangle$
$|\phi^+\rangle$
$|\phi^-\rangle$
$|\psi^+\rangle$
$|\psi^-\rangle$
teleportation

Universality

A set S of gates is universal iff there exists a product of gates in

Example $\{\text{CNOT}, \{SU(2)\}\}$

Def (Approximately universal set of gates) A set of gates S is approxima

any unitary $\hat{U} \in SU(2^n)$ can be approximated with any given finite accuracy.

ex. $\{\hat{H}, \text{CNOT}, \hat{R}_{\pi/4}\}$, $\{\hat{H}, \hat{R}_{\pi/2}, \text{TOFFOLI}\}$

...ct of gates in S that yields any unitary $\hat{U} \in SU(2^n)$

...is approximately universal if there exists a finite product of gates such that

...nite accuracy.